Hodge – de Rham (Dirac) operators on classical and quantum spheres

Alessandro Zampini
I.N.F.N. – Napoli

Geometry and Physics 2017

- The aim of this talk is to describe the setting and the main results of a research line developed together with G. Marmo, F. Di Cosmo, J.M. Perez-Pardo.

- It describes a formulation for the YM equations and a Dirac – Kähler (Hodge – de Rham) operator on a class of quantum spheres. Such a Dirac operator acts upon spinors introduced *algebraically*, i.e. with no needs to define a principal spinor bundle on them.

- Gauge theory describes interactions in physics by requiring the dynamics of a system to be invariant with respect to the *local* action of a symmetry group.

- Its geometric formulation is based on the notion of principal (and associated) bundles.

$$\pi : P \xrightarrow{G} M, \qquad M \sim P \backslash G$$
$$\rho : G \to \mathrm{GL}(E),$$
$$\Gamma_{\mathcal{E}} \ni \psi : M \to E \qquad \text{(matter fields)}$$

$$A \in \mathfrak{g} \otimes \Lambda^1(M) \qquad \text{(vector potential)}$$
$$D\psi = \mathrm{d}\psi + A \wedge \psi,$$
$$D^2\psi = F \wedge \psi, \qquad F = \mathrm{d}A + A \wedge A \qquad \text{(curvature)}$$

- On (M, g) with gauge group G

$$S = -\frac{1}{2}\int_M d\mu \, \mathrm{Tr}\,(\mathrm{F} \wedge \star\,\mathrm{F}\,)$$

gives the YM action, whose extremals are the Yang-Mills equations

$$\mathfrak{D}\, F = 0, \qquad\qquad \mathfrak{D}(\star\, F) = 0$$

with

$$\mathfrak{D}\, T = \mathrm{d}T + [[A, T]], \qquad T \in \mathrm{End}(E) \otimes \Lambda^k(M) \qquad \text{(covariant derivative)}$$

$$\star : \Lambda^k(M) \to \Lambda^{N-k}(M), \qquad \star^2 = (-1)^{k(N-k)}(\mathrm{sign}\,(g))$$

- The action is invariant under a gauge transformation $G \in \mathcal{F}(M) \otimes \mathrm{Aut}(E)$,

$$A \quad \mapsto \quad A' = G\, A\, G^{-1} + (\mathrm{d}G)G^{-1}$$

- Along a basis of $\Lambda^1(M)$ the YM equations are 2nd order nl PDE with higher order term

$$\Box = \star \mathrm{d} \star \mathrm{d} = \mathrm{div} \circ \mathrm{grad}$$

(the Laplace-Beltrami operator)

- On the Cartan algebra $(\Lambda(M), \wedge, \mathrm{d}, i_X)$ one represents the Clifford product

$$\mathrm{d}x^a \vee \mathrm{d}x^b + \mathrm{d}x^b \vee \mathrm{d}x^a = 2\, g^{ab}$$

with $\mathrm{d}x^a \vee \mathrm{d}x^b = \mathrm{d}x^a \wedge \mathrm{d}x^b + g^{ab}$. The set

$$(\Lambda(M), g, \wedge, \vee, \mathrm{d}, i_X)$$

is the Kähler - Atiyah algebra on (M, g).

- From $\mathrm{d}\,\phi = \mathrm{d}x^a \wedge \nabla_a \phi$ for $\phi \in \Lambda^k(M)$, following [K] one defines

$$\begin{aligned}\mathcal{D}\phi = \mathrm{d}x^a \vee \nabla_a \phi &= \mathrm{d}\phi + (-1)^{N(k-1)} \star \mathrm{d} \star \phi \\ &= \mathrm{d}\phi - \mathrm{d}^\dagger \phi\end{aligned}$$

where the duality is

$$\langle \mathrm{d}\alpha \mid \beta \rangle = \langle \alpha \mid \mathrm{d}^\dagger \beta \rangle + \int_M \mathrm{d}(\alpha \wedge \star \beta).$$

with respect to the scalar product

$$\langle \phi \mid \phi' \rangle \,=\, \int_M \phi \,\wedge\, \star\phi'.$$

- The square gives the Laplace - Beltrami operator

$$\mathcal{D}^2\phi \,=\, -(\mathrm{d}\,\mathrm{d}^\dagger \,+\, \mathrm{d}^\dagger\,\mathrm{d})\phi$$

- The restriction of $\mathcal{D}$ to irreducible modules $I \subset \Lambda(M)$ gives the action of the so called Dirac - Kähler operator. Elements in I are called *spinors*. The action $\mathrm{d}x^a \vee$ upon I gives the corresponding γ^a. The operator $\mathcal{D}$ is defined on any orientable manifold.

- The KD operator is *not* the spin (Atiyah) Dirac operator $\not{D}$ on (M, g). Only upon sections of bundles associated to a spin bundle $\pi \,:\, P \stackrel{\mathrm{Spin}(g)}{\rightarrow} M$ it is

$$\not{D}\psi \,=\, \gamma^r(\partial_r \,+\, \frac{1}{4}\eta_{as}\Gamma_{rb}^{\ s}\gamma^a\gamma^b)\psi$$

- On the group manifold $SU(2)$ with $g = \delta_{ab}\omega^a \otimes \omega^b$ (the CK metric, with $\star\omega^a = -\mathrm{d}\omega^a$) and $\mathrm{d}f = (L_a f)\omega^a$ the KD operator acts on a 4-dim spinor space

$$\mathcal{D}\psi = \begin{pmatrix} 0 & L_+ & L_z & L_- \\ L_- & L_z - i & -L_- & 0 \\ L_z & -L_+ & -i & L_- \\ L_+ & 0 & L_+ & -i - L_z \end{pmatrix} \psi$$

with $\mathrm{sp}(\mathcal{D}) = \{-i(j+1), ij, \pm i\sqrt{j(j+1)}\}, \qquad j = 1/2, 1, 3/2, \ldots.$

- The spin Dirac operator acts on a 2-dim spinor space as

$$\not{D}\,\psi = (\sigma^a L_a - \frac{3i}{4})\psi$$

with $\mathrm{sp}(\not{D}) = \{i(j - 1/4), \; -i(j + 3/4)\}$.

- On the principal bundle (with $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(3) = \mathrm{SU}(2)$)

$$\pi_P : \mathrm{Spin}(4)/\mathrm{Spin}(3) \to S^3$$

the vector potential given by a multiple of the Maurer-Cartan form

$$A = \lambda X_a \otimes \omega^a = \lambda g^{-1}\mathrm{d}g$$

gives

$$\mathfrak{D}(\star F) = \lambda(\lambda - 1)(\lambda - \frac{1}{2})\epsilon_{ab}{}^{c} X_c \otimes (\omega^a \wedge \omega^b)$$

so that $A = 1/2\, g^{-1}\mathrm{d}g$ solves the YM eqs on S^3.

- With $\pi : \mathbb{R}^4 \backslash \{0\} \to S^3$, one proves that $\pi^*(A)$ gives the *meron* solution of YM eqs. [D-A,FF76]

- Is it possible to define a KD operator in a quantum group setting, and use it to solve YM equations?

- In the spirit of Gelfand duality, (following [Wo]) a compact quantum group $G = (A, \Delta)$ is separable unital C^*-algebra with a (dense) coproduct Δ.

- As quantum group $\mathrm{SU}_q(2)$ consider the Hopf $(S, \varepsilon, \Delta, *)$, polynomial *unital* $*$-algebra (with $q \in \mathbb{R}$) generated by

$$\gamma = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \qquad \begin{array}{l} ac = qca \quad ac^* = qc^*a \quad cc^* = c^*c \\ a^*a + c^*c = aa^* + q^2cc^* = 1. \end{array}$$

- The dually paired Hopf universal envelopping algebra to $\mathrm{SU}_q(2)$ is $\mathcal{U}_q(\mathfrak{su}(2)) = \{K^\pm, E, F = E^*\}$

$$K^\pm E = q^\pm E K^\pm \qquad K^\pm F = q^\mp F K^\pm \qquad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

- A gauge theory on a classical manifold M requires:
  1. a differential structure on M and G (gauge group),
  2. a notion of connection (vector potential, covariant derivatives),
  3. a metric tensor on M.

- In the algebraic formulation of a gauge theory over $\mathcal{A} = \mathrm{SU}_q(2)$ we assume:
  1. the formulation à la Woronowicz for differential calculi,
  2. covariant derivatives are derivations on finite projective modules over $\mathcal{A}$, whose elements describes sections of vector bundles (matter fields),
  3. symmetric tensors and $\star$-Hodge on $\mathcal{A}$.

- On $\mathrm{SU}_q(2)$ a family of left-covariant 3D $*$-calculi (i.e. $\mathrm{d}(x^*) = (\mathrm{d}x)^*$)

$$\mathrm{d}x = \sum_a (X_a \triangleright x)\omega_a, \qquad X_a \in \mathcal{U}_q(\mathfrak{su}(2)), \quad (a = \pm, z),$$

- For any of such calculi we have a quantum Lie algebra $\mathcal{X}_q$ (with a braided commutator depending on σ) and a Maurer-Cartan structure equation

$$[X_a, X_b] = f_{ab}{}^{c} X_c, \qquad \mathrm{d}\omega_a = -\frac{1}{1+q^2} f_{bc}{}^{a} \omega_b \wedge \omega_c$$

- A Maurer-Cartan form exists (although the differential calculus is only left-covariant [Wo]):

$$\gamma^{-1}\mathrm{d}\gamma = X_a \otimes \omega_a \in \mathcal{X}_q \otimes \Omega^1(\mathrm{SU}_q(2))$$

with $\mathcal{X}_q$ the quantum Lie algebra corresponding to the calculus.

- For any 3d left covariant exterior algebra it is possible to define a symmetric tensor and

$$\star\,:\,\Lambda^k(\mathrm{SU}_q(2))\,\to\,\Lambda^{3-k}(\mathrm{SU}_q(2))$$

with

$$\star\,(1)=\tau,\qquad \star(\tau)=1,\qquad \star^2(\omega_a)=A\omega_a,\qquad \star(\omega_a)=\mu\mathrm{d}\omega_a,$$

$$\text{with}\qquad \lim_{q\to1}A=1,\qquad\qquad \lim_{q\to1}\mu=-1$$

and a duality

$$\langle\phi\mid\phi'\rangle\;=\;\int_\tau\phi'\wedge(\star\phi),$$

$$\langle\mathrm{d}^\dagger\phi\mid\phi'\rangle\;=\;\langle\phi\mid\mathrm{d}\phi'\rangle,\qquad\qquad \mathrm{d}^\dagger\phi=(-1)^k\star\mathrm{d}(\star\phi)$$

which we use to define a KD operator on $\mathrm{SU}_q(2)$ as

$$\mathcal{D}(\phi)\;=\;\mathrm{d}\phi\;-\;(-1)^k\star\mathrm{d}(\star\phi)$$

- We limit to say that the spectrum of the quantum $\mathcal{D}$ is purely discrete, and is a deformation (in q) of the classical one. For such operator one can prove the Hodge decomposition theorem

$$\Lambda(\mathrm{SU}_q(2)) \,=\, \mathrm{d}(\Lambda) \,\oplus\, \mathrm{d}^\dagger(\Lambda) \,\oplus\, \ker(\mathcal{D}^2)$$

- What about the YM eqs on $\mathrm{SU}_q(2)$? It is possible to prove that

$$A \,=\, \frac{1}{2}\gamma^{-1}\mathrm{d}\gamma \qquad \Rightarrow \qquad \mathfrak{D}(\star F) \,=\, 0$$

- We consider $\mathbb{R}^4_q$ as the $*$-algebra generated by x_1, x_2, x_3, x_4 with

$$x_i x_j = q x_j x_i \qquad\qquad i<j,\ i+j \neq 5,$$
$$x_2 x_3 = x_3 x_2, \qquad x_1 x_4 - x_4 x_1 = (q^{-1} - q) x_2 x_3$$

and $x_1 * q x_4, \quad x_2^* = x_3.$

- It has a positive central group like element

$$S \;=\; q x_1 x_4 + q^2 x_2 x_3$$

The localisation $\mathbb{R}^4_q \backslash \{0\}$ comes by adding $\mathbb{R}^4_q$ an invertible central generator $r = r^*$ with $Sr^{-2} = 1$.

- The set $\mathbb{R}^4_q \backslash \{0\}$ gives the Hopf $*$-algebra $GL_q(1, H)$ of quantum invertible quaternions, and $\mathrm{SU}_q(2)$ is a quantum subgroup of it.

- On $GL_q(1, H)$ it is possible to define all the relevant geometric structures needed, and to prove that the vector potential A is transformed into a solution of the YM eqs. This is the *quantum meron*.