Geometric description of open quantum systems

Geometry & Physics - Remembering Giuseppe Morandi
Bologna, November 24-25, 2017

Jesús Clemente-Gallardo
BIFI-Departamento de Física Teórica
Universidad de Zaragoza

Universidad Zaragoza

Email: jcg@unizar.es

Universidad Zaragoza

Contents

References:
J. F. Cariñena, J. Clemente-Gallardo, J. A. Jover-Galtier and G. Marmo,
Tensorial dynamics on the space of quantum states, J. Phys. A:Math. Theor.,
50, 365301, 2017
J. A. Jover-Galtier, Open quantum systems: geometric description, dynamics
and control, Ph. Thesis, U. Zaragoza 2017

Geometric formulation of Quantum Mechanics

Let us consider a finite dimensional quantum system. Heinsenberg formalism will be defined on a C^*–algebra $\mathcal{A}$, finite dimensional, and therefore isomorphic to $M_n(\mathbb{C})$, with the Frobenius norm $|A|^2 = \mathrm{Tr}(A^\dagger A)$.

We can now consider only the set of physical observables. The set of hermitian operators is isomorphic to the Lie algebra of the unitary group $\mathcal{O} \approx \mathfrak{u}(n)$. As we have a (non-degenerate) scalar product

$$\langle A|B \rangle = \mathrm{Tr}(AB); \qquad \forall A, B \in \mathcal{O}$$

we can identify $\mathcal{O}$ with $\mathcal{O}^*$

Lie-Jordan algebra

A vector space endowed with a Jordan algebra structure $\circ$ and a Lie structure $[\cdot, \cdot]$, such that $\forall a, b, c \in \mathcal{L}$:

- Leibnitz $[a, b \circ c] = [a, b] \circ c + b \circ [a, c]$
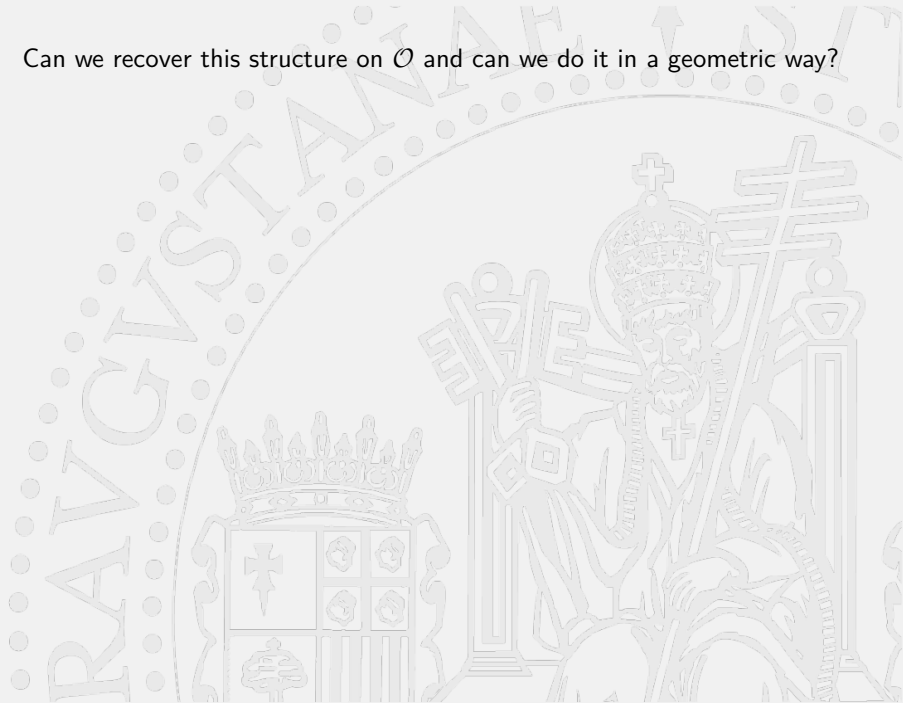- $(a \circ b) \circ c - a \circ (b \circ c) = \hbar^2[b, [c, a]]$ where $\hbar \in \mathbb{R}$.

Lie-Jordan Banach (LJB) algebras

A Lie-Jordan algebra $\mathcal{L}$ endowed with a norm $\| \cdot \|$ such that $\mathcal{L}$ is complete and satisfies

- $\|a \circ b\| \leq \|a\| \|b\|$
- $\|[a, b]\| \leq |\hbar|^{-1} \|a\| \|b\|$
- $\|a^2\| = \|a\|^2$
- $\|a^2\| \leq \|a^2 + b^2\|$

for any $a, b \in \mathcal{L}$.

Can we recover this structure on $\mathcal{O}$ and can we do it in a geometric way?

Can we recover this structure on $\mathcal{O}$ and can we do it in a geometric way?

Indeed. If we consider first the linear functions on $\mathcal{O}^*$

Definition of tensor fields

We can consider two tensors encoding relevant algebraic structures of $\mathcal{O}$:

$$R_\xi(df_A, df_B) = \langle \xi, (AB + BA) \rangle$$

and

$$\Lambda_\xi(df_A, df_B) = \langle \xi, [A, B] \rangle$$

R is a symmetric tensor and Λ is the canonical Lie-Poisson tensor for the unitary algebra. We can extend trivially the definition from linear to general differentiable functions on $\mathcal{O}^*$.

These tensor fields allow us to consider the notion of Hamiltonian vector field associated with an observable A (X_A) and the corresponding gradient vector field (Y_A).

Definition

We can consider the set of real linear functions on $\mathcal{O}^*$ defined as

$$\mathcal{F}_{\mathcal{O}}(\mathcal{O}^*) = \{f_A : \mathcal{O}^* \to \mathbb{R}|\ f_A(\xi) = \xi(A)\}$$

Theorem

$(\mathcal{F}_{\mathcal{O}}, R, \Lambda)$ is a LJ algebra.

$$f_A \circ f_B = R(df_A, df_B) = f_{A \circ B}; \qquad \{f_A, f_B\} = \Lambda(df_A, df_B) = f_{[A,B]}$$

Summary

The algebraic properties encoding the main aspects of the quantum system can be encoded in the Lie-Jordan algebra structure which combines the commutator and anti-commutator structures from the space of observables. But we encode them in two contravariant tensor fields R and Λ, and this gives us the possibility of considering nonlinear objects.

The space of physical states

Definition

The **set of density matrices** $\tilde{\mathcal{S}}$ of a system corrresponds to the subset of $\mathcal{O}$ defined by the convex combinations of rank-one projectors on the Hilbert space. Analogously, $\rho \in \mathcal{O}$ is a density matrix iff

$$\mathrm{Tr}\rho = 1, \qquad \rho \geq 0.$$

Analogously, we can also define them in terms of the functions $\mathcal{F}(\mathcal{O}^*)$ as

$$\mathcal{S} = \{\rho \in \mathcal{O}^* | f_I(\rho) = 1; \ f_{A^2}(\rho) \geq 0; \ \forall A \in \mathcal{O}\} \subset \mathcal{O}^*$$

$$\Pi = \{f_{A^2}(\rho) \geq 0\}$$

$$\widehat{S} = \{f_I(\rho) = 1\}$$

$$\mathcal{S}$$

$$\mathcal{O}^*$$

$$0$$

We adapt the notation from Grabowski, Kus and Marmo and denote by D_Λ and D_R the generalized distributions on $\mathcal{O}^*$ of Hamiltonian and gradient vector fields, respectively.

Proposition GKM

The distribution $D_1 = D_\Lambda + D_R$ on $\mathcal{O}^*$ is involutive and can be integrated to a generalized foliation $\mathcal{F}_1$, whose leaves correspond to the orbits of the action of the general linear group $GL(m, \mathbb{C})$ on $\mathcal{O}^*$, $m = \dim\mathcal{O}$, defined by $(T, \xi) \mapsto T\xi T^\dagger$.
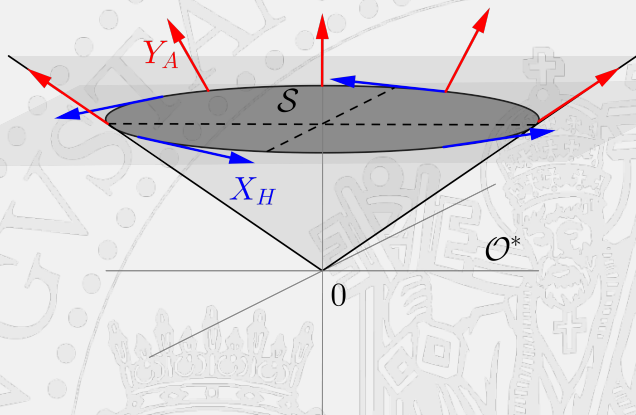
Proposition

Let $\mathcal{P}(\mathcal{O})$ denote the set of real positive linear functionals $\zeta : \mathcal{O} \to \mathbb{C}$, i.e. such that

$$\zeta(a^*) = \overline{\zeta(a)}, \quad \zeta(a^*a) \geq 0, \, \forall a \in \mathcal{O}.$$

The set $\mathcal{P}(\mathcal{O})$ is a subset of $\mathcal{O}^*$. Furthermore, it is a stratified manifold,

$$\mathcal{P}(\mathcal{O}) = \bigcup_{k=0}^{n} \mathcal{P}^k(\mathcal{O}),$$

where the stratum $\mathcal{P}(\mathcal{O})^k$ is the set of rank k operators in $\mathcal{P}(\mathcal{O})$. Each stratum $\mathcal{P}(\mathcal{O})^k$ is a leaf of the foliation $\mathcal{F}_1$ corresponding to the joint distribution, union of Hamiltonian and gradient vector fields.

Proposition

The set of states $\mathcal{S}$ is a stratified manifold,

$$\mathcal{S} = \bigcup_{k=1}^{n} \mathcal{S}^k, \quad \text{where} \quad \mathcal{S}^k = \mathcal{P}(\mathcal{O})^k \bigcap \{\xi \in \mathcal{O}^* | \xi(I) = 1\}.$$

Some considerations:

- Let us consider the foliation of $\mathcal{O}^*$ defined by the gradient vector field Y_I. As $Y_I \in D_1$, any leaf that intersects $\mathcal{P}(\mathcal{O})$ belongs completely to $\mathcal{P}(\mathcal{O})$.

- Notice that the functional $0 \in \mathcal{P}(\mathcal{O})$ is a fixed point of Y_I. Removing it, we obtain a regular foliation by Y_I of $\mathcal{P}_0(\mathcal{O}) := \mathcal{P}(\mathcal{O}) - \{0\}$.

- We can thus define the corresponding quotient manifold identifying points in the same leaf; two points ζ, ζ' are equivalent if $\zeta = c\zeta'$, with $c > 0$. The set of states $\mathcal{S}$ is the section of this fibration defined by the elements of trace equal to one.

We are interested in the characterization of geometrical objects in $\mathcal{S}$ as objects in $\mathcal{P}(\mathcal{O})$ that are projectable with respect to the fibration

$$\pi_{\mathcal{P}}(\zeta) = \frac{1}{f_I(\zeta)}\, \zeta, \quad \zeta \in \mathcal{P}_0(\mathcal{O}).$$
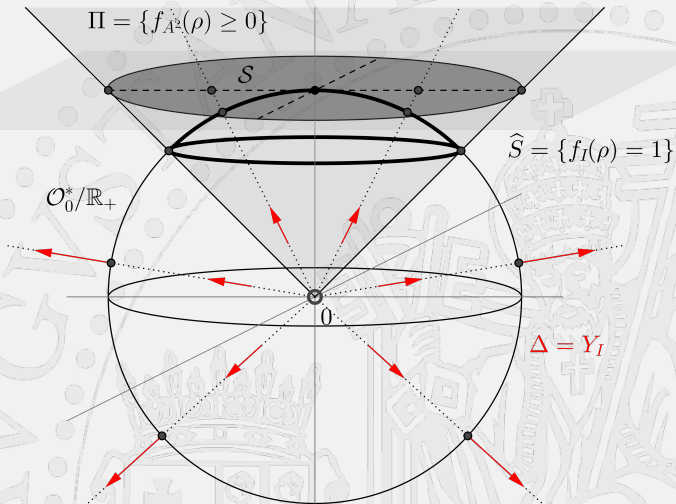
Definition

Let us consider a set of expectation value functions defined, from the linear ones, in the form

$$e_A(\rho) := \pi_{\mathcal{P}}^*(f_a|_{\mathcal{S}})(\zeta) = \frac{f_a(\zeta)}{f_I(\zeta)}, \quad \zeta \in \mathcal{P}_0(\mathcal{O}), \quad a \in \mathcal{O}.$$

Theorem

We can define thus a symmetric and a skewsymmetric tensors on the submanifold $\mathcal{S}$ as

$$R_{\mathcal{S}}(de_A, de_B) = e_{A \circ B} - e_A e_B = Cov(A, B)$$

$$\Lambda_{\mathcal{S}}(de_A, de_B) = \Lambda(de_A, de_B) = e_{[A, B]}$$

The set of expectation value functions $\mathcal{E}_{\mathcal{S}}$ becomes a Lie-Jordan algebra

$$e_A \circ_{\mathcal{S}} e_B := e_{A \circ B}; \qquad [e_A, e_B]_{\mathcal{S}} = e_{[A, B]};$$

and its complexification an associative algebra

$$e_A \star_{\mathcal{S}} e_B = \frac{1}{2} e_A \circ_{\mathcal{S}} e_B + \frac{i}{2} [e_A, e_B]_{\mathcal{S}} = e_{AB}$$
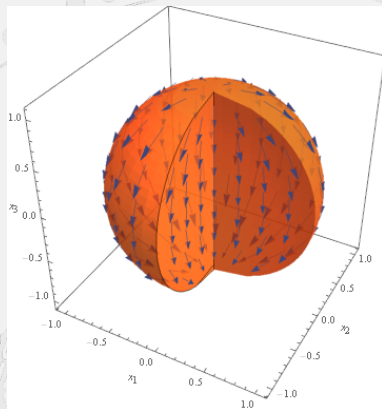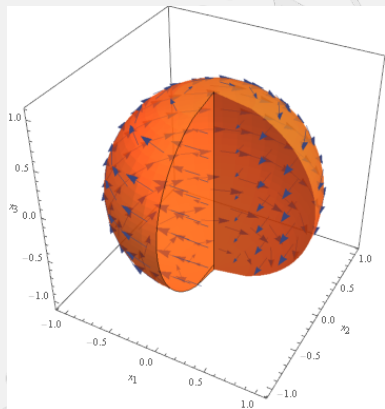
As a simple application, let us consider a simple example: consider a single qubit, a magnetic vector field and the operator

$$H = \vec{B}\vec{\sigma}.$$

If we consider the Hamiltonian and gradient vector fields on the Bloch sphere we find

$$X_H = \epsilon_{jkl} x^j B^k \frac{\partial}{\partial x^l}$$

$$Y_H = B^k \frac{\partial}{\partial x^k} - (\vec{x}\vec{B}) x^k \frac{\partial}{\partial x^k}$$

There have been interesting dynamical models in the last forty years aiming to describe effective or ab-initio dissipative phenomena

- Metriplectic formulation by Kaufman (1984) and Morrison (1986): dissipation introduced through entropic effects

$$\dot{\rho} = [H, \rho] + S \odot \rho = X_H + Y_S$$

  It is also related to Rajeev (2007) construction of complex valued Hamiltonian.

- Gisin (1981): nonlinear effects in Quantum Mechanics. As even being non-linear, the dynamics preserves the spectrum, it must be a nonlinear combination of Hamiltonian vector fields:

$$\dot{\rho} = [\rho, [\rho, H]] = \sum_k f_k X_k$$

- Brody-Holm-Ellis (2007, 2008): linear dynamics through a double bracket to reproduce the state of a canonical ensemble:

$$\dot{G} = [H, [H, G]] = (H \circ G) \circ G - H^2 \circ G = K + Y_{H^2}$$

  For a finite dimensional system the trace is conserved and thus K must compensate the effect of the gradient vector field.

GKS and Lindblad determined, in 1976, the form of the infinitesimal generator of a markovian dynamics on the set of states.

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \frac{1}{2} \sum_{j=1}^{n^2} ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]) =$$

$$- i[H, \rho(t)] + \frac{1}{2} \sum_{j=1}^{n^2} ([V_j^\dagger V_j, \rho(t)]_+ + \frac{1}{2} \sum_{j=1}^{n^2} V_j \rho(t) V_j^\dagger$$

This equation defines a vector field Z_L on $\mathcal{S}$:

$$\frac{d\rho(t)}{dt} = Z_L(\rho).$$

We can characterize the different terms from a geometrical point of view and write

$$Z_L = X_H + Y_J + K$$

where

- X_H is a Hamiltonian vector field with respect to the Poisson tensor Λ_S
- Y_J, is the gradient vector field associated with the function $J = \sum_{j=1}^{n^2} V_j^\dagger V_j$ by the symmetric tensor R_S.
- K is the vector field associated to the action of the Kraus operators

$$K(\rho) = \sum_{j=1}^{n^2} V_j \rho V_j^\dagger$$

Dynamics on the space of tensors

We can encode the evolution in a transformation of the algebraic structures of our LJB system. Therefore we shall consider the following equations

$$\frac{d}{dt}\Lambda(t) = L_{Z_L}\Lambda(t); \qquad \Lambda(0) = \Lambda_{\mathcal{S}}$$
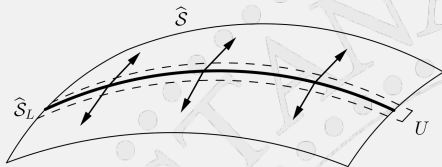
$$\frac{d}{dt}R(t) = L_{Z_L}R(t); \qquad R(0) = R_{\mathcal{S}}$$

The system we are interested in is the limit:

$$R_\infty = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} e^{-tL_{Z_L}}R_{\mathcal{S}}; \qquad \Lambda_\infty = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} e^{-tL_{Z_L}}\Lambda_{\mathcal{S}}$$

Question

Does $(R_\infty, \Lambda_\infty)$ define a LJB algebra? This is the dual question to the one analyzed in Chruściński et al, 2012.

Theorem (Jover)

Consider a set of vector fields $W_1, W_2, ...$ which generate the tangent space to $\mathcal{S}$ at the limit manifold $\mathcal{S}_L$. Then, the contraction T_∞ of the flow T_t on the space of tensor fields on $\mathcal{S}$ exists if and only if there exists asymptotic limits for all the tensors

$$\mathcal{L}_{W_j} T_t$$

This result is particularly useful when used on a set of symmetries of the dynamical vector field.

Example: 2-level systems

Let us consider the phase damping of a qubit, given by the following Kossakowski-Lindblad operator
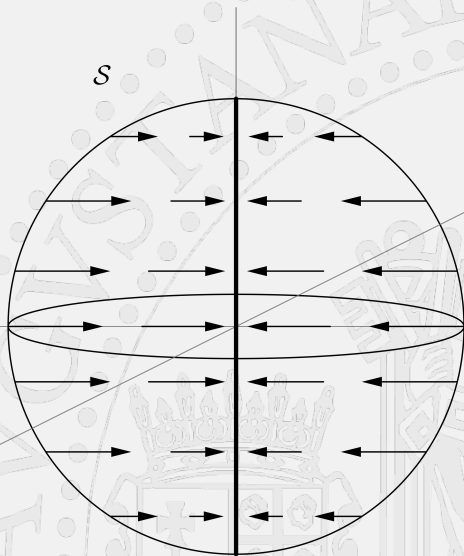
$$L\rho = -\gamma(\rho - \sigma_3\rho\sigma_3).$$

The vector field Z_L associated to this operator is:

$$Z_L = -2\gamma\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right).$$

By computing the Lie derivatives with respect to this vector field of Λ_S and R_S, we obtain the coordinate expressions of the families $\Lambda_{S,t}$ and $R_{S,t}$:

$$\Lambda_{S,t} = e^{-4\gamma t}x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1},$$

$$R_{S,t} = e^{-4\gamma t}\left(\frac{\partial}{\partial x_1}\otimes\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\otimes\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\otimes\frac{\partial}{\partial x_3} - \sum_{j,k=1}^{3}x_jx_k\partial x_j\otimes\partial x_k.$$

In this case, the asymptotic limits $t \to \infty$ of the families do exist.

Proposition

The phase damping evolution of a qubit defines a contraction of the Lie-Jordan algebra of functions on the space of states, determined by the following products:

$$\{x_1, x_3\}_\infty = -x_2, \quad \{x_2, x_3\}_\infty = x_1, \quad \{x_1, x_2\}_\infty = 0,$$
$$(x_1, x_1)_\infty = (x_2, x_2)_\infty = 0, \quad (x_3, x_3)_\infty = 1.$$

The Lie algebra $(\mathrm{span}(x_1, x_2, x_3), \{\cdot, \cdot\}_\infty)$ is isomorphic to the Euclidean Lie algebra. The pair $(\mathrm{span}(x_1, x_2, x_3, 1), (\cdot, \cdot)_\infty)$ is a Jordan algebra. The triple $(\mathrm{span}(x_1, x_2, x_3, 1), (\cdot, \cdot)_\infty, \{\cdot, \cdot\}_\infty)$ is a Lie-Jordan algebra.

The model of decoherence for massive particles is given by

$$L(\rho) = -\gamma[X, [X, \rho]],$$

where X is the position operator. This model can be discretized by considering a finite number $d = 3$ of positions $\vec{x}_m$ along a circle. The positions are given by

$$\vec{x}_m = (\cos\phi_m, \sin\phi_m), \quad \phi_m = \frac{2\pi m}{d}, \quad m, = 1, 2, \ldots, d.$$

The operator L in the basis of eigenstates of the position operator takes the form

$$L|m\rangle\langle n| = -\gamma\, |\vec{x}_m - \vec{x}_n|\, |m\rangle\langle n| = -4\gamma \sin^2\left(\frac{\pi(m-n)}{d}\right)|m\rangle\langle n|,$$

for $m, n = 1, 2, \ldots, d$.

On the other hand, the pure decoherence of a d-level system is given by

$$L(\rho) = -\frac{1}{d} \sum_{k=1}^{d-1} \gamma_k(\rho - U_k \rho U_k^*), \quad \gamma_k > 0, \ k = 1, 2, \ldots, d-1,$$

where U_k are the unitary operators given by

$$U_k = \sum_{l=1}^{d-1} \lambda^{-k(l-1)} P_l, \quad \lambda = e^{\frac{2\pi i}{d}},$$

and P_l are the 1-dimensional projectors $|l\rangle\langle l|$.

The evolutions of a 3-level system by either the decoherence model of massive particles or the pure decoherence model define a contraction of the Lie-Jordan algebra of functions. The Poisson and the Jordan brackets of the contracted algebras are

$$\{x_1, x_3\}_\infty = -x_2, \ \{x_2, x_3\}_\infty = x_1,$$

$$\{x_4, x_3\}_\infty = -\frac{1}{2}x_5, \ \{x_5, x_3\}_\infty = \frac{1}{2}x_4, \ \{x_4, x_8\}_\infty = -\frac{\sqrt{3}}{2}x_5, \ \{x_5, x_8\}_\infty = \frac{\sqrt{3}}{2}x_4,$$

$$\{x_6, x_3\}_\infty = \frac{1}{2}x_7, \ \{x_7, x_3\}_\infty = -\frac{1}{2}x_6, \ \{x_6, x_8\}_\infty = -\frac{\sqrt{3}}{2}x_7, \ \{x_7, x_8\}_\infty = \frac{\sqrt{3}}{2}x_6,$$

$$(x_3, x_3)_\infty = \frac{2}{3} + \frac{1}{\sqrt{3}}x_8, \ (x_8, x_8)_\infty = \frac{2}{3} - \frac{1}{\sqrt{3}}x_8,$$

$$(x_1, x_8)_\infty = \frac{1}{\sqrt{3}}x_1, \ (x_2, x_8)_\infty = \frac{1}{\sqrt{3}}x_2, \ (x_3, x_8)_\infty = \frac{1}{\sqrt{3}}x_3, \ (x_4, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_4,$$

$$(x_5, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_5, \ (x_6, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_6, \ (x_7, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_7,$$

$$(x_4, x_3)_\infty = \frac{1}{2}x_4, \ (x_5, x_3)_\infty = \frac{1}{2}x_5, \ (x_6, x_3)_\infty = -\frac{1}{2}x_6, \ (x_7, x_3)_\infty = -\frac{1}{2}x_7.$$

The triple $(\mathrm{span}(x_1, \ldots, x_8, 1), (\cdot, \cdot)_\infty, \{\cdot, \cdot\}_\infty)$ is a Lie-Jordan algebra.

Thanks for your attention