Qubit Losses in Topological Color Codes

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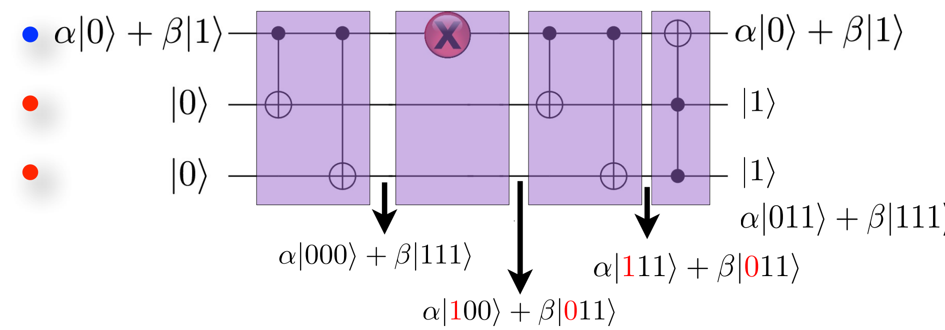
Physics and Geometry

Remembering Giuseppe Morandi

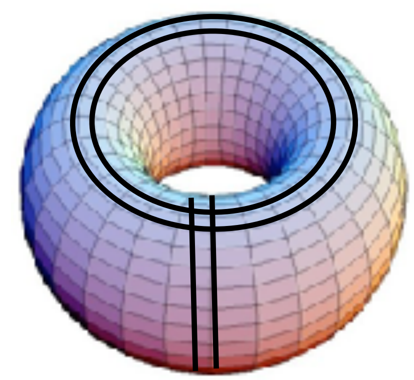
Outline of the talk

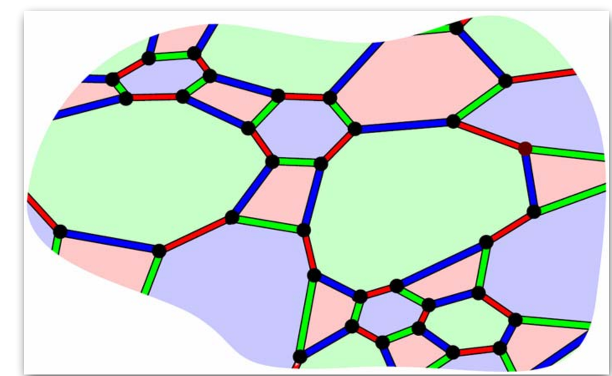
1 - Introduction to standard
quantum error correction (QEC)



2 - Introduction to topological codes

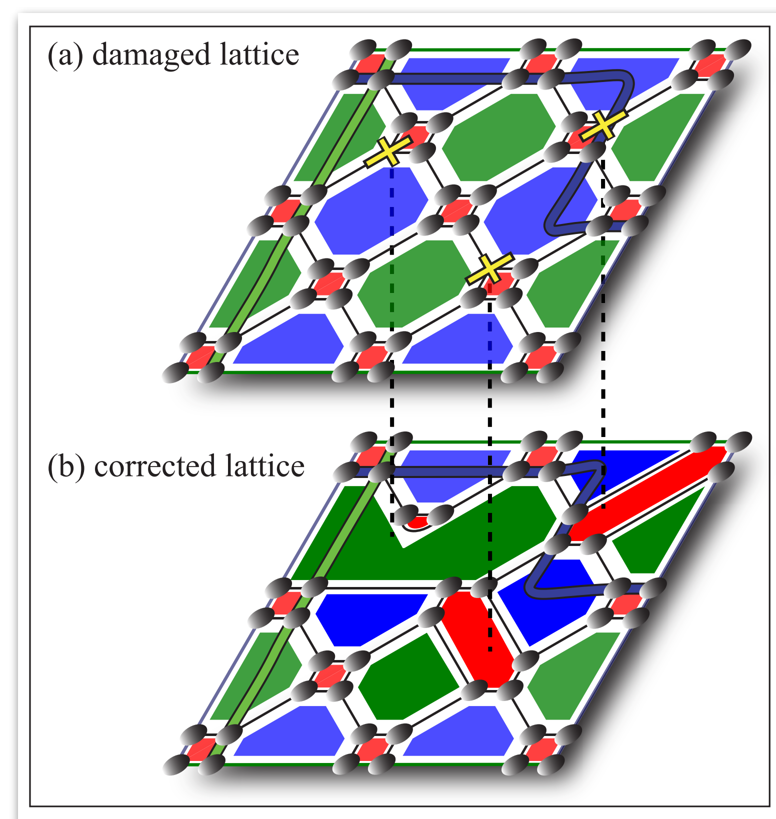
- General properties
- Topological error correction





3 - Losses in topological codes

- Toric code
- New results for the color code



A quantum computer is ...

classical input →  → classical output

a computer ...
which works based on the laws of
quantum physics

Central ingredients:

- quantum **superposition** principle
- quantum mechanical **entanglement**

Basic unit in **classical** information: the bit

0

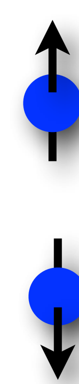
or

1

Basic unit in **quantum** information:
two-level system = quantum bit (**qubit**)

—— $|1\rangle$

—— $|0\rangle$



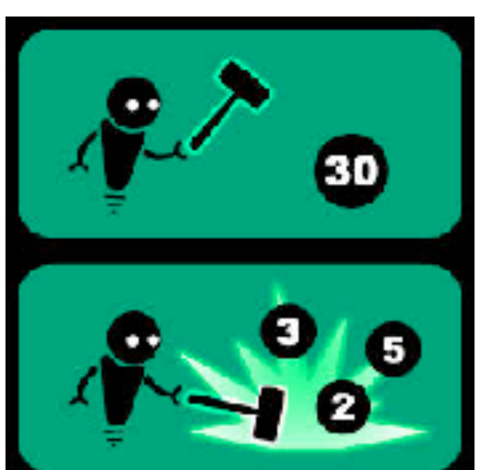
$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

with $c_0, c_1 \in \mathbb{C}$

$$p_0 = |c_0|^2,\ p_1 = |c_1|^2$$

Why should we build a large-scale and fault-tolerant quantum computer?

prime factoring
(Shor‘s alg.)



data base search
(Grover‘s alg.)

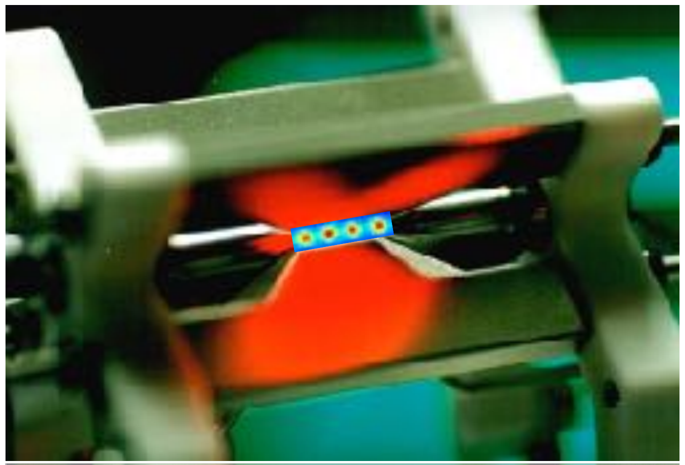
...

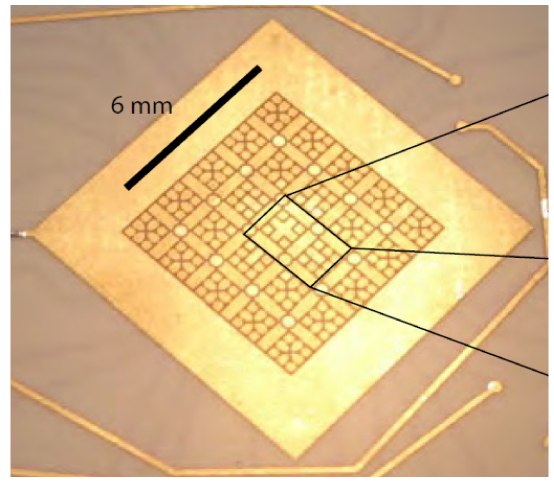
many-body
quantum
systems

universal quantum
simulator

Which physical platform?

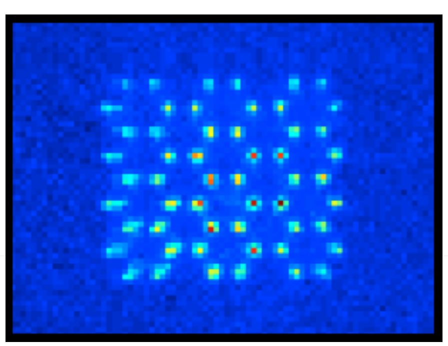
From 1D to 2D **Ion Traps**

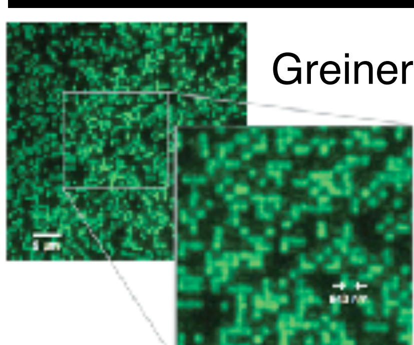




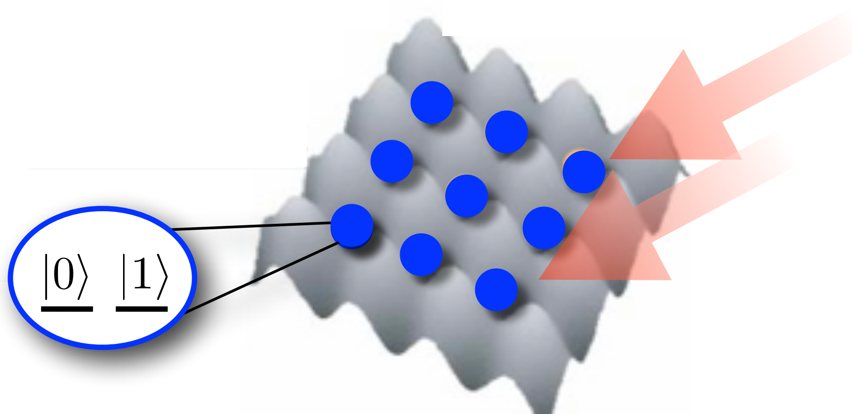
Blatt, Schmidt-Kaler, Schätz, Wineland... and many more groups

Cold Rydberg atoms





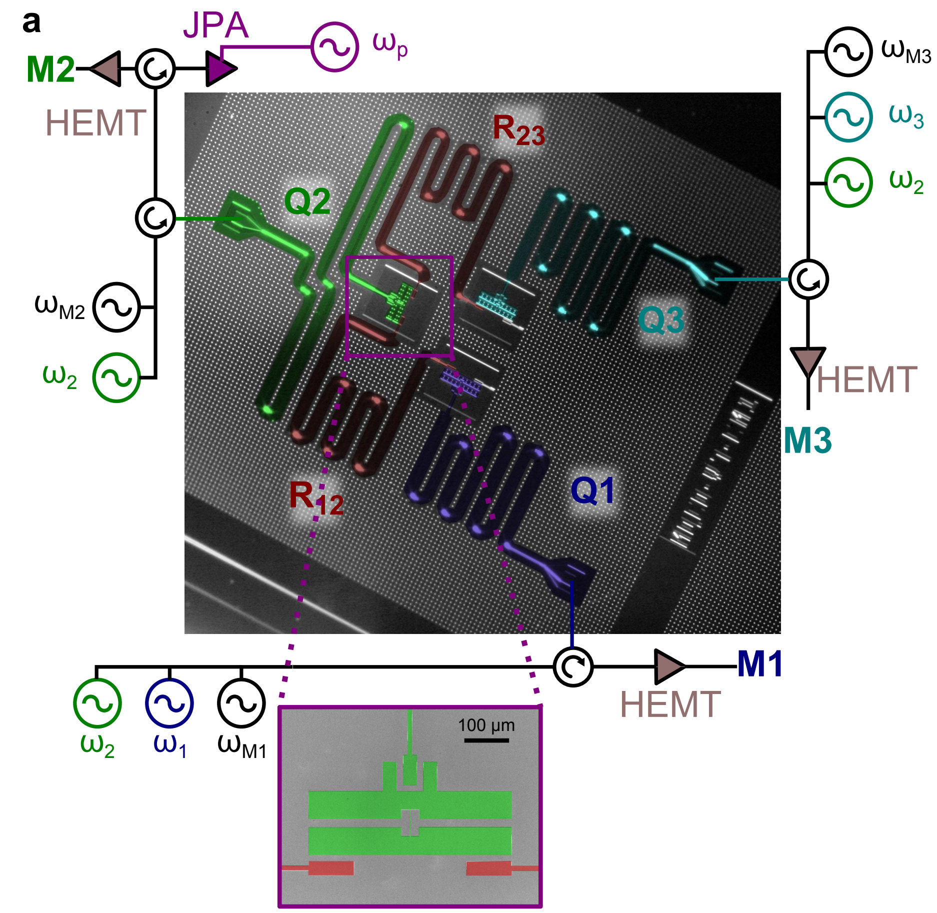
Greiner, Kuhr, Bloch groups (2011)



Weimer, Müller et al. (2010)

first 2-qubit Rydberg gates: Saffman, Browaeys & Grangier groups (2009)

super-conducting qubits



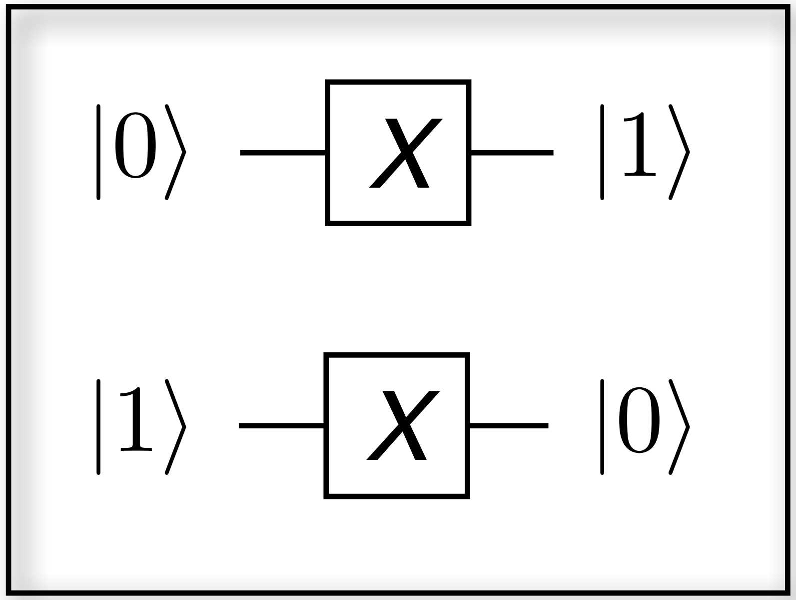
Chow, Gambetta et al., BBN,
G. Kirchmair group (Innsbruck), ...

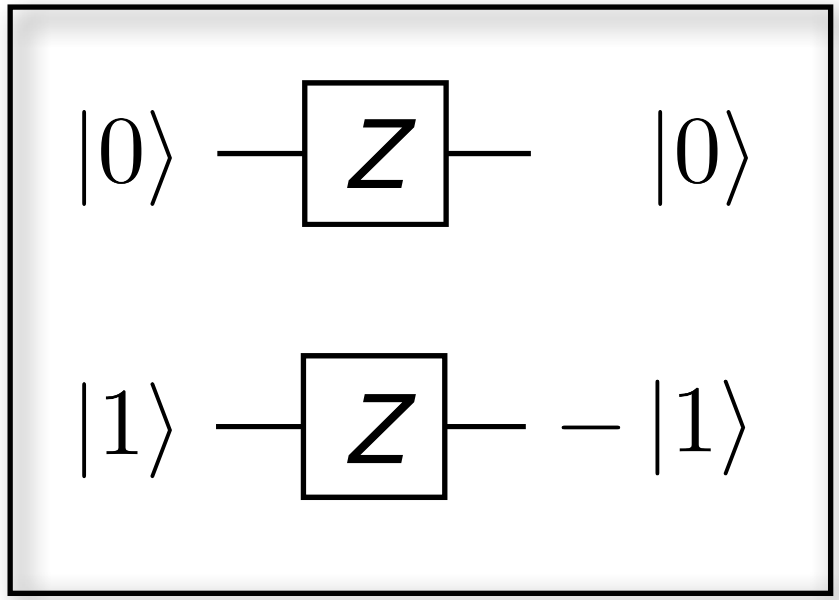
- Photons
- NV centres
- Quantum Dots

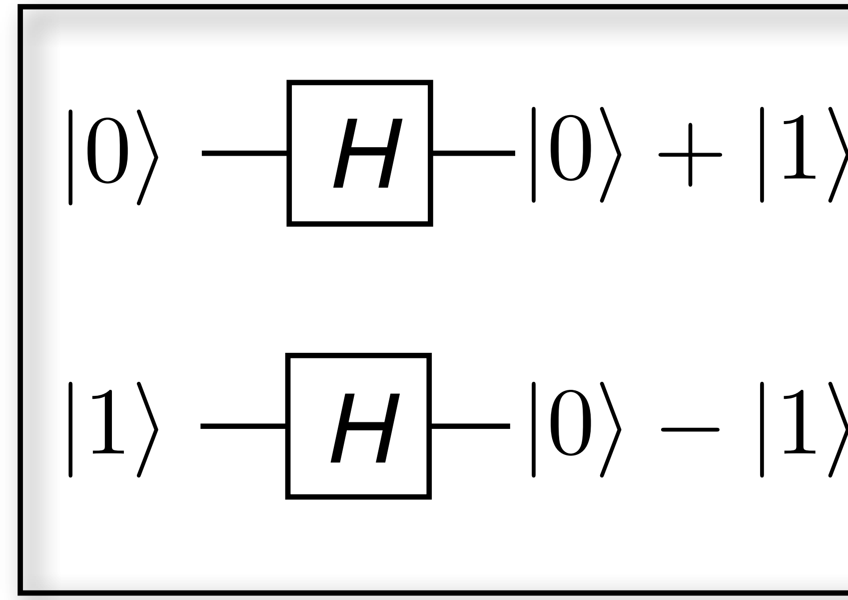
…

Operations on qubits

Single qubit gates

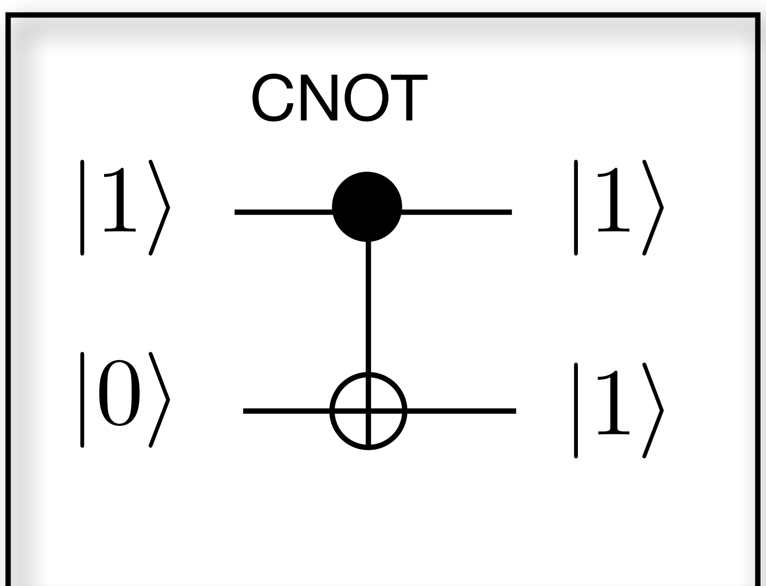


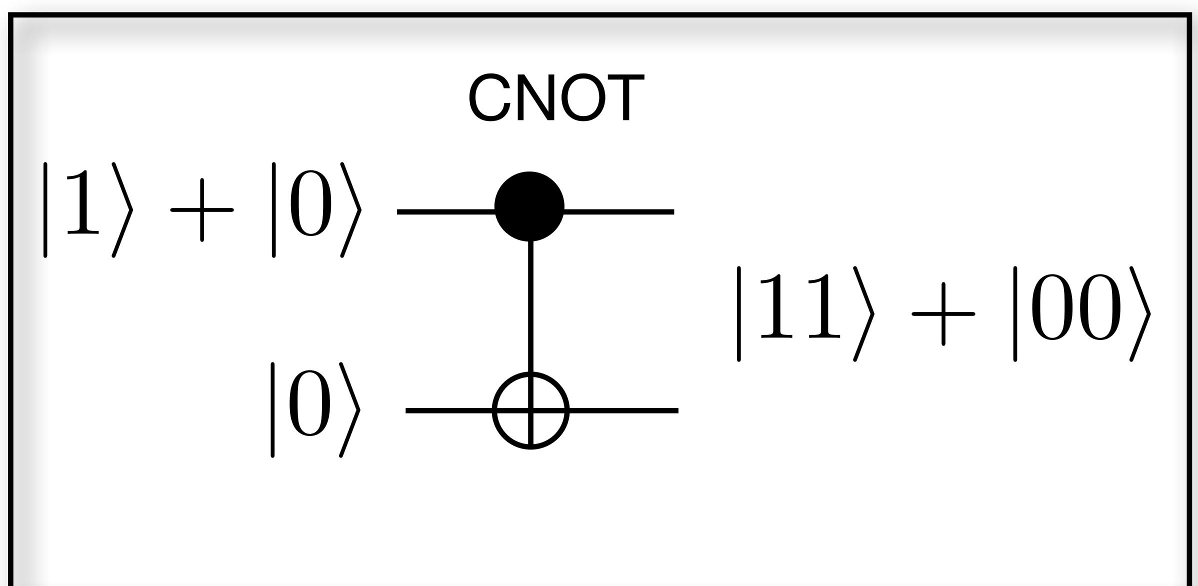




An arbitrary single qubit gate and CNOT form an universal set of quantum gates

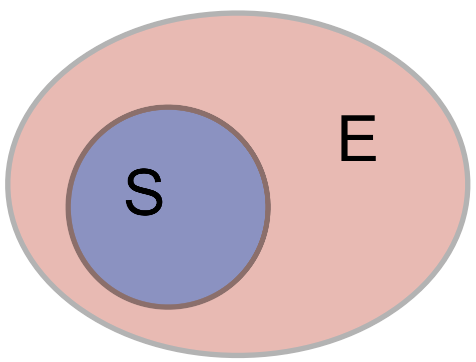
Two-qubit gates



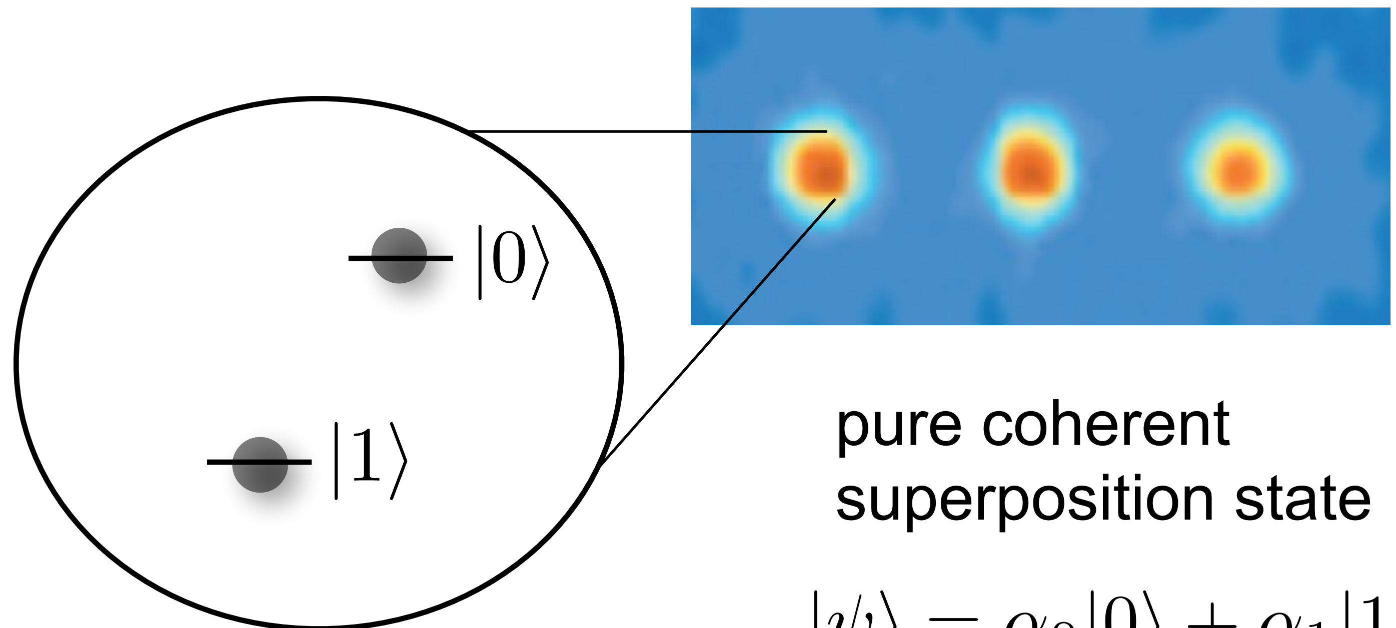


Main obstacle towards quantum computers: decoherence & errors

- Coupling to the environment causes decoherence:



example:
magnetic field
fluctuations



pure coherent superposition state

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

dephasing

$$\alpha_0 e^{i\phi_0}|0\rangle + \alpha_1 e^{i\phi_1}|1\rangle$$

$$\rho = |\alpha_0|^2|0\rangle\langle 0| + |\alpha_1|^2|1\rangle\langle 1|$$

classical mixture, phase information lost

error channels

$|0\rangle$ — X — $|1\rangle$

$|1\rangle$ — X — $|0\rangle$

X-error (bit flip)

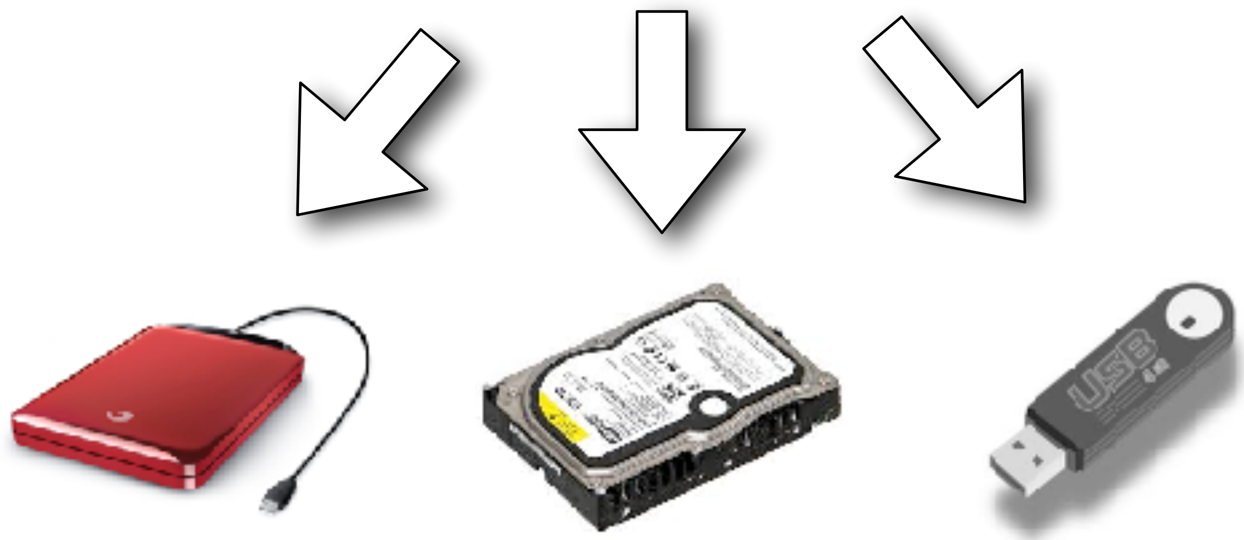
- similarly: Z-error (phase flip) $\quad |0\rangle \to |0\rangle$, $\quad |1\rangle \to -|1\rangle$

- other channels: amplitude damping, **qubit loss**, ...

$$\boxed{\rho \mapsto \varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger}$$

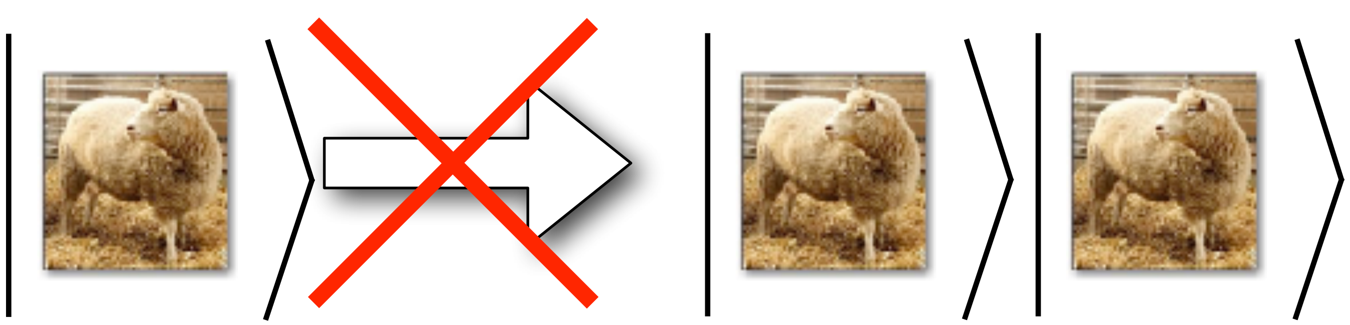
Need for error correction: naive approach ... fails

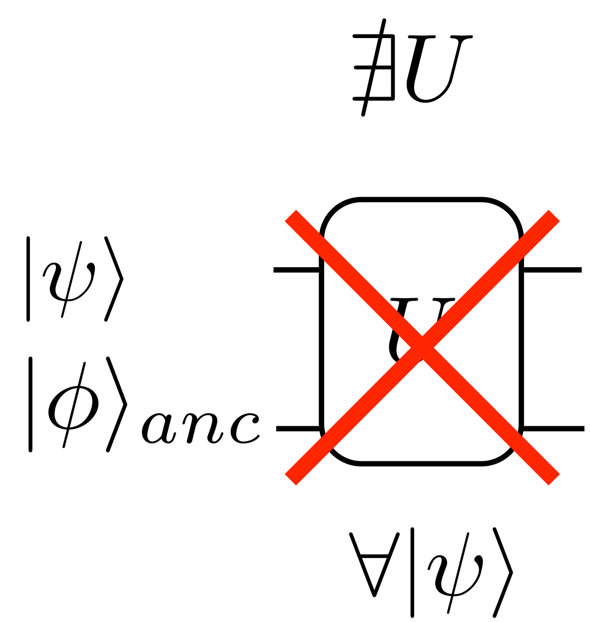
- protection by redundancy  ...011010...



...011010... **...010010...** ...011010... recover: ...011010...

- not possible in this straightforward way: **no-cloning theorem for quantum states**





- quantum states can‘t be copied,
  but keep idea of **redundancy** and **majority vote**...

Quantum error correcting codes

- 3-qubit bit-flip code

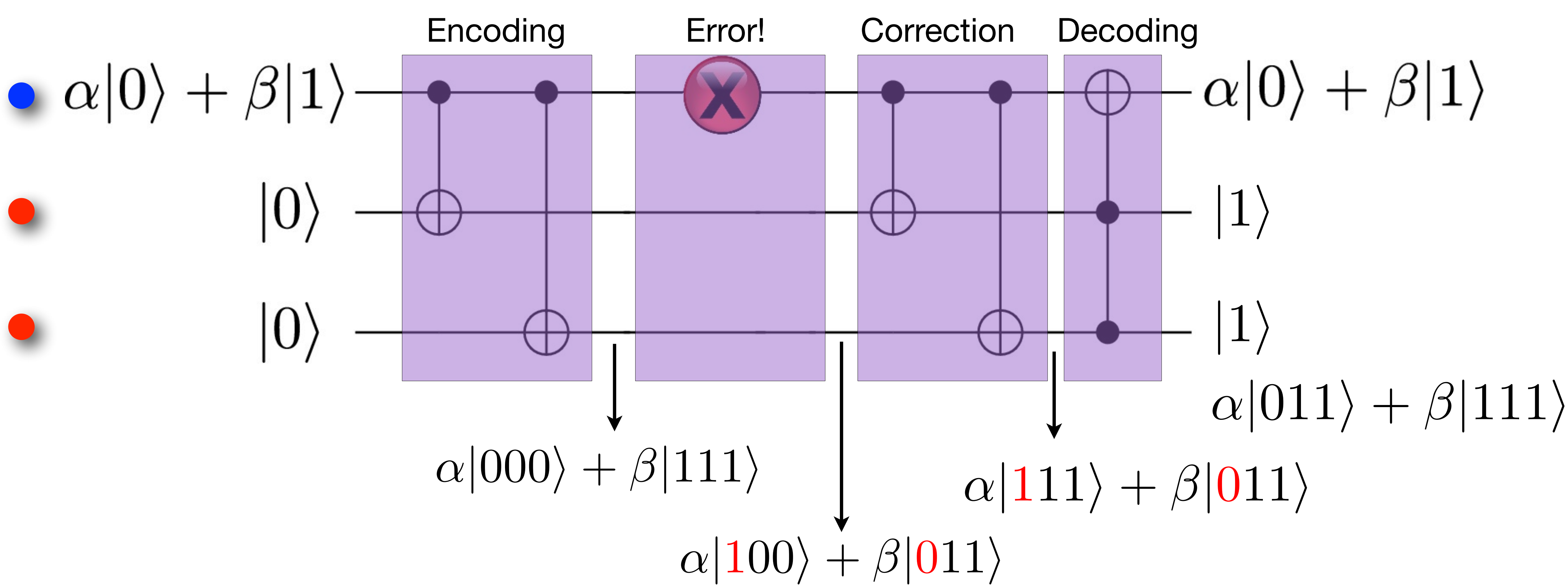
P. Shor 1995

- use two ancilla qubits
- code corrects 1 bit-flip error

- Encoding
- Detection of the error
- Correction
- Decoding

Encoding

$$|0\rangle \rightarrow |0_L\rangle = |000\rangle$$
$$|1\rangle \rightarrow |1_L\rangle = |111\rangle$$



Quantum Error Correcting (QEC) codes: Stabiliser codes

D. Gottesman (1996)

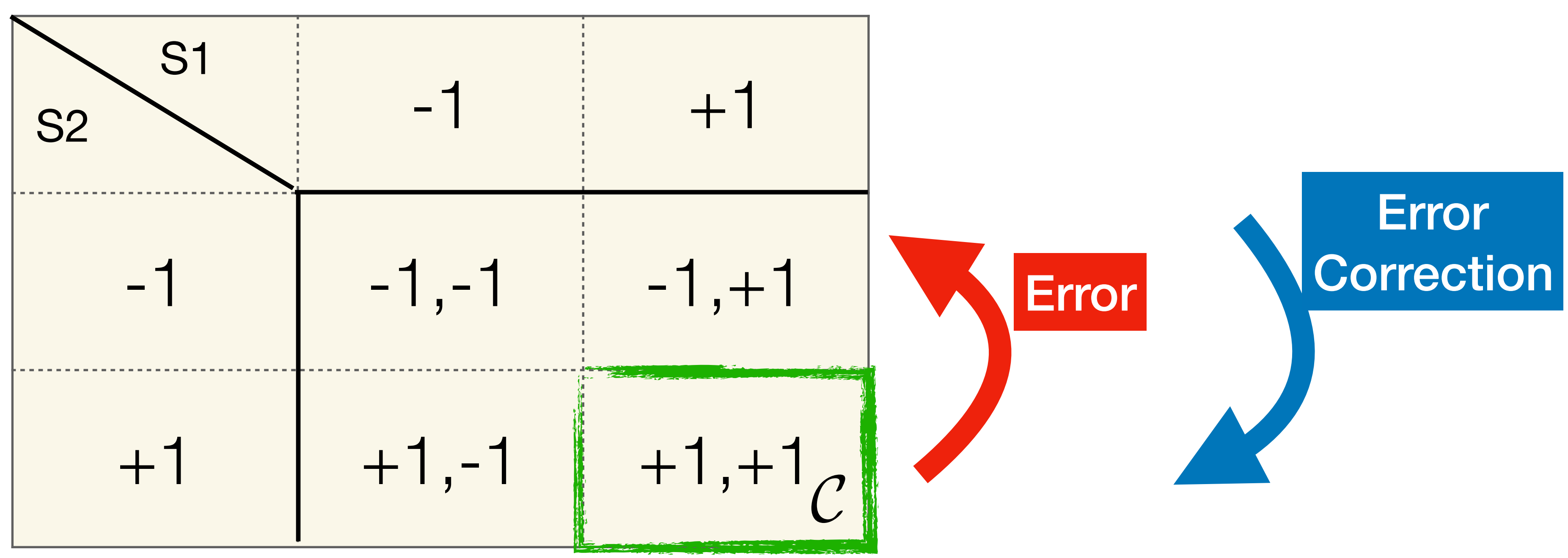
Pauli Group $\mathcal{P}$

- Product of single Pauli on *n* qubits

Stabilizer group $\mathcal{S}$

- Operators in $\mathcal{P}$ that mutually commute (Abelian subgroup of $\mathcal{P}$)
- Defines the code space $\mathcal{C}$ as the common +1 eigenspace of all its elements

The code space stores the logical state(s)



An error brings the logical state out of $\mathcal{C}$

Quantum Error Correcting (QEC) codes: Stabiliser codes

D. Gottesman (1996)

- 3-qubit bit-flip code

$$|0\rangle \rightarrow |0_L\rangle = |000\rangle$$
$$|1\rangle \rightarrow |1_L\rangle = |111\rangle$$

is an example of a stabiliser code that corrects only one error.

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

code space is fixed by a set of (**commuting**) stabiliser generators $\{Z_1Z_2,\, Z_2Z_3\}$

$$Z_1Z_2|\psi\rangle = +|\psi\rangle$$
$$Z_2Z_3|\psi\rangle = +|\psi\rangle$$

logical operators

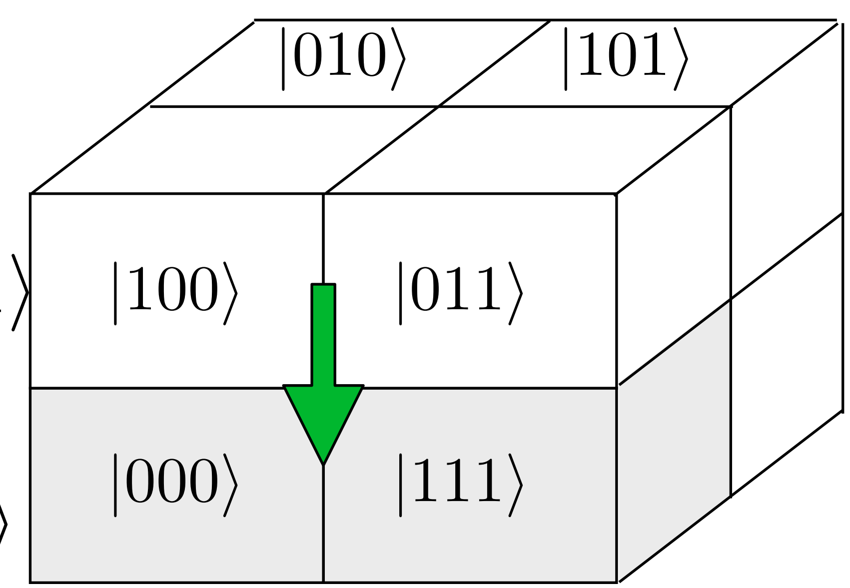
$$\bar{Z} = Z_1Z_2Z_3$$
$$\bar{X} = X_1X_2X_3$$

$$\boxed{\begin{aligned} Z_1Z_2|\phi_{\mathrm{err}}\rangle &= -|\phi_{\mathrm{err}}\rangle \\ Z_2Z_3|\phi_{\mathrm{err}}\rangle &= +|\phi_{\mathrm{err}}\rangle \end{aligned}}$$

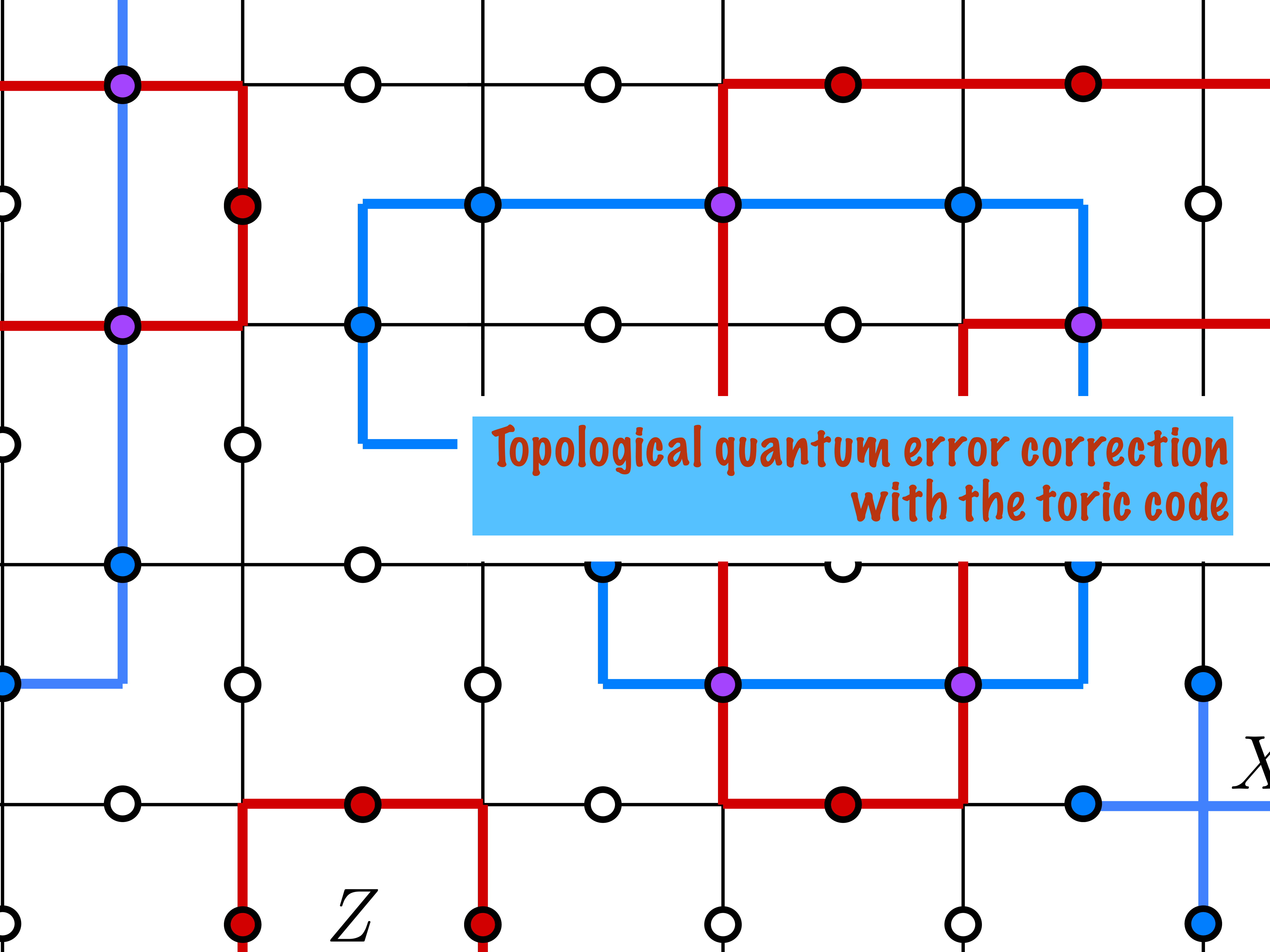
X_1 error

$$\alpha|100\rangle + \beta|011\rangle$$

$$\alpha|000\rangle + \beta|111\rangle$$

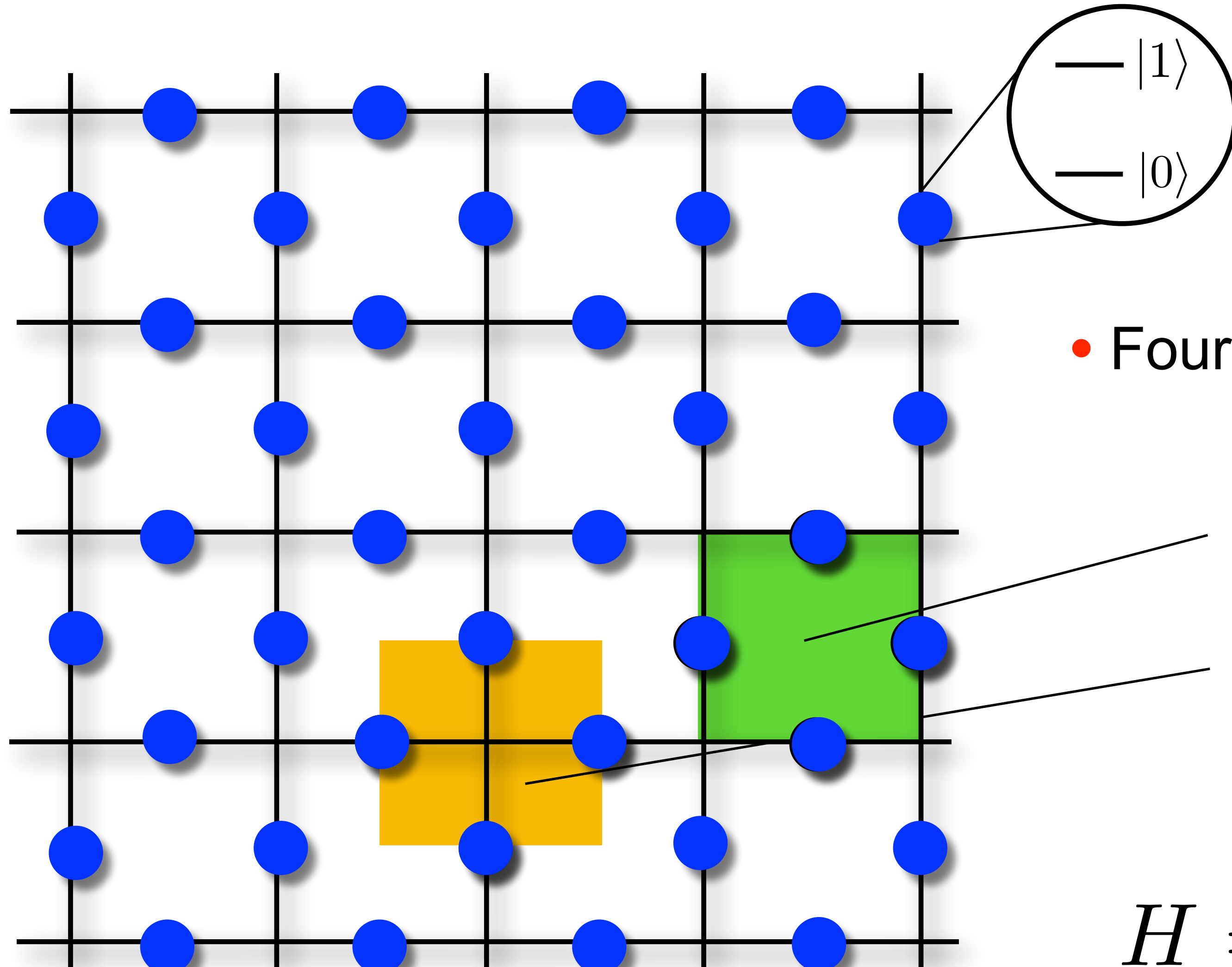


Topological quantum error correction with the toric code



Kitaev's toric code

A. Yu. Kitaev, *Annals of Physics*, **303** (2003)



- Spin-1/2 (qubits) located on the edges of a two-dimensional square lattice

- Four-qubit stabiliser generators

$$S_z = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$$

$$S_x = \sigma_5^x \sigma_6^x \sigma_7^x \sigma_8^x$$

$$H = -J \sum_{\square} S_z - J \sum_{+} S_x$$

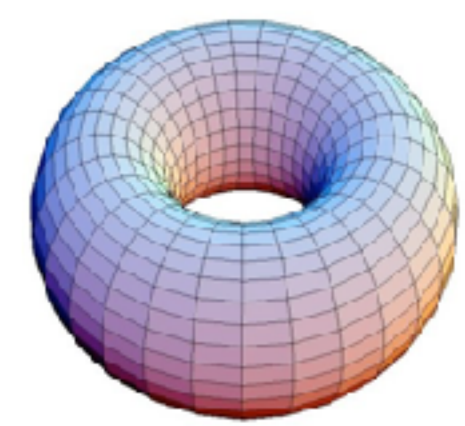
- all terms commute: $[S_x, S_z] = 0$ for all stabilisers ➡ Hamiltonian exactly solvable.

- The ground state manifold coincides with the code space

Ground state degeneracy of the toric code

$$H = -J \sum_{\square} S_z - J \sum_{+} S_x$$

- ground state $|K\rangle$ with $\{S_x^{(i)}|K\rangle = |K\rangle, S_z^{(j)}|K\rangle = |K\rangle\}$ for all X- and Z-stabilisers

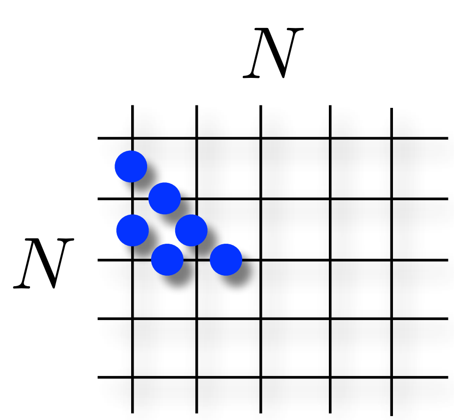


- **ground state degeneracy**, which depends on the
  - boundary conditions
  - and the **topology** of the surface on which the lattice lives
- on a **torus**: periodic boundary conditions:

N^2 plaquettes, $2N^2$ physical qubits

$N^2 - 1$ indep. X-stabilizers, $N^2 - 1$ Z-stabilizers,

since $\prod_{\square} S_z = \prod_{+} S_x = \mathbf{1}$

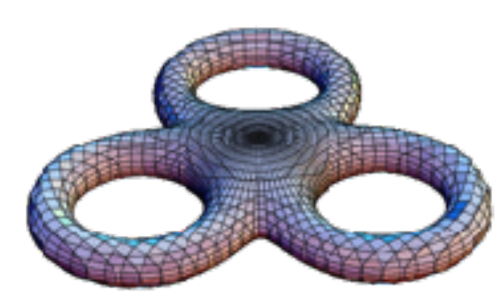


- $2N^2 - 2(N^2 - 1) = 2$ conditions missing:
- encode **two logical qubits** in the ground state manifold

For a different manifold

$\chi = 2 - 2g$

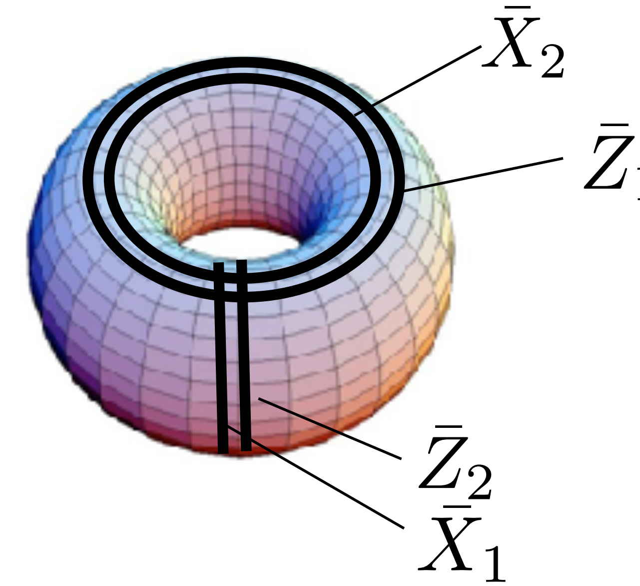
$2g$ logical qubits



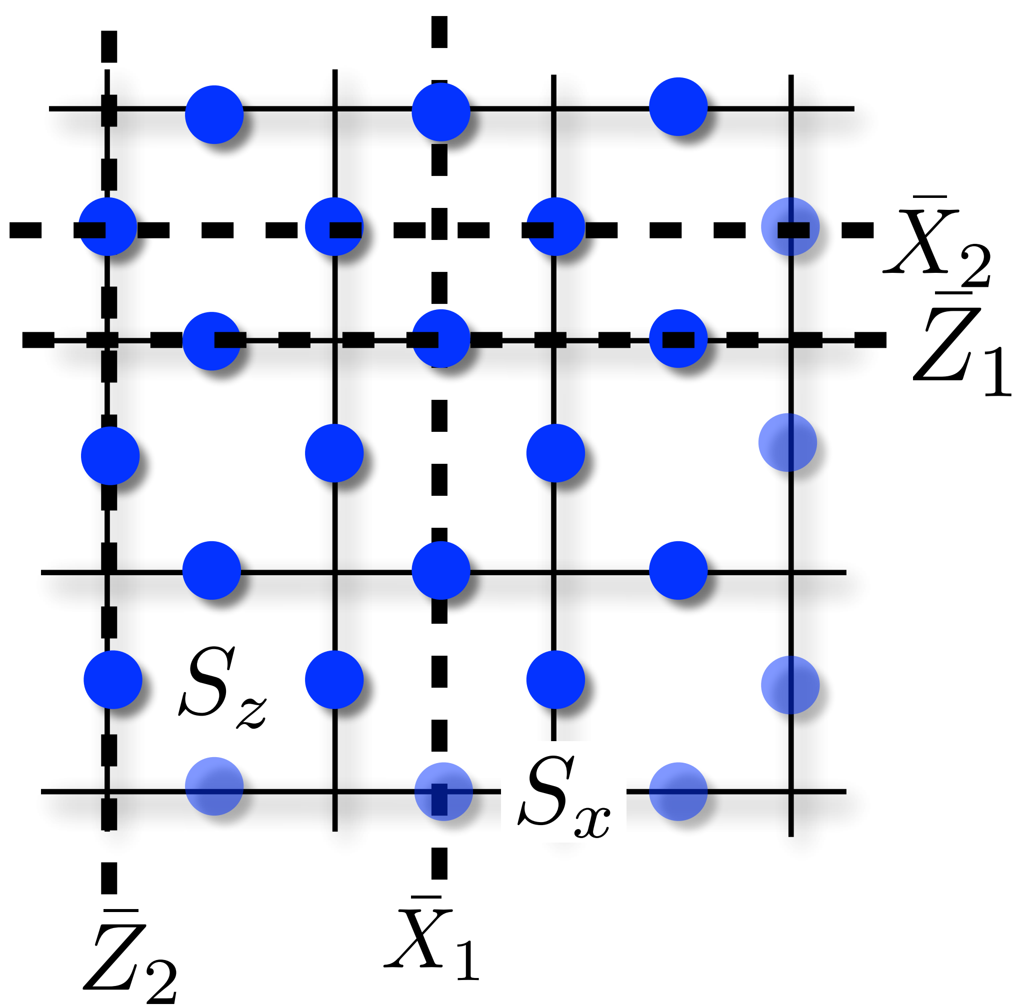
Logical qubits

Logical operators

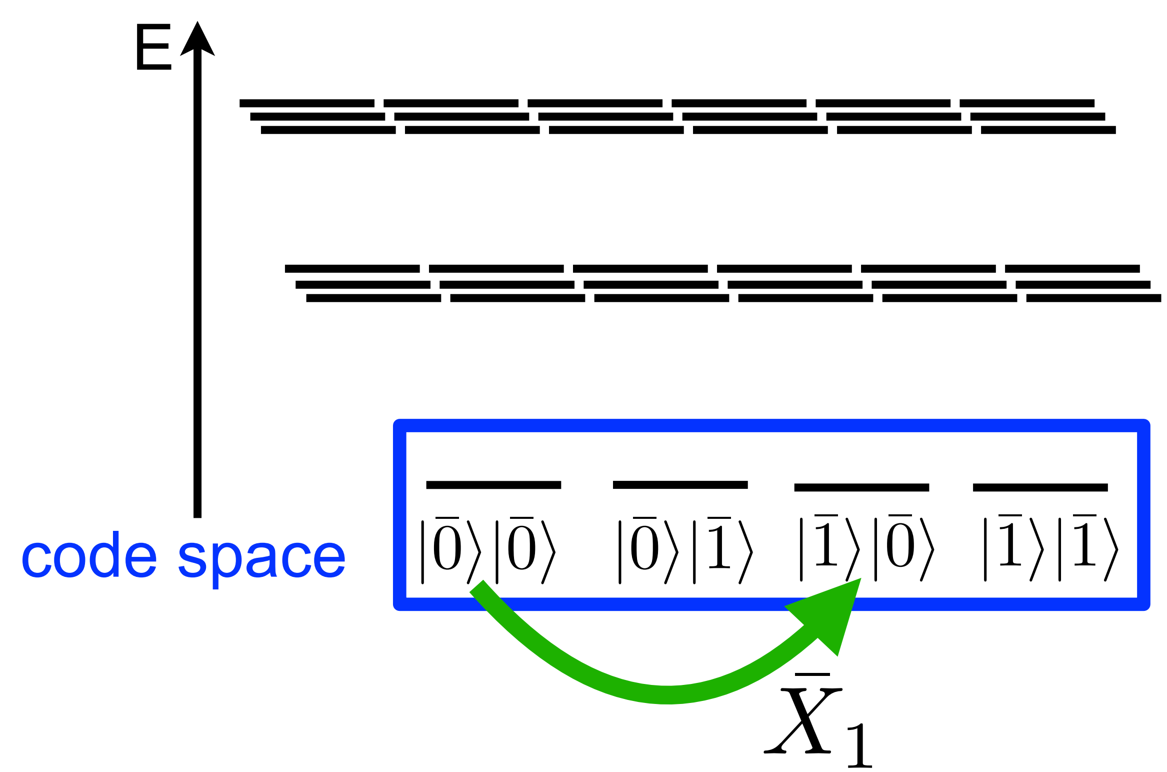
- must commute with all stabilisers
- must be independent
- must respect the usual AC relations



The stabilisers always excite an even number of qubits

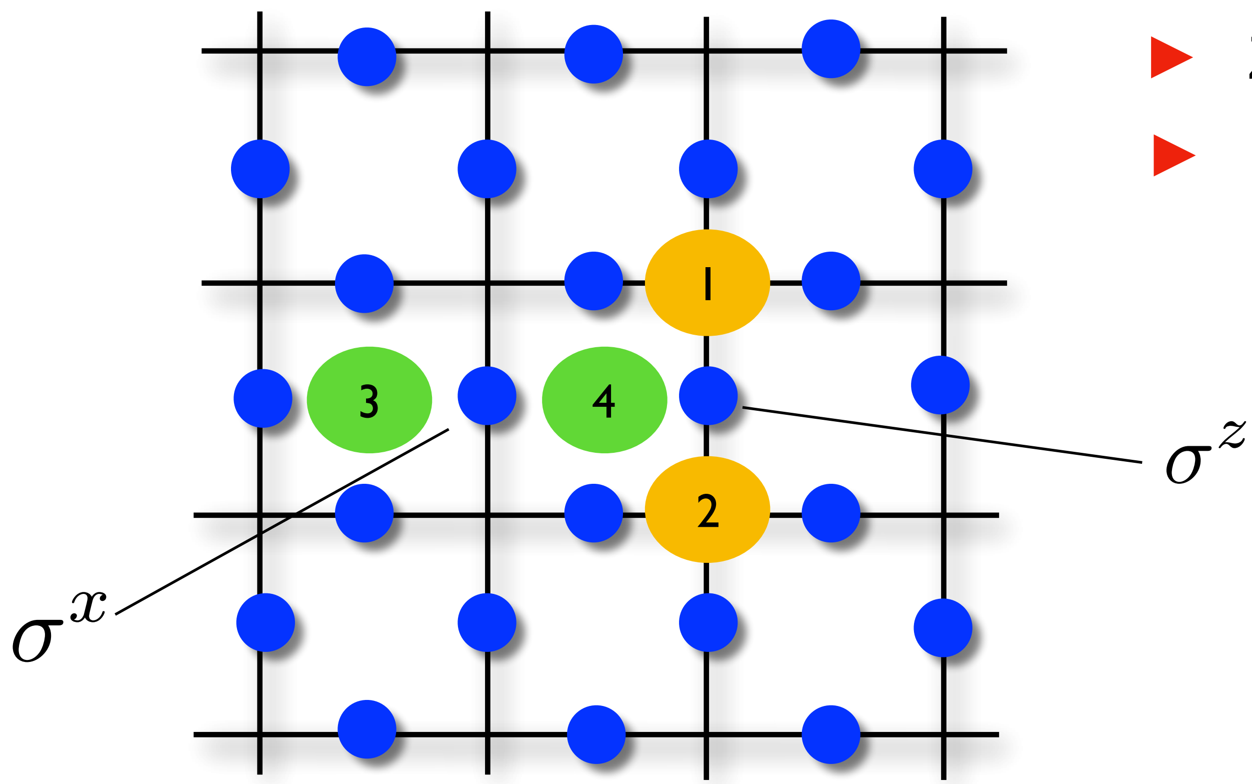


Logical operators are strings that **percolate** through the lattice and act non trivially in the code space



Errors in the toric code

- errors = gapped **excitations** = **local** stabilizer violations

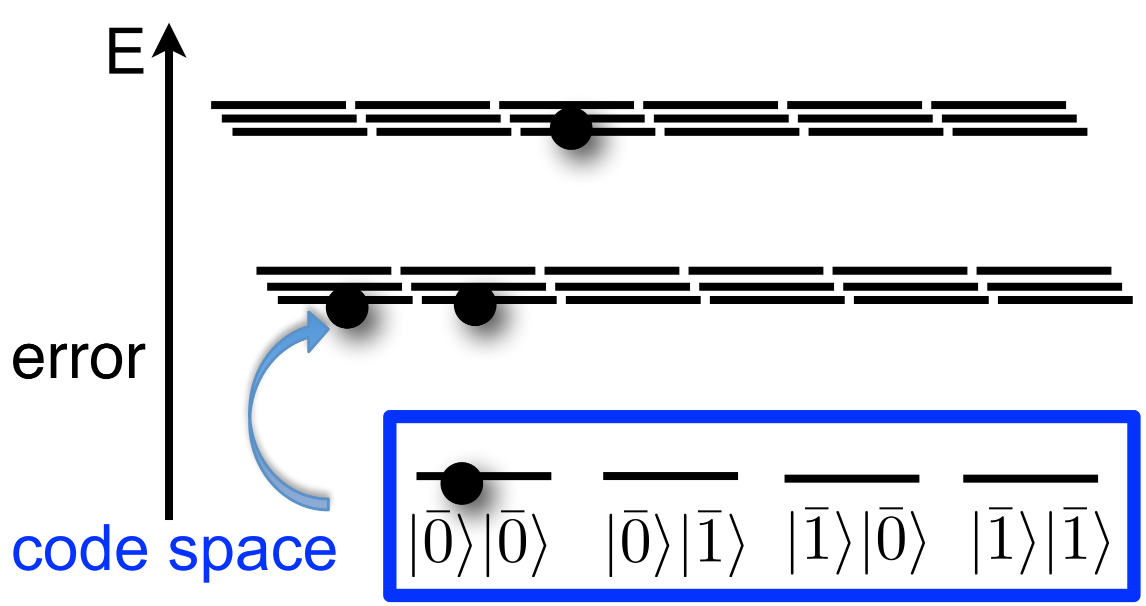


▶ Z-type errors are detected by X-type stabiliser

▶ X-type errors are detected by Z-type stabiliser

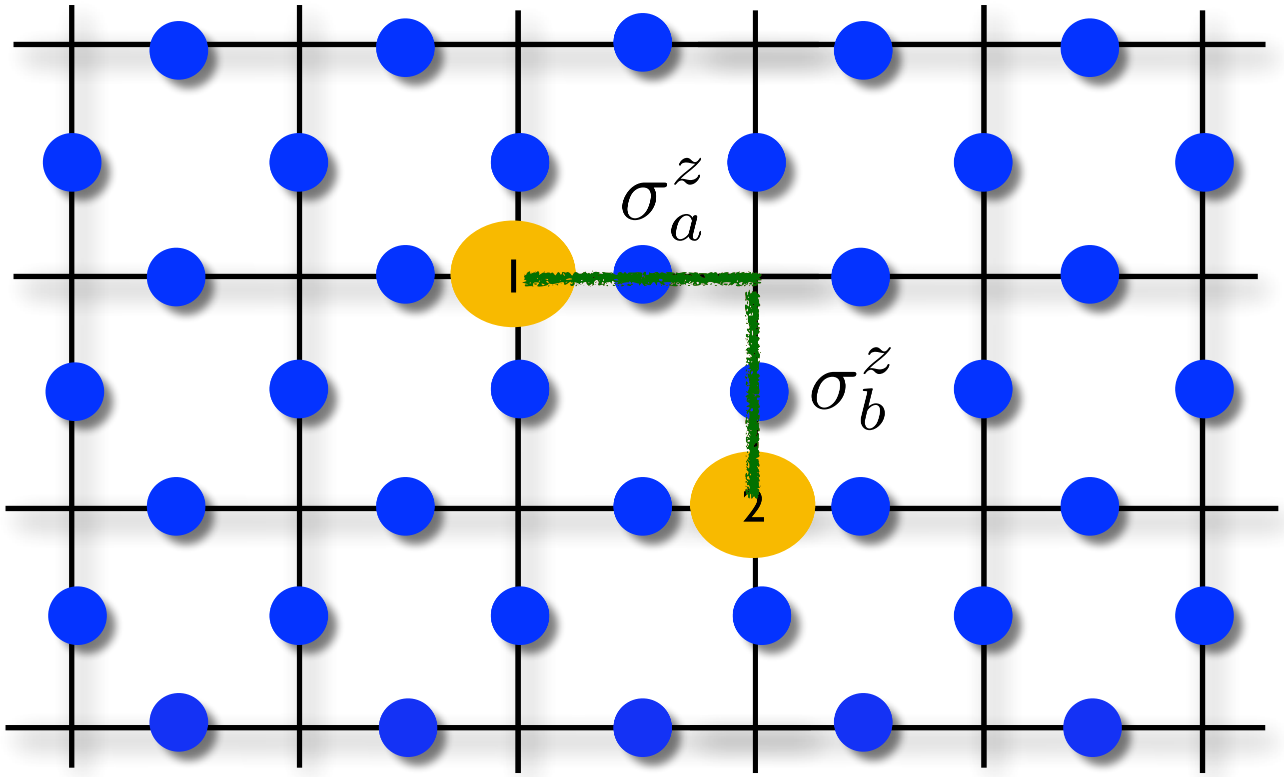
▶ Two excitations have a non trivial statistics:
They are Abelian anyons

The state picks up a phase -1 if
1 is moved around 4



The goal of quantum error correction is to find the chain of errors (or one equivalent) that has produced the excitations detected

Errors in the toric code - 2

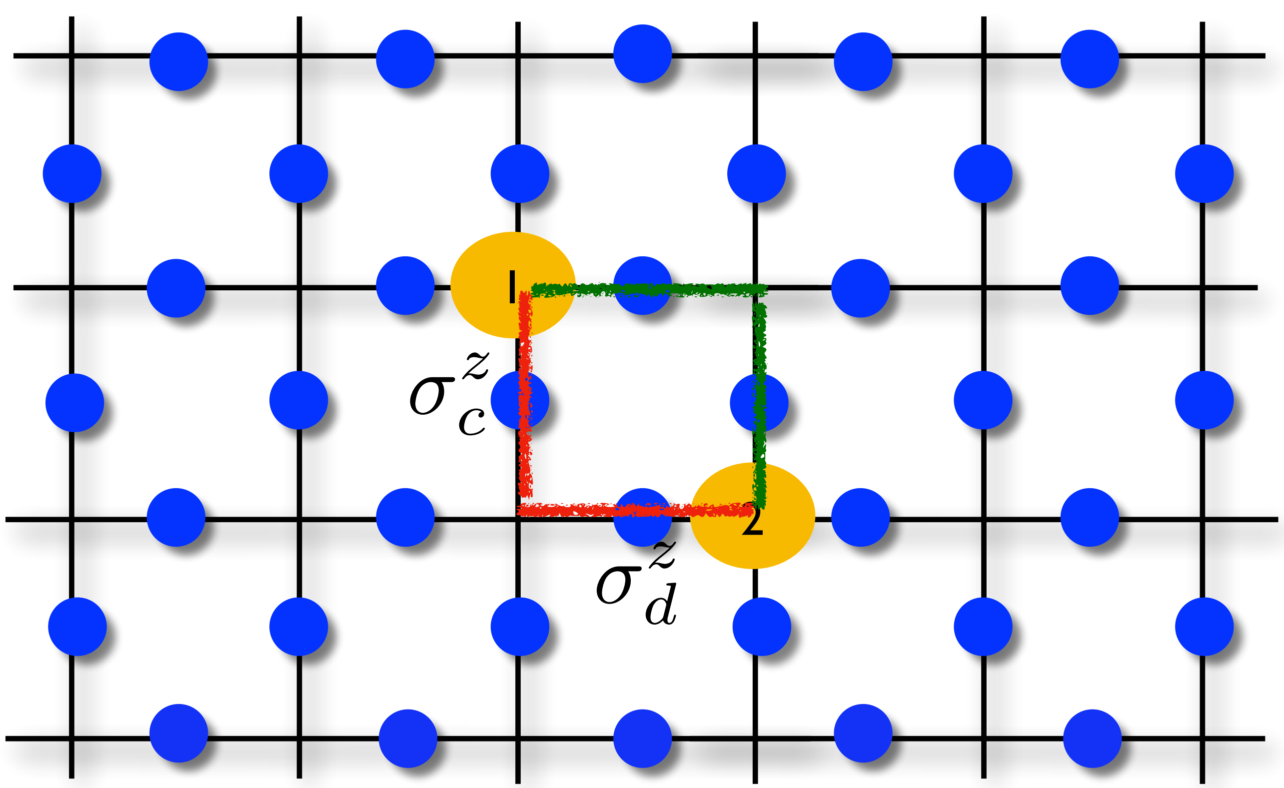


Two excitations detected

possible chain of errors $E = \sigma_a^z \sigma_b^z$

correct it by applying $C = \sigma_a^z \sigma_b^z$

$$CE = \mathbf{1}$$



What if the chain of errors was

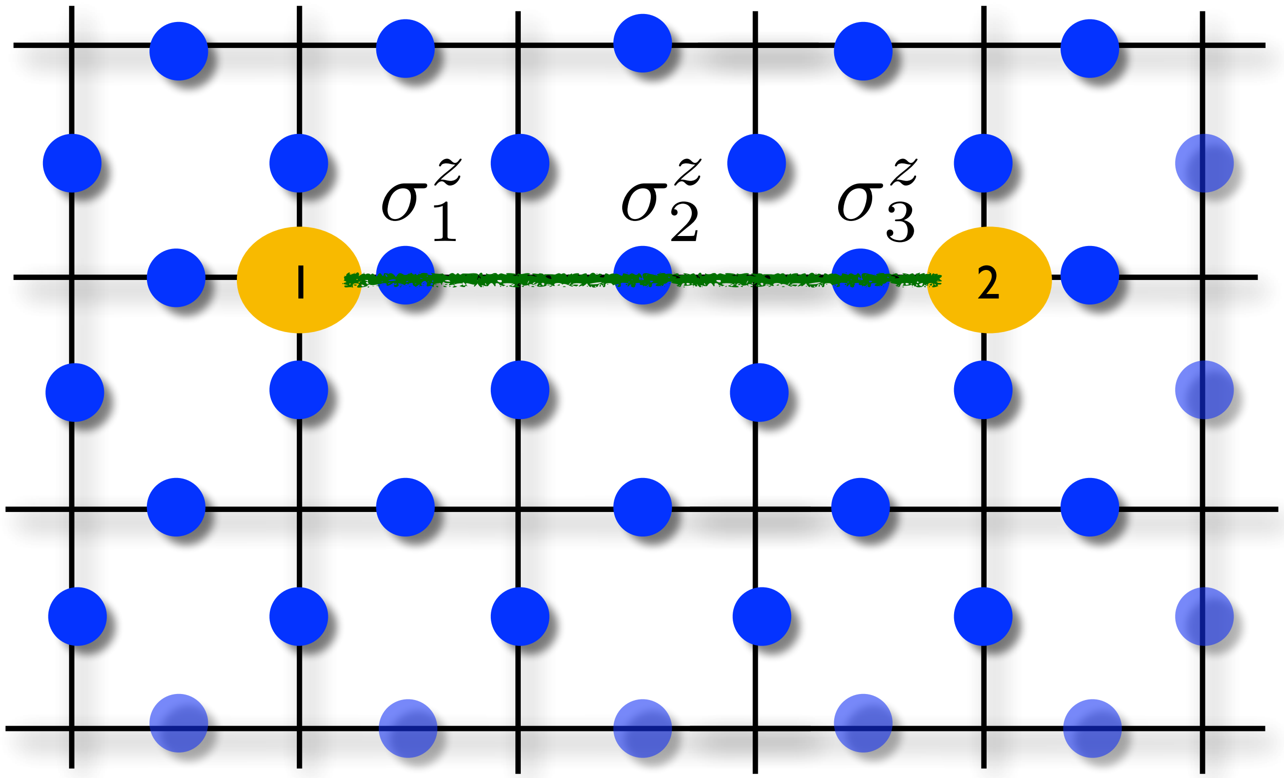
$$E = \sigma_c^z \sigma_d^z$$

Applying $C = \sigma_a^z \sigma_b^z$ to the error

$$CE = \sigma_a^z \sigma_b^z \sigma_c^z \sigma_d^z$$

Z stabiliser

Errors in the toric code - 3



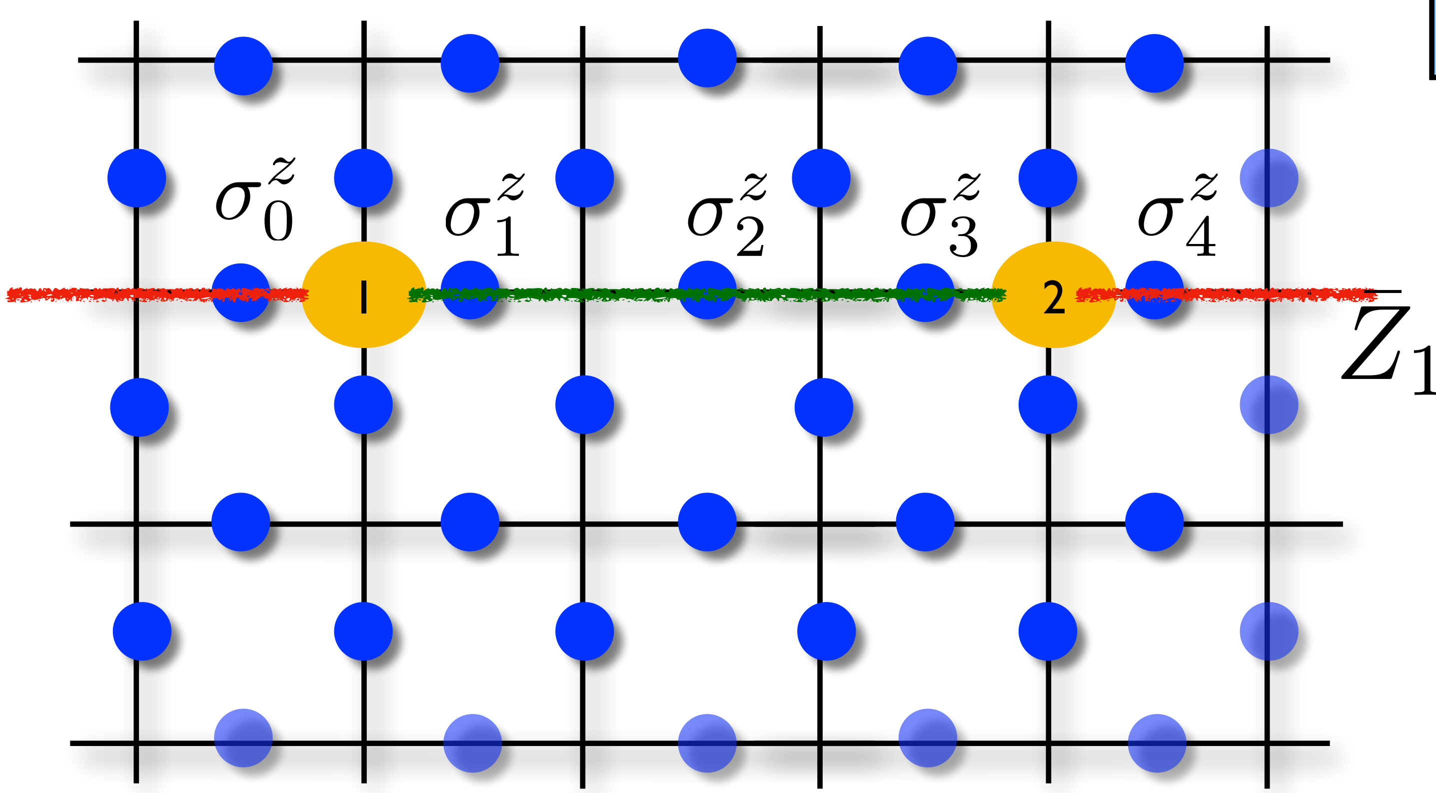
Two excitations detected

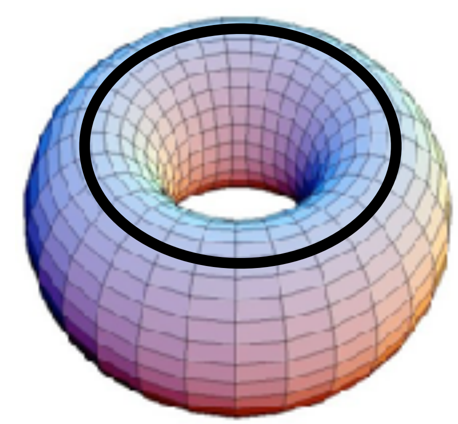
Correction $C = \sigma_1^z \sigma_2^z \sigma_3^z$

The error was $E = \sigma_0^z \sigma_4^z$

Error + Correction = Logical *Z* operator

Correction failed





A toric code *L* x *L* can correct *(L - 1) / 2* errors successfully

Losses in quantum codes

A loss can be

- a qubit that flies away from the trap
- a qubit that leaks out of the computational basis

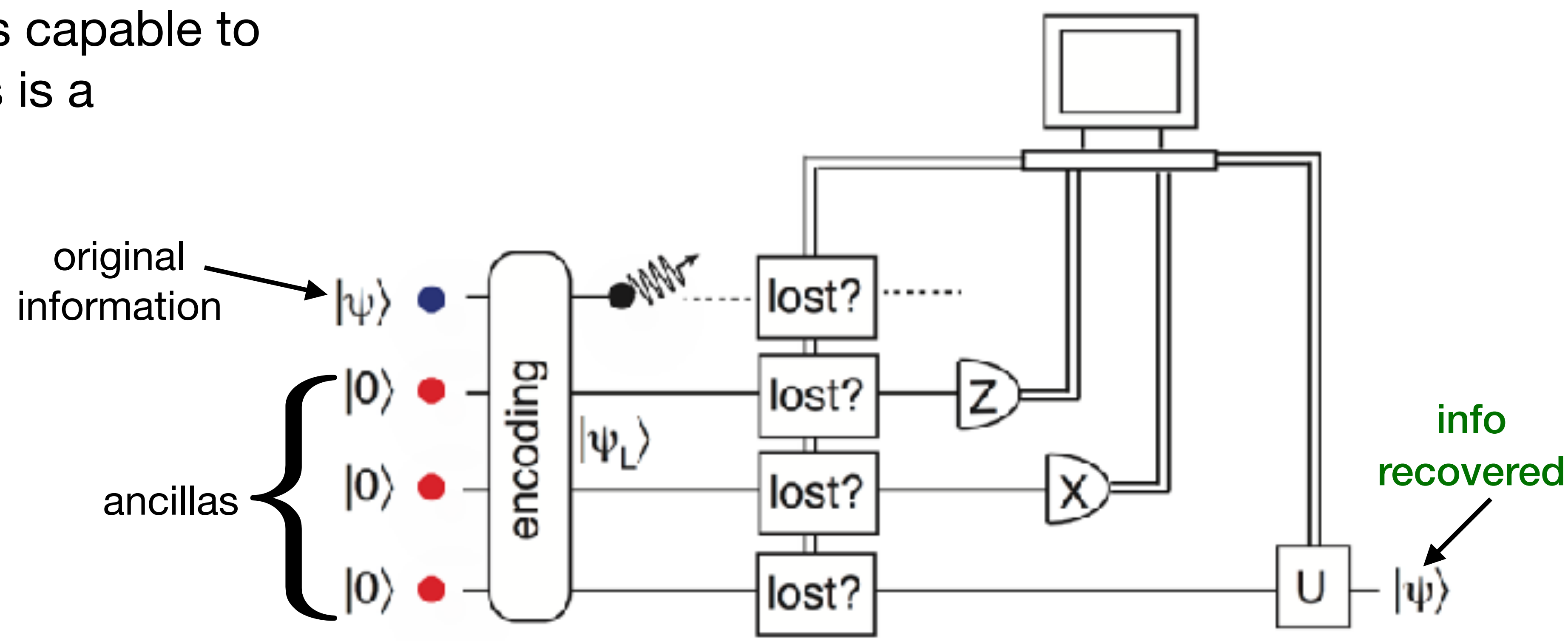
Detectable,
but inaccessible

Losses and leakage can damage the performance of (topological) QEC codes

Challenges:

- Find a protocol to deal with qubit losses
- Understand robustness of the code used

The smallest system that is capable to correct for one loss is a 4-qubit code

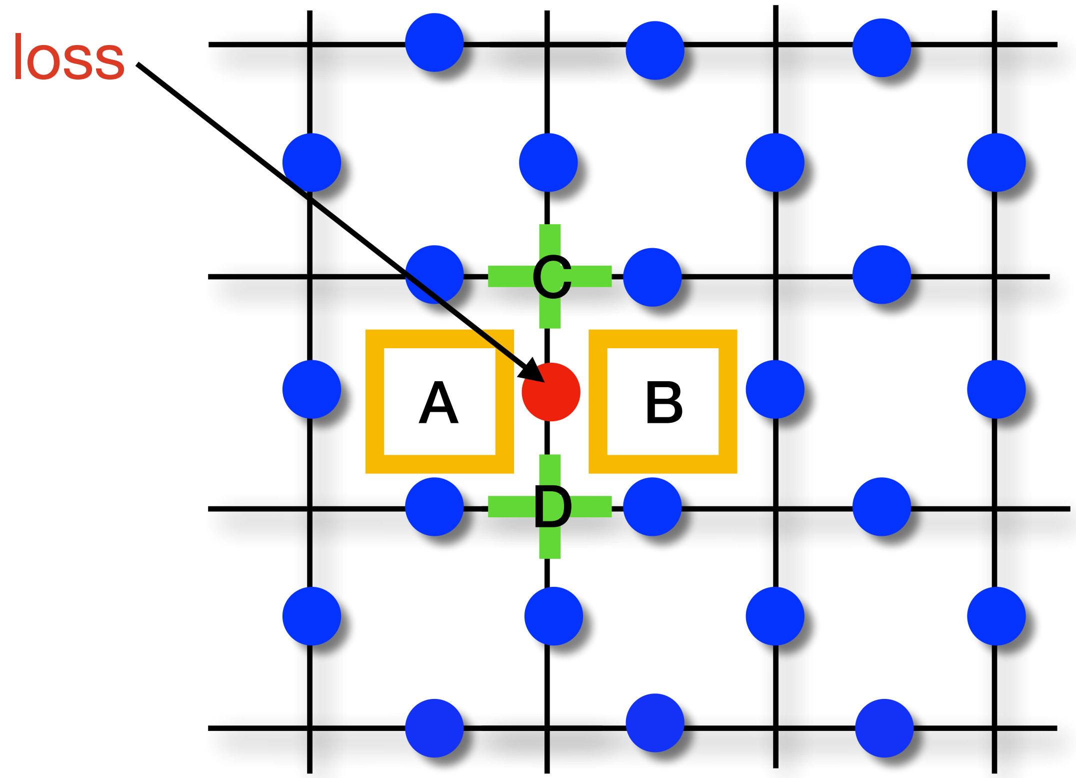


Grassl, et al PRA **56** (1997)

Losses in the toric code

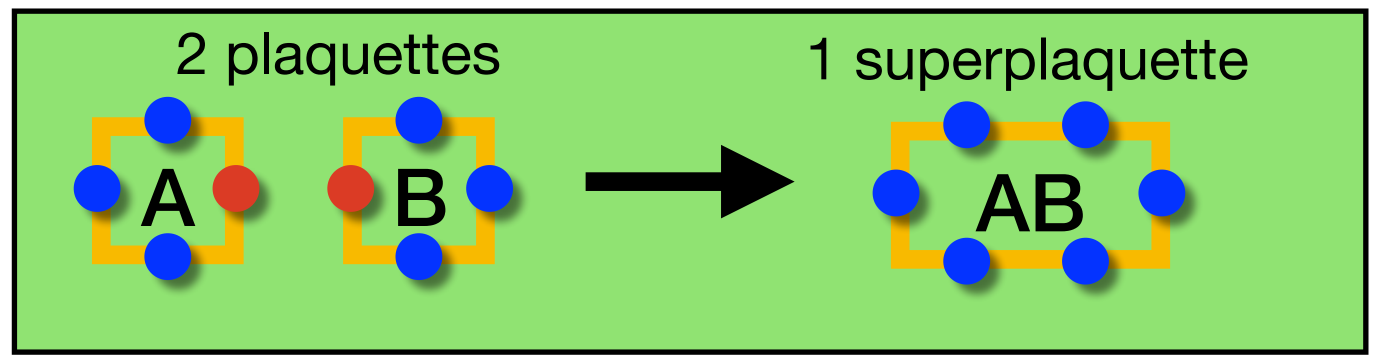
Idea!

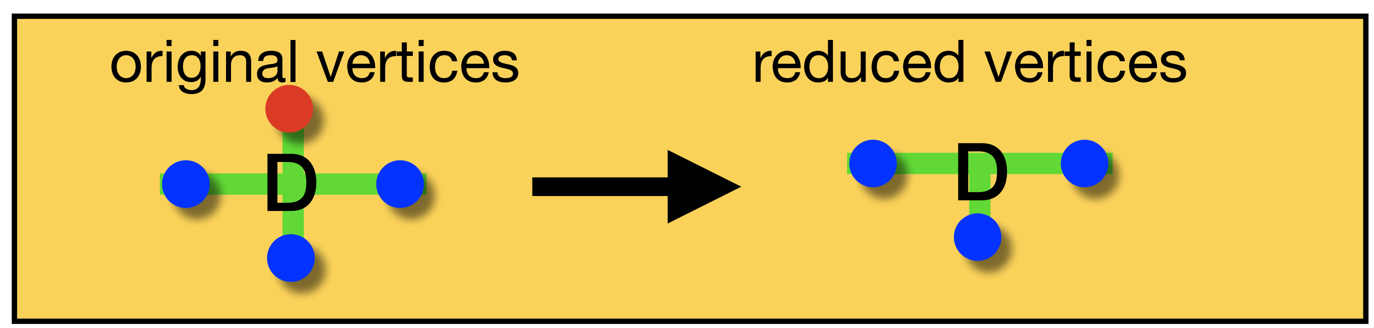
Use the freedom in defining the plaquettes/vertices and the logical operators



The loss affects

- two plaquettes generators
- two vertices generators





p probability of losing a qubit

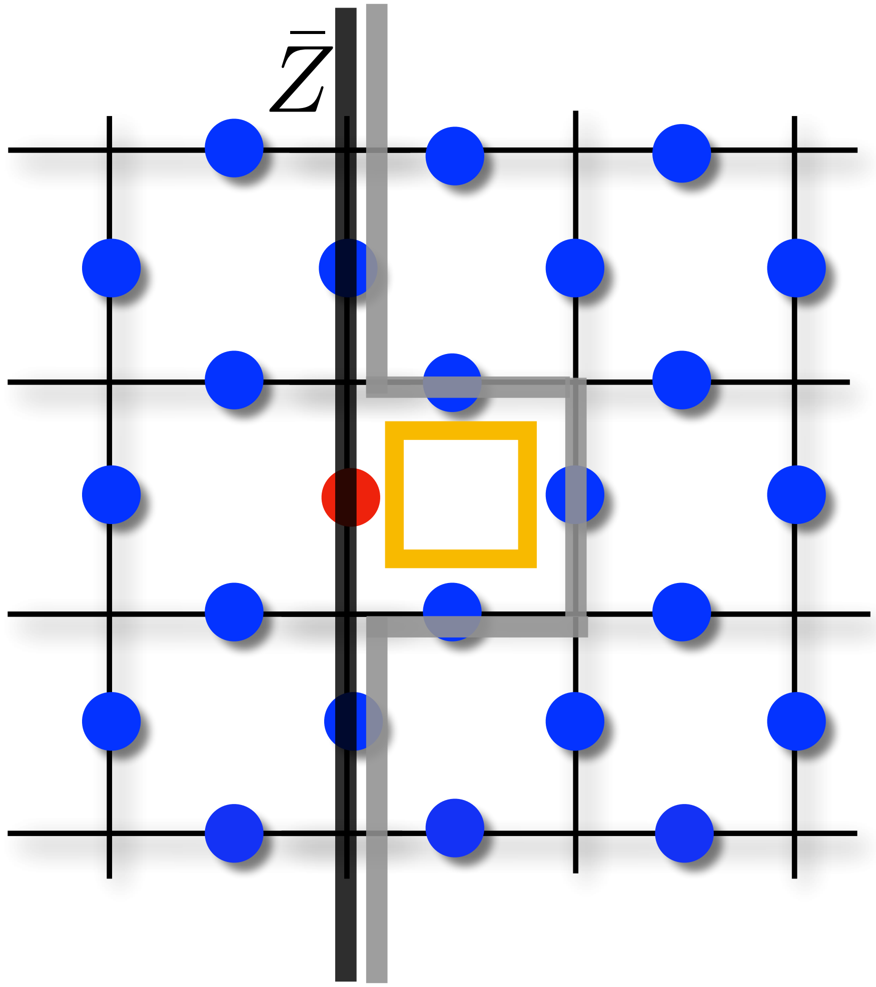
The new defined operators still mutually commute

What about the logical operators?

T. Stace, S. Barrett, A. Doherty, PRL **102**, 200501 (2009)

Losses in the toric code - 2

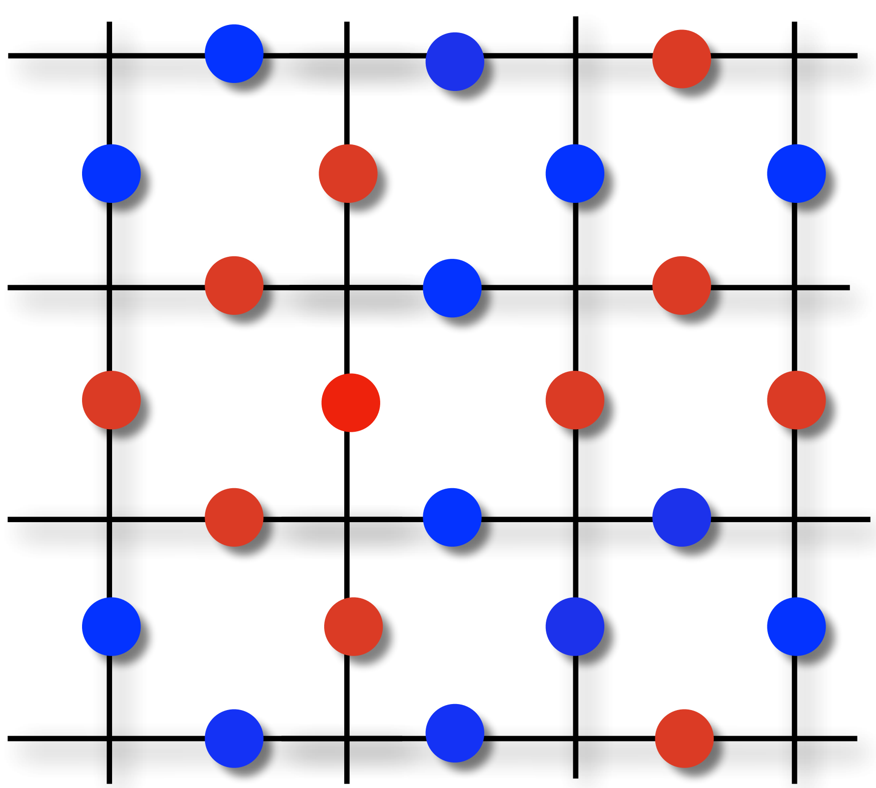
T. Stace, S. Barrett, A. Doherty, PRL **102**, 200501 (2009)



We can use the stabiliser group to redefine the logical operators that go through the loss qubits

After all the losses are detected, check if a logical operator can be defined on the remaining qubits

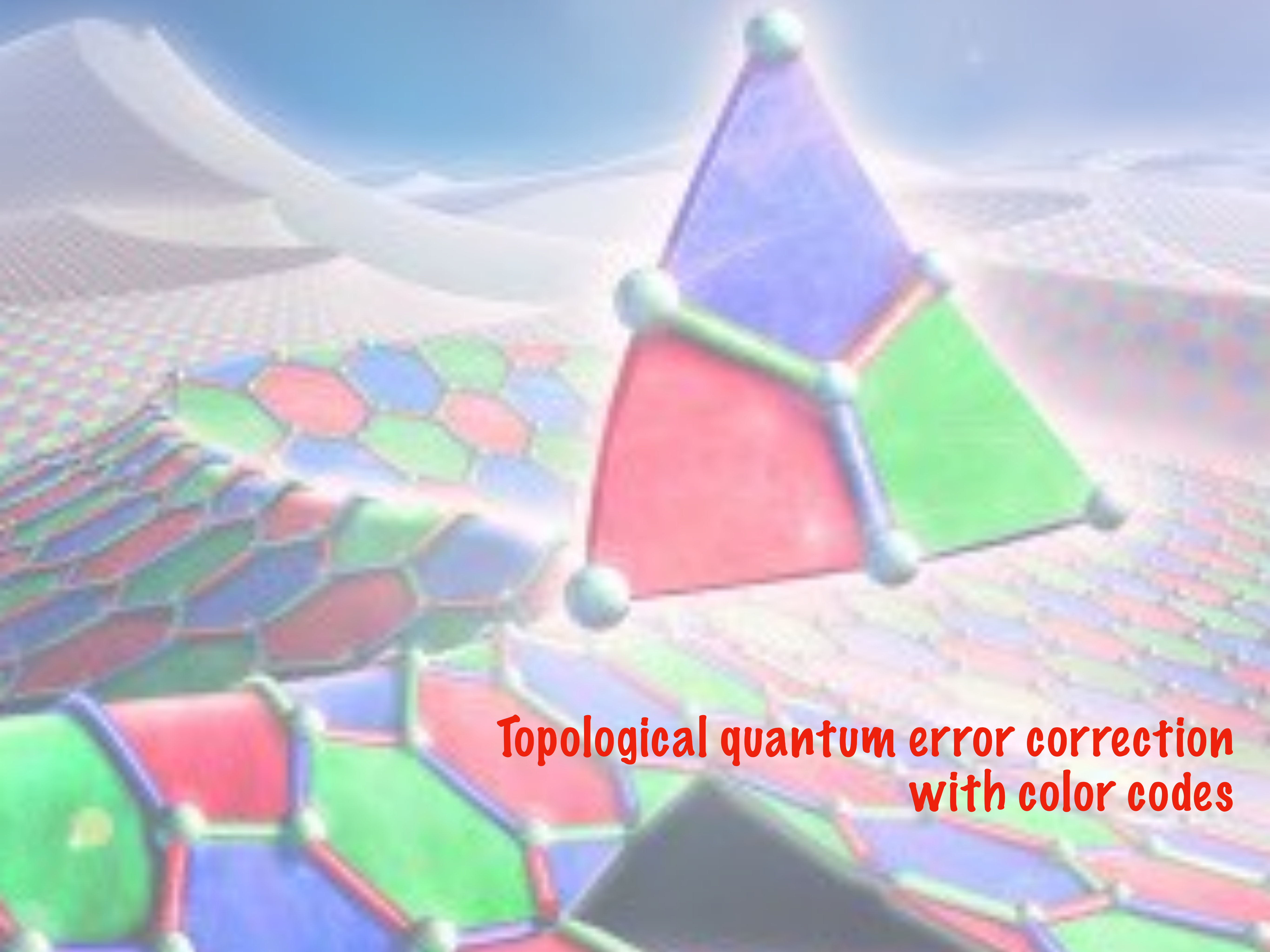
How many losses can be tolerated?



no **percolating** path ⟷ no logical operator

The threshold for losses is given by the bond percolation critical value

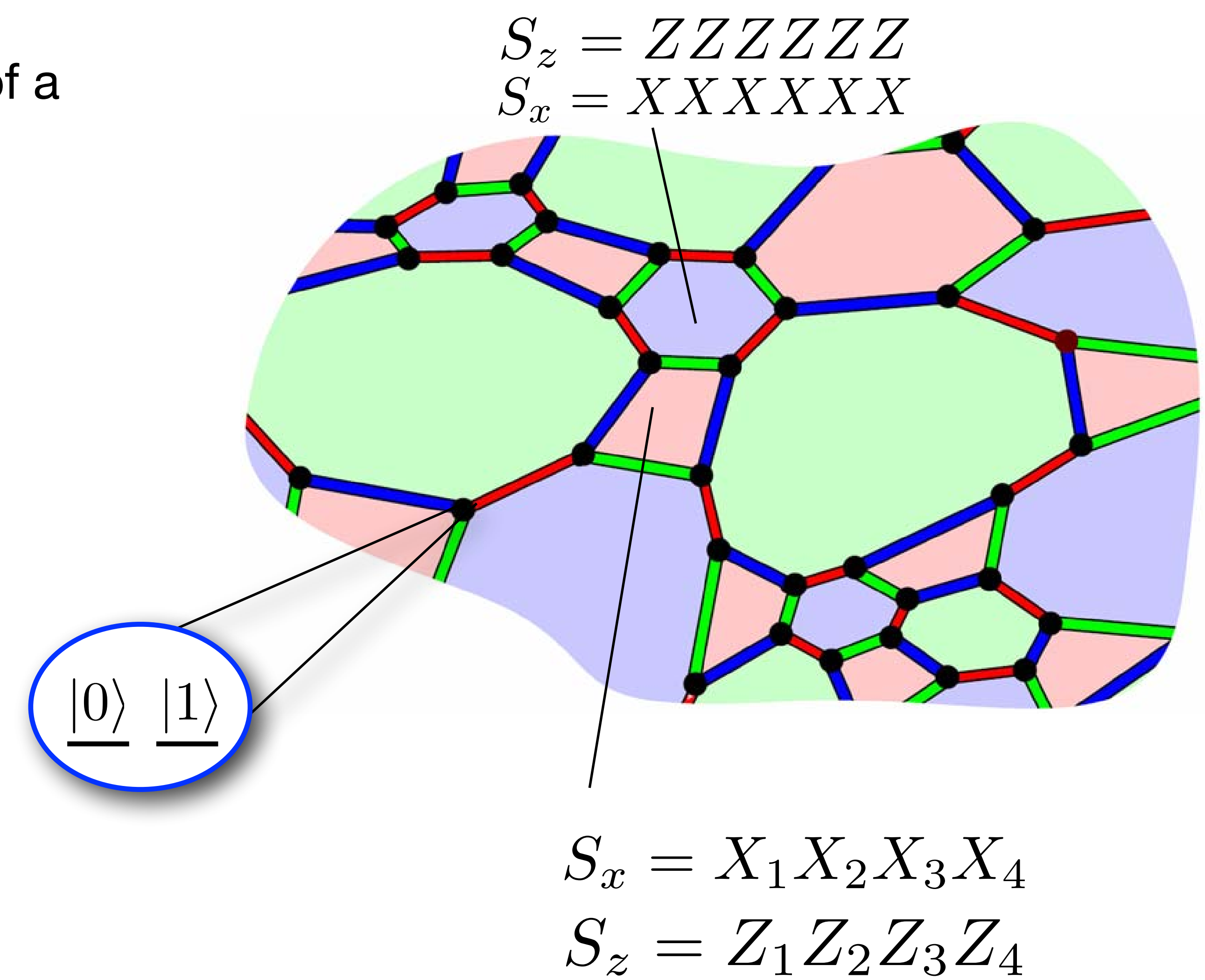
for the square lattice $p_c = 1/2$

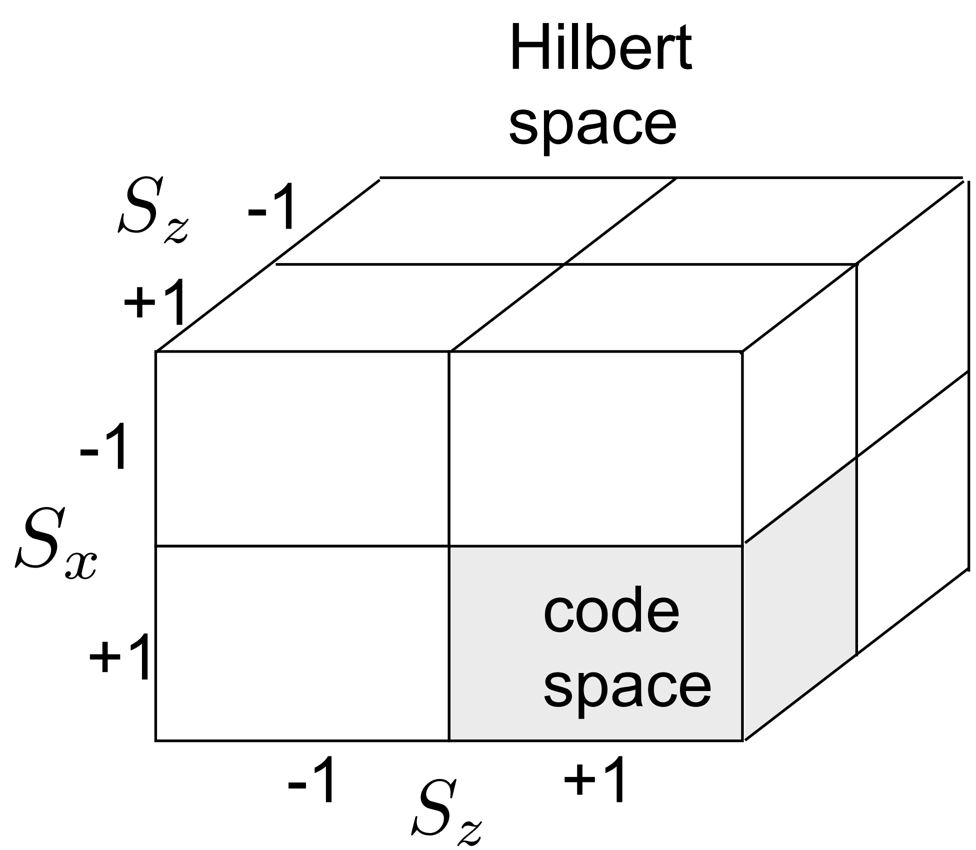


Topological quantum error correction with color codes

Topological color codes: definition and properties

- Defined on a trivalent 3-colorable lattice:
  the neighbours of each plaquette are of a complementary color
- Qubits on vertices of the lattice
- *X,Z* types of **plaquette** operators





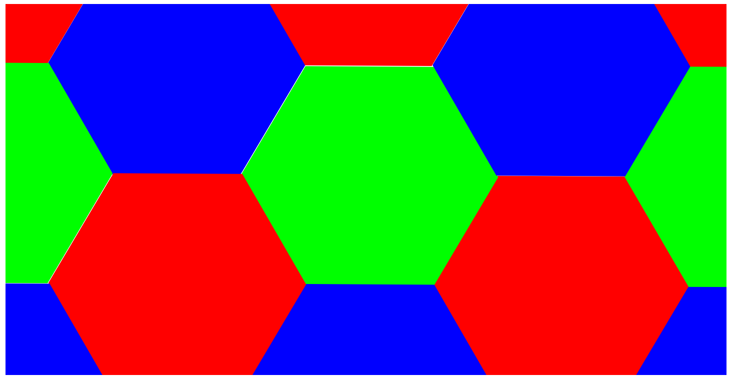
- all stabilisers commute: $[S_x, S_z] = 0$

Code space:

- ground state(s) $|K\rangle$ with $\{S_x^{(i)}|K\rangle = |K\rangle, S_z^{(j)}|K\rangle = |K\rangle\}$ for all X- and Z-stabilisers

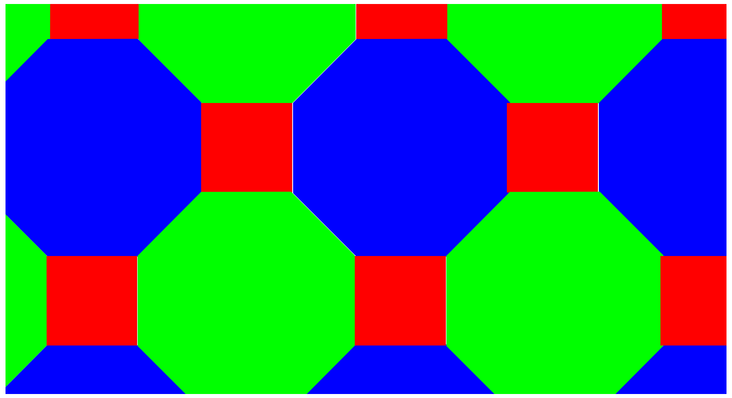
H. Bombin & M. A. Martin-Delgado. PRL **97** , 180501 (2006)
H. Bombin & M. A. Martin-Delgado. PRL **98** , 160502 (2007)

Color codes on 2D regular lattices

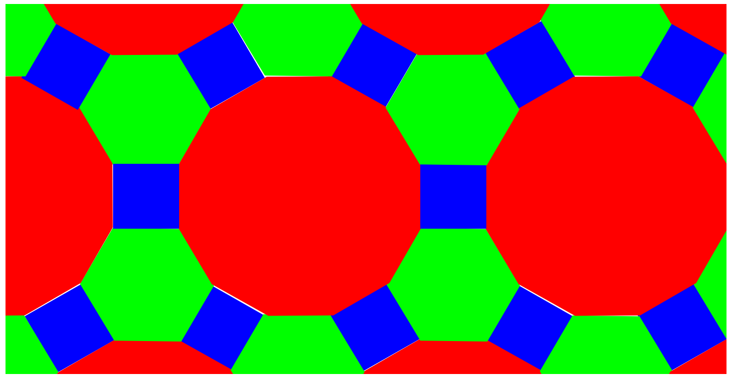


6.6.6 code

(hexagonal)



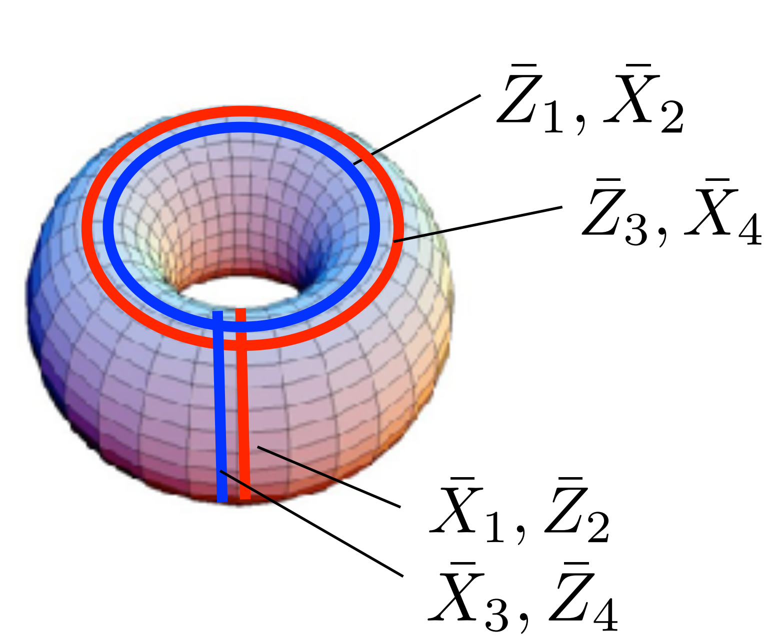
4.8.8 code

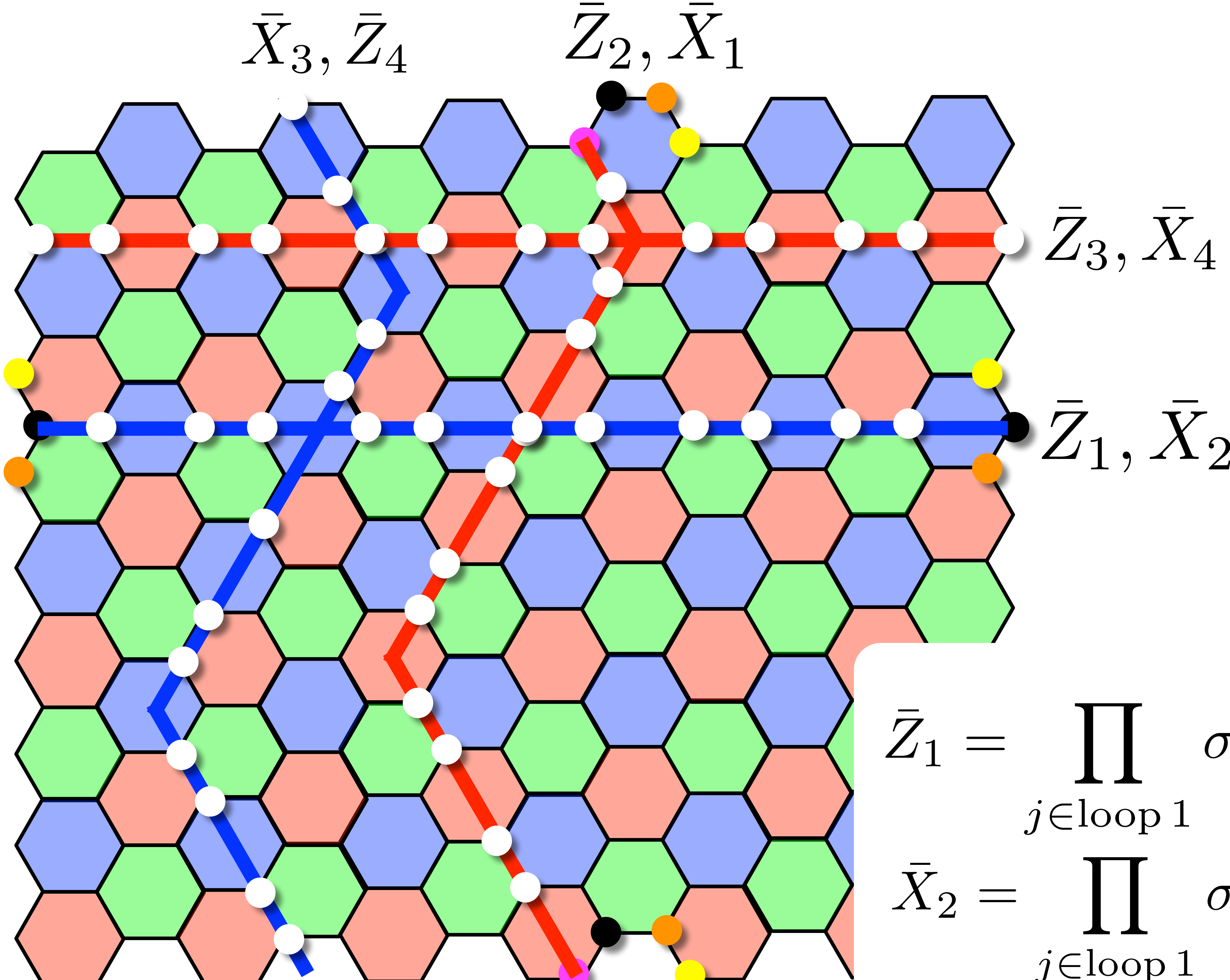


4.6.12 code

Color code and its logical operators

- Color code on a torus (periodic boundary conditions)
- doubling the number of logical qubits (compared to Kitaev)
- string operator of third color: linearly dependent



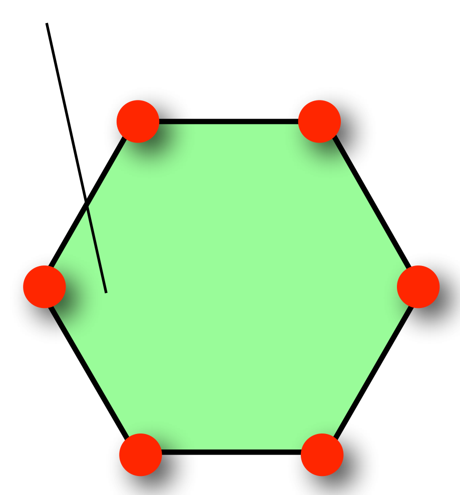


$$\bar{Z}_1 = \prod_{j \in \text{loop } 1} \sigma_j^z$$

$$\bar{X}_2 = \prod_{j \in \text{loop } 1} \sigma_j^x$$

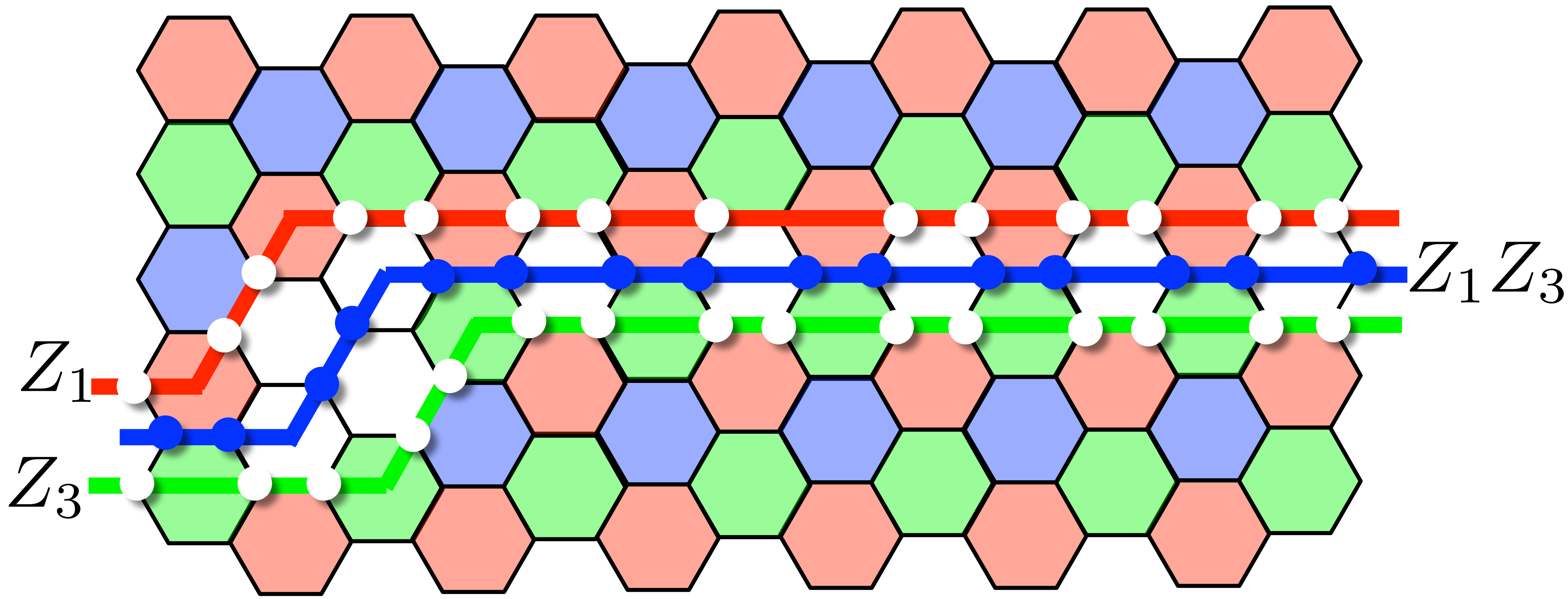
Remember:
qubits on sites,
only X- and
Z-plaquette
stabilisers

$$S_z = ZZZZZZ$$
$$S_x = XXXXXX$$



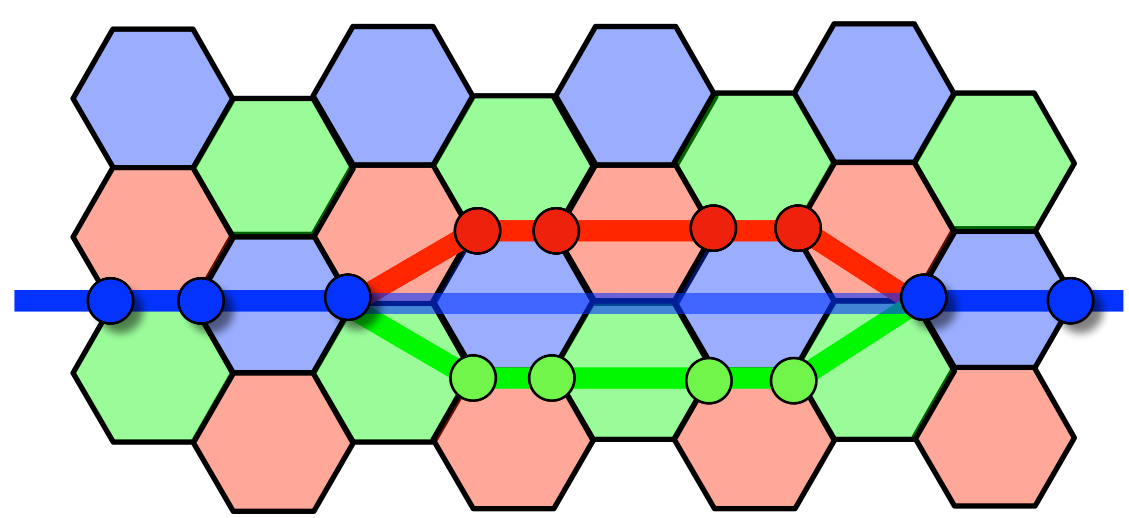
Color code and its logical operators

- third (blue) string is not linearly independent, but just a tensor product of logical operators



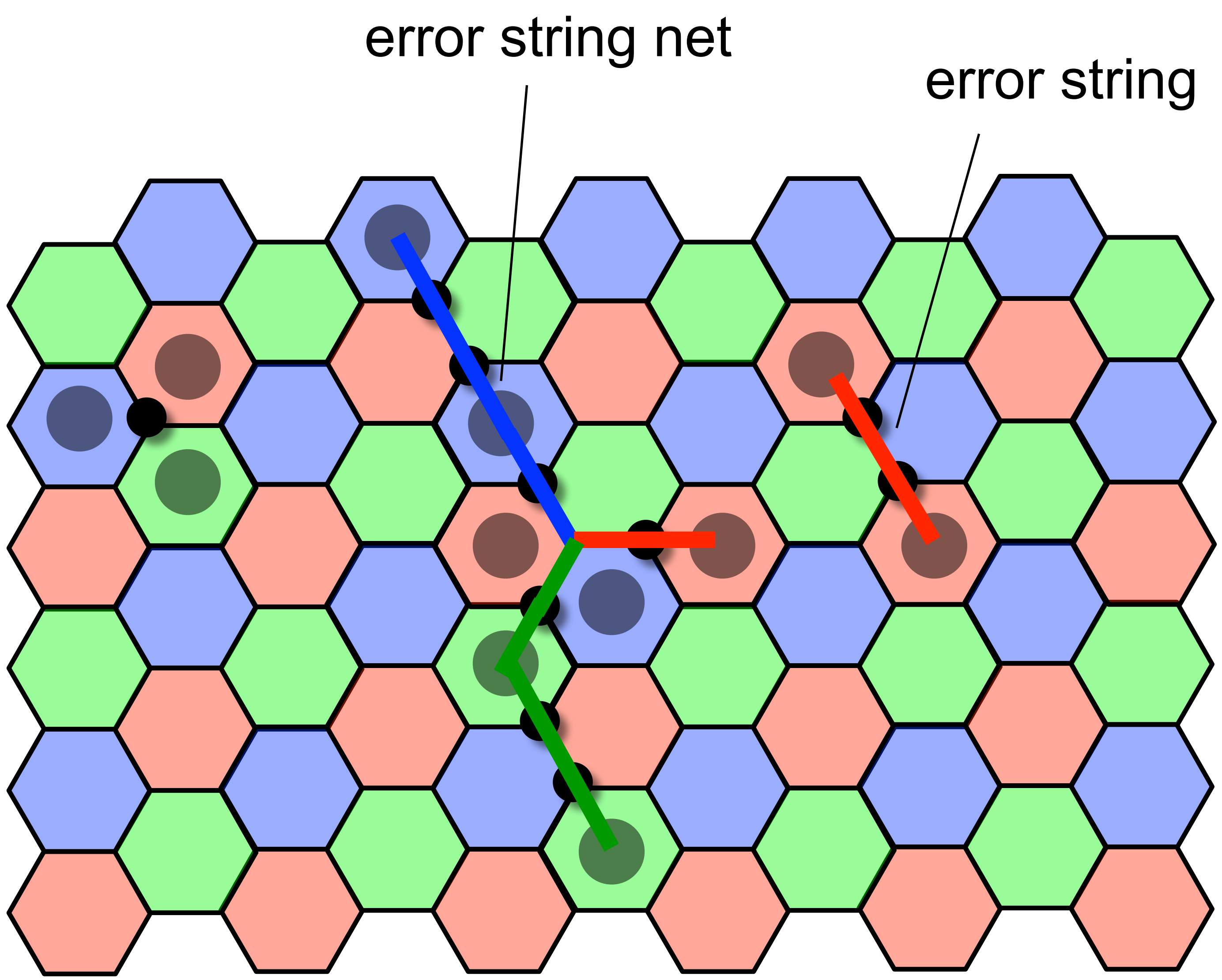
new!

- Branching



Quantum error correction in color codes

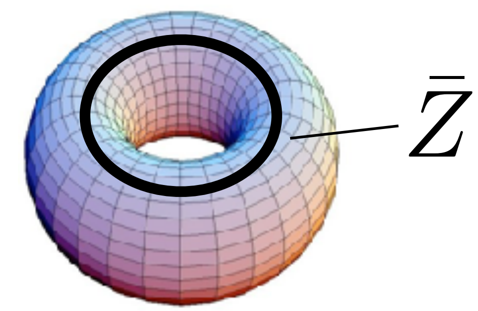
- more complex error behavior than in toric code

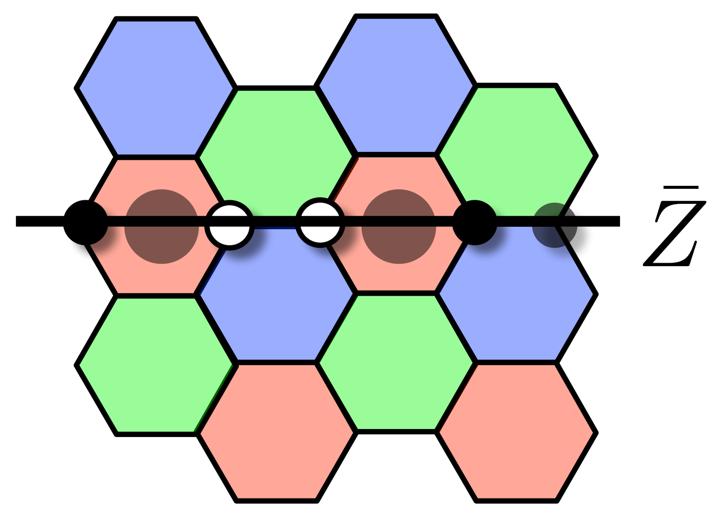


- Error correction:

derive from measured error syndrome the class of most probable physical error scenario

derive (efficiently) a good recovery operation





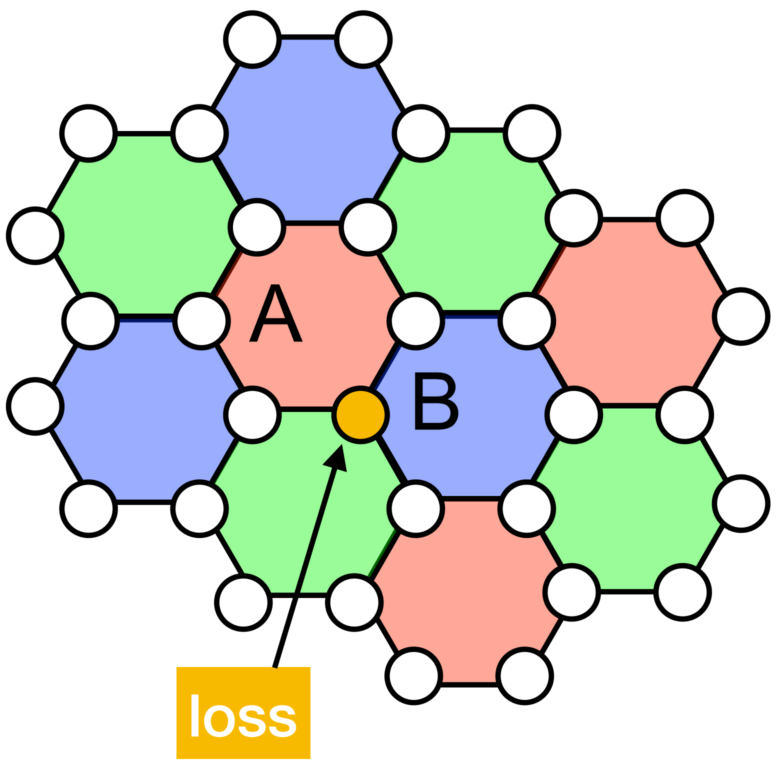
Failure: logical error occurs if **Errors + Correction** include a **non-contractible path**

Losses in color codes

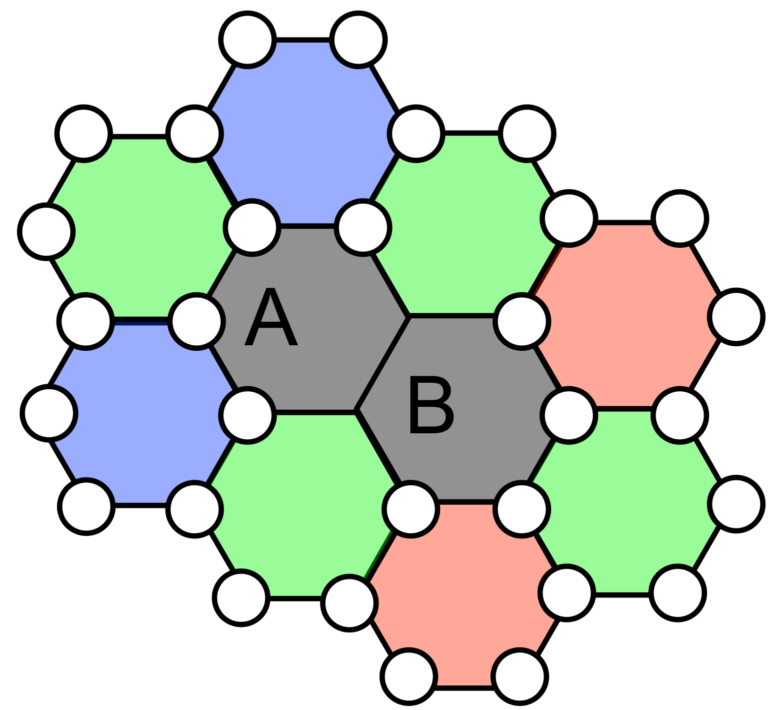
Losses in color codes

Challenge!

Cure all the losses and keep the three-colorability



- Multiply A and B plaquettes?



What colour AB?

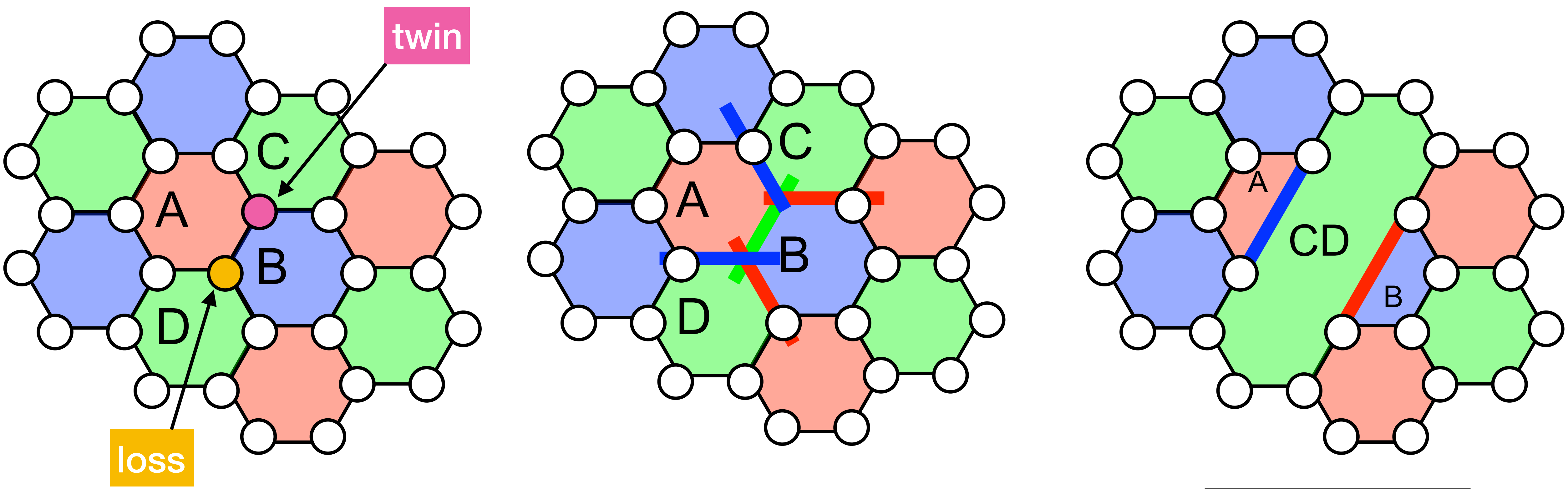
Losses in color codes

Challenge!

Cure all the losses and keep the three-colorability

Solution!

Choose a neighbouring twin qubit



Pick a twin qubit randomly

Remove 5 links

Add 2 new links

Redefine the plaquettes
Merged super-plaquette CD
Reduced plaquettes A,B

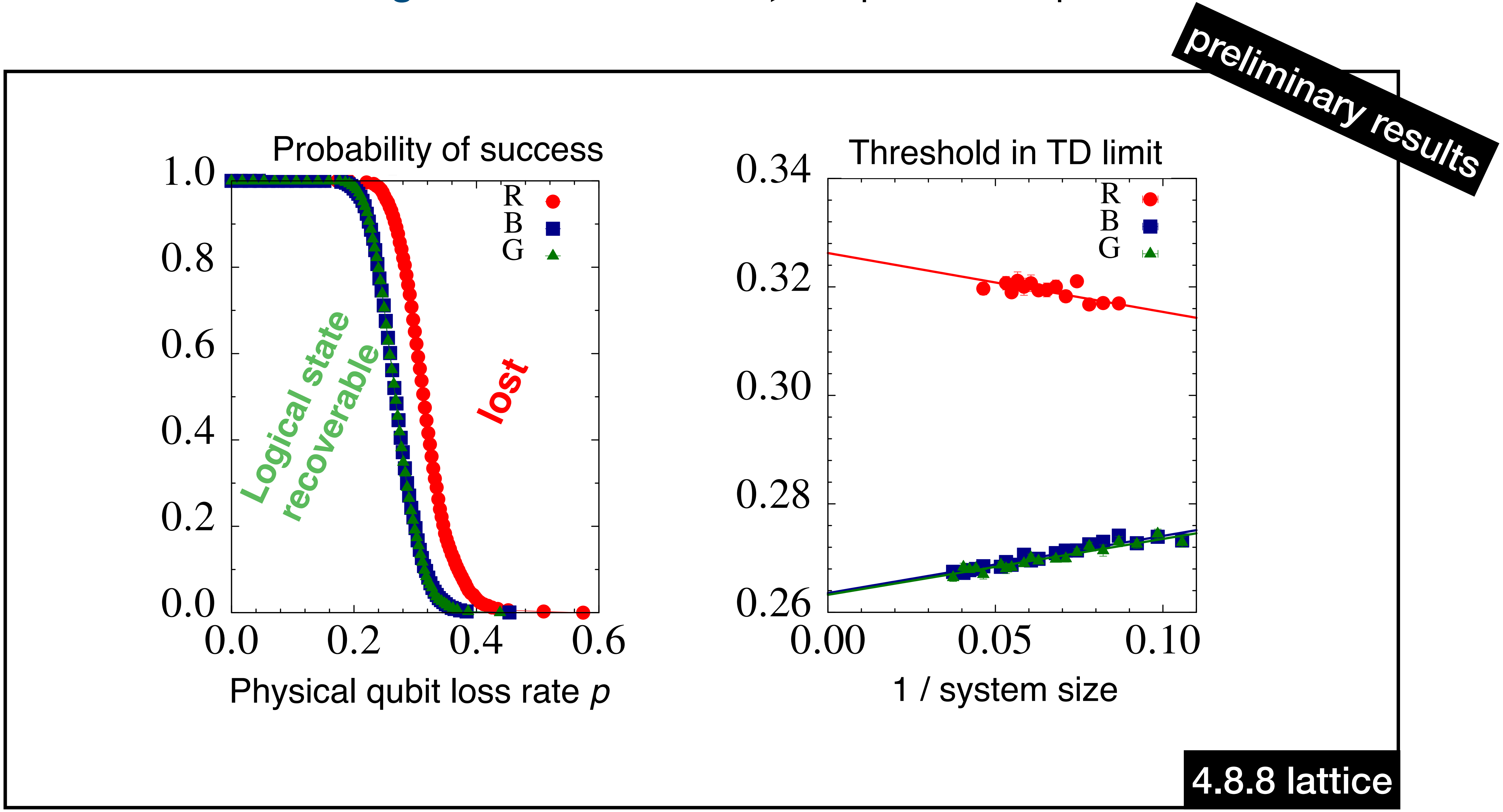
Why does it work?

- The Euler characteristic is still the same
  $\chi = F + V - E$

$$-1f - 2v - (-5 + 2)e = 0$$

Losses in color codes

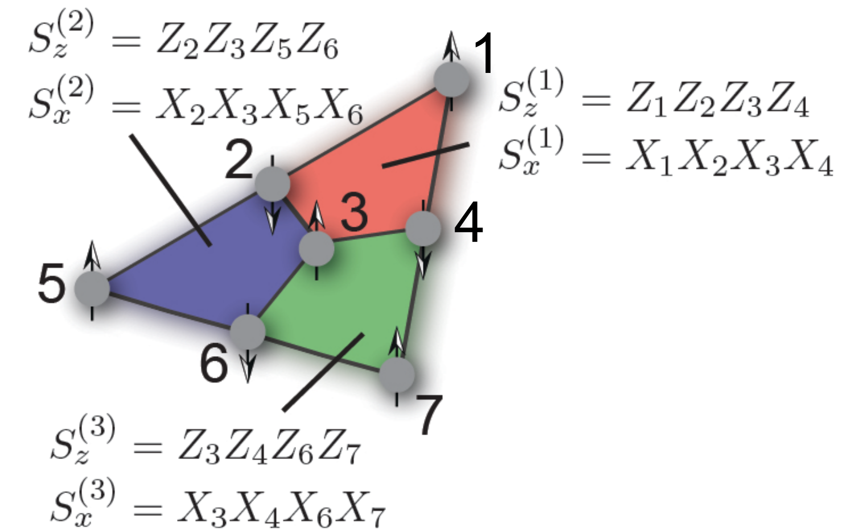
How robust is the code against losses? → percolation problem



Outlook & Conclusions

What's left?

- Take into account the possibility for an operator to branch into two operators of complementary colors
- Implement a strategy for choosing the twin qubits
- Look at an experimental realisation of the color code like the one in Nigg, Muller et al., *Science* **234**, (2014)
- …



What we've seen:

- Error and losses can affect quantum computers but can be cured with success
- Quantum error correcting codes can be realised in condensed matter topological systems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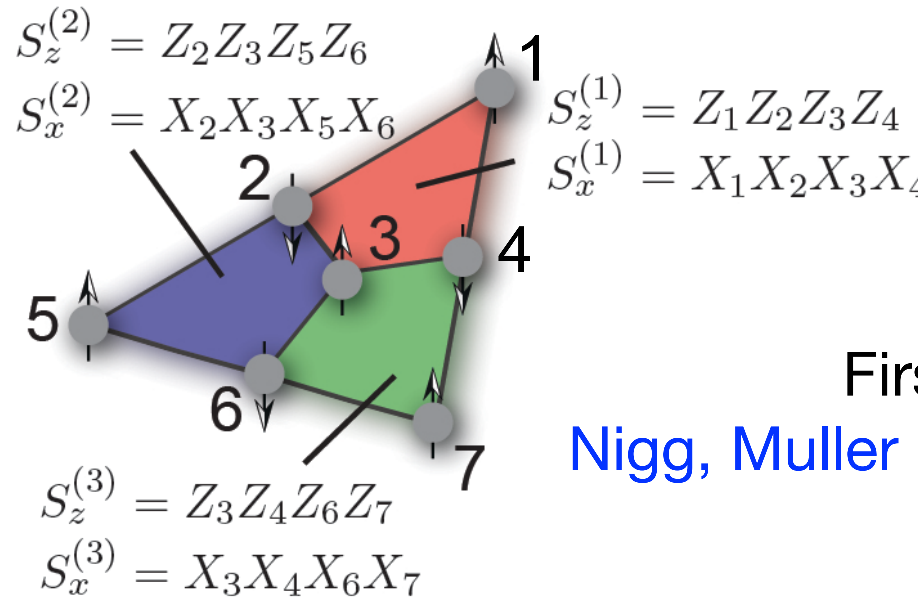
Thank you!
Yes, that’s Wales!

Color codes in trapped ions

- System of 7 ions

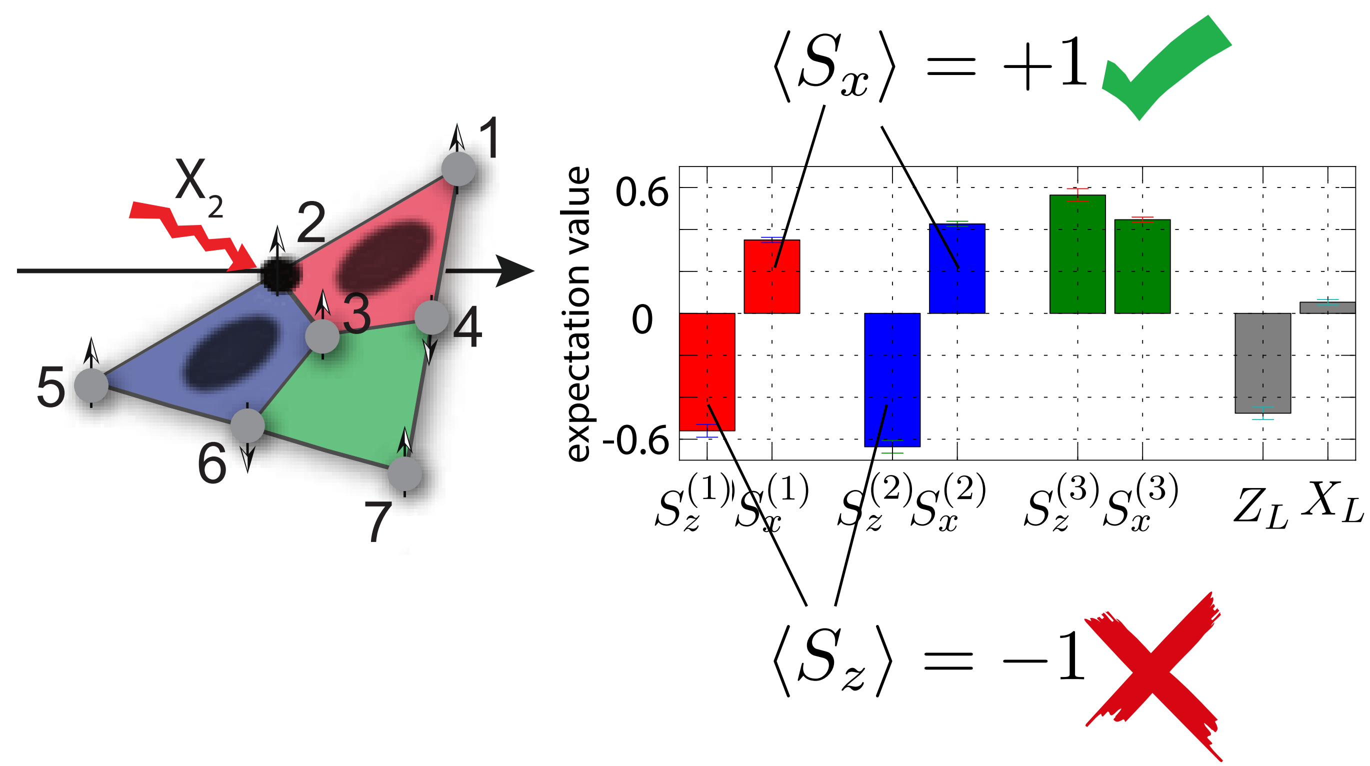
  3+3 stabiliser generators

  1 logical qubit



First experiment in
Nigg, Muller et al., *Science* **234**, (2014)

- Initialize the system in the code space
- Mimic up to two errors
- Measure the stabilisers
- Correct the error(s)



Error threshold in the toric code

How many errors we can effectively correct in a toric code?

- (*L* - 1)/2 if they lie on straight lines!

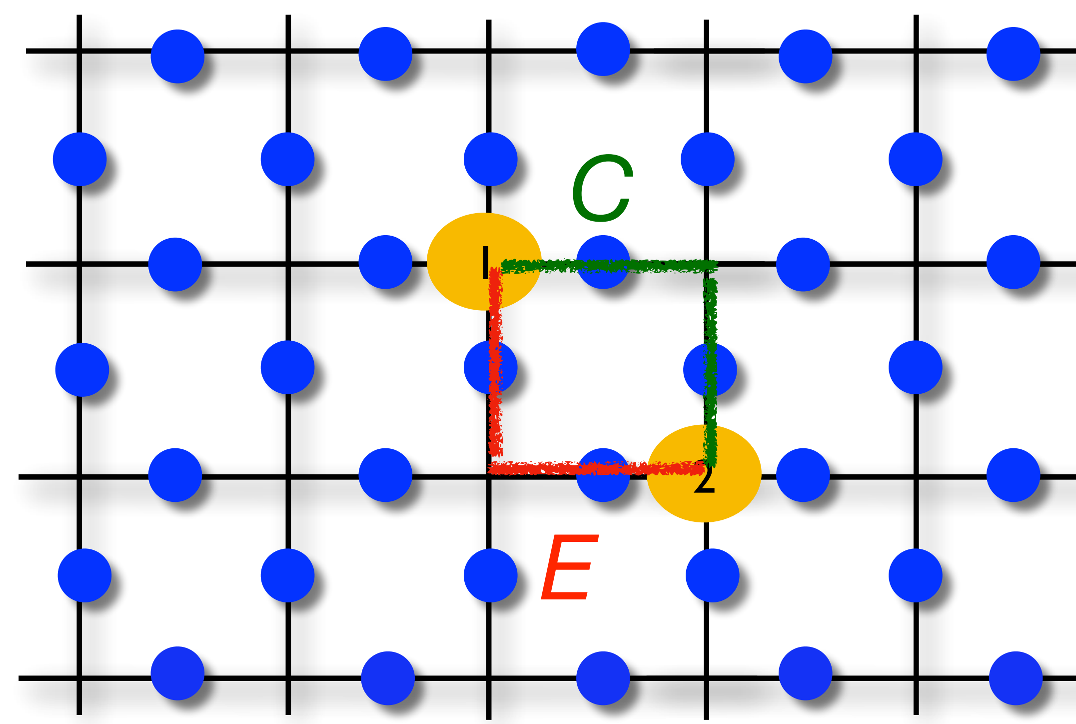
- If they have random positions

Correcting for errors → Find a cycle such that C + E = trivial cycle

Phase transition in a
Random Bond Ising Model

$$H_{\mathrm{RBIM}} = -J \sum_{\langle i,j \rangle} \eta_{ij} \sigma_i \sigma_j$$

random Ferro/Antiferro
coupling related to E + C



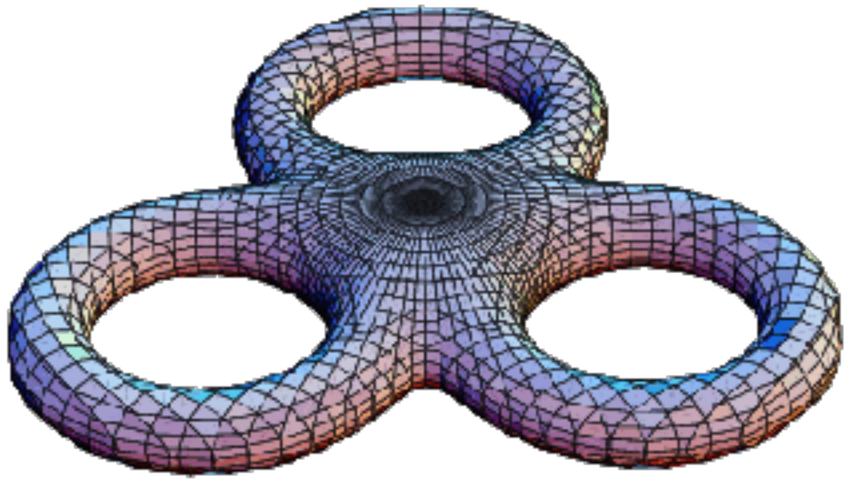
RB Ising	Ordered phase	Disordered phase	
Toric code	Correctable	Non-correctable	*error probability*

phase transition p=10%

Ground state degeneracy of the toric code - 2

What if the system has a different topology?

- Physical qubits occupy the edges *E*
- *X*-stabilisers on vertices *V*
- *Z*-stabilisers on faces *F*



As before $\prod_{\square} S_z = \prod_{+} S_x = \mathbf{1} \longrightarrow$ One X and one Z stabiliser generator not independent

There will be

E	physical qubits
$V-1$	*X*-stabilisers
$F-1$	*Z*-stabilisers

Degrees of freedom left =

$$= E - (F-1) - (V-1)$$

$$= 2 - (F+V-E) =$$

$$= 2 - \chi$$

Euler characteristic

In a torus with genus $g \longrightarrow \chi = 2 - 2g \longrightarrow 2g$ logical qubits