Giuseppe Morandi and me

PhD In Bologna years 1994-96: very fruitful and formative period. Collaboration continued for some years after.

What I have learned from him

Follow the beauty (and don't care too much about the experiments)

Giuseppe had a great esthetic sense which guided him in deciding what was worth to be investigated and what was not. To some extent "beautiful" or elegant were synonyms of "interesting" or "important" to him.

Timeo hominem unius libri

Giuseppe was a man of many books (he was really fond of them) and a deeply learned scholar. At a time when people were still not so obsessed by citations (and thus did not demand them explicitly), he was extremely attentive to other people's work.

Do not oversell your work

Giuseppe preferred understatements to emphatic claims.

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Quantal phase factors accompanying adiabatic changes

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Transition temperature of a superfluid Fermi gas throughout the BCS-BEC crossover

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The BCS-BEC crossover

Gas of fermions interacting via an attractive potential:

Weak attraction: Cooper pairs form at low temperature according to BCS theory. Largely-overlapping **pairs form and condense at the same temperature** (T_c).

Strong attraction: the pair-size shrinks and pair-formation is no longer a cooperative phenomenon. Non-overlapping pairs (composite bosons) undergo Bose-Einstein condensation at low temperature. **Pair-formation and condensation** become **unrelated**.

How does the system evolves from one regime to the other one?

The interaction potential

The standard BCS effective potential (constant attraction $-V$ within a shell of width ω_D about the Fermi energy E_F) is **not suitable** for studying the BCS-BEC crossover (Fermi surface meaningless in the BEC limit).

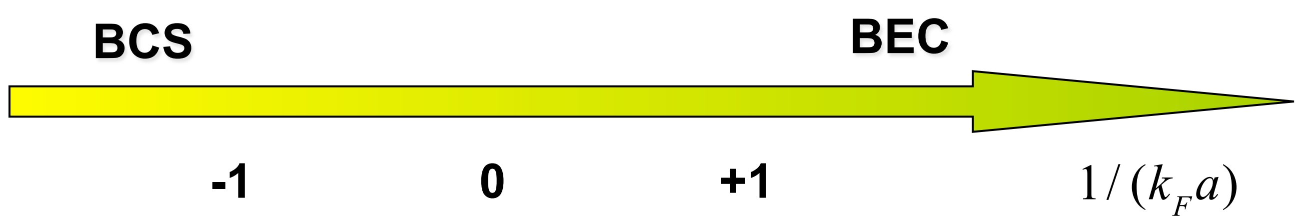
Two body attractive potential $V(\mathbf{k}\text{-}\mathbf{k}')$ is OK. Often approximated by a separable potential $V(k,k')$. For example NSR (or Yamaguchi) potential

$$V(k,k') = -V_0 \; w(k)w(k') \qquad\qquad w(k) = 1/\sqrt{1+(k/k_0)^2}$$

For $k_0 >> k_F \sim n^{1/3}$(diluteness condition) the attractive potential is effectively short-ranged and can be parametrized completely in terms of the s-wave **scattering length** a :

$$\frac{1}{V_0} = \frac{m}{4\pi a} - \int_{|\mathbf{k}|<k_0} \frac{d\mathbf{k}}{(2\pi)^3}\frac{m}{\mathbf{k}^2} \qquad [\text{example for a sharp cutoff } w(k) = \Theta(k_0 - k)]$$

Diluteness condition normally satisfied in experiments with **ultracold Fermi atoms** for which the dimensionless effective **coupling parameter** $1/(k_F a)$ can be tuned from BCS to BEC limit by using appropriate Fano-Feshbach resonances.



BCS critical temperature throughout the BCS-BEC crossover

BCS equation for T_c (for contact potential): $-\dfrac{m}{4\pi a} = \Pi_{\mathrm{pp}}(0)$ (1)

where $\Pi_{\mathrm{pp}}(0) \equiv \int \dfrac{d\mathbf{k}}{(2\pi)^3}\left(\dfrac{\tanh(\xi_{\mathbf{k}}/2T_c)}{2\xi_{\mathbf{k}}} - \dfrac{m}{\mathbf{k}^2}\right)$ with $\xi_{\mathbf{k}} \equiv \varepsilon_{\mathbf{k}} - \mu$.

With **weak-coupling assumption** $T_c << \mu$ one gets

$\Pi_{\mathrm{pp}}(0) = \dfrac{m k_\mu}{2\pi^2}\ln\left(\dfrac{8e^{\gamma-2}\mu}{\pi T_c}\right)$ where $k_\mu \equiv \sqrt{2m\mu}$. Eq. (1) then yields:

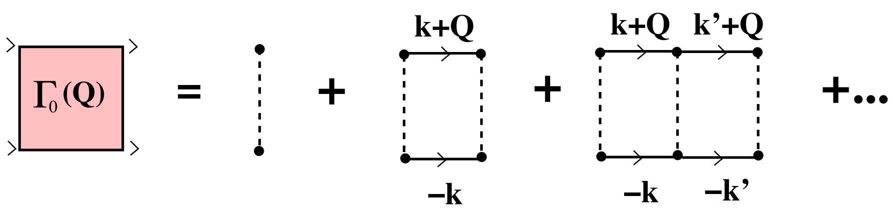
$$T_c = \frac{8e^{\gamma-2}\mu}{\pi}\exp\left(\frac{\pi}{2k_\mu a}\right) = \frac{8e^{\gamma-2}E_F}{\pi}\exp\left(\frac{\pi}{2k_F a}\right)$$ where $\mu = E_F$ for weak coupling.

For stronger coupling T_c becomes of order of (or larger than) T_F : necessary to use **number equation** to adjust chemical potential.
BCS number equation at T_c (and above) reduces to free Fermi gas relation

$$n = 2\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\exp(\xi_{\mathbf{k}}/T_c) + 1} \qquad (2)$$

Eagles was first to notice (1969) that simultaneous solution of (1) and (2) yields in the BEC limit a **"pairing" temperature** T^* rather than critical temperature for condensation.

Interlude: a threefold role for ladder diagrams



$$Q \equiv (\mathbf{Q}, \Omega_\nu)$$

The sum of ladder diagrams $\Gamma_0(Q)$ has a threefold role:

- The **BCS equation for T_c** corresponds to the equation $\Gamma_0^{-1}(Q=0)=0$ (**Thouless criterion**): the two-particle Green's function diverges at zero center-of-mass momentum and frequency.

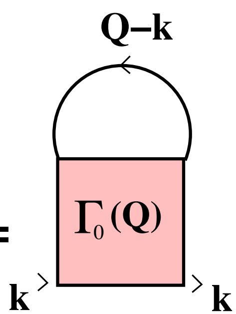
- In the **BEC limit**, $\Gamma_0(Q)$ becomes (except for numerical factors) the **free propagator of molecules** (composite bosons).

- For a **contact potential**, the series of ladder diagrams **replaces the bare interaction in diagrammatic theory.** This is because the "bare" potential strength V_0 vanishes as the momentum cutoff $k_0 \to \infty$. Every particle-particle rung in the ladder series is ultraviolet divergent and compensates a vanishing V_0, making Γ_0 finite for $k_0 \to \infty$.

Nozières-Schmitt-Rink approach to the BCS-BEC crossover

Consider the simplest self-energy which can be constructed with ladder series:

$\Sigma(\mathbf{k}) =$

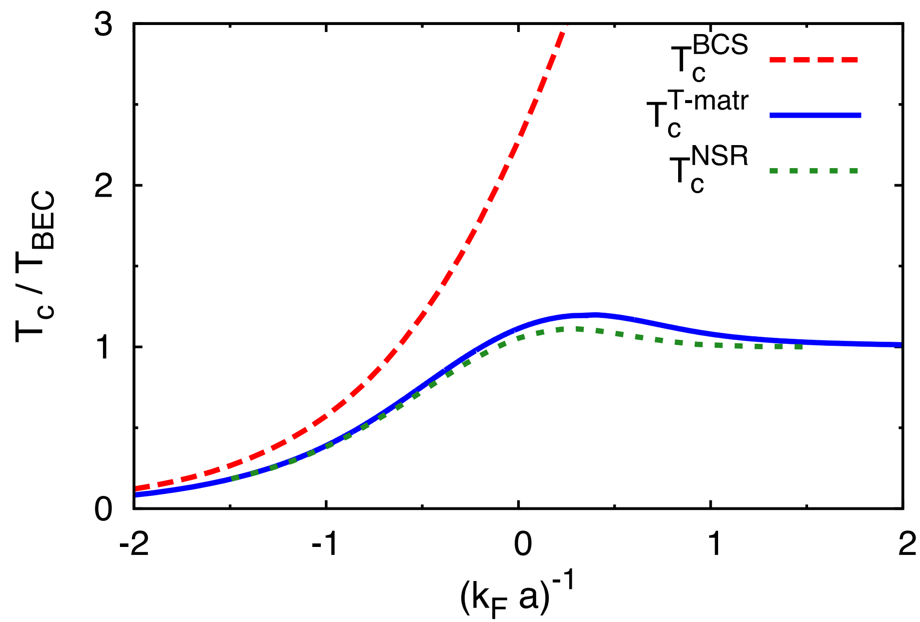


Use number equation n=Tr G, where $G = G_0 + G_0\,\Sigma\, G_0$ coupled with the standard BCS equation for critical temperature.

In this way T_c approaches BE critical temperature in the BEC limit.
True also when the Dyson's equation

$$G^{-1} = G_0^{-1} - \Sigma$$

is used to calculate G and then n (T-matrix self-energy approach).



NSR theory in the weak-coupling limit

Common belief: The NSR curve for T_c interpolates between the BCS and BE critical temperatures. **Not really true**: in the BCS limit the NSR T_c approaches the BCS critical temperature **divided** by a factor $e^{1/3}$.

Reason? In weak coupling: $\Gamma_0 \to -4\pi a_F / m \Longrightarrow \Sigma(k) \to 2\pi n a_F / m \equiv \Sigma_0$.
The number equation in NSR approach then yields: $\mu = E_F + 2\pi n a / m$

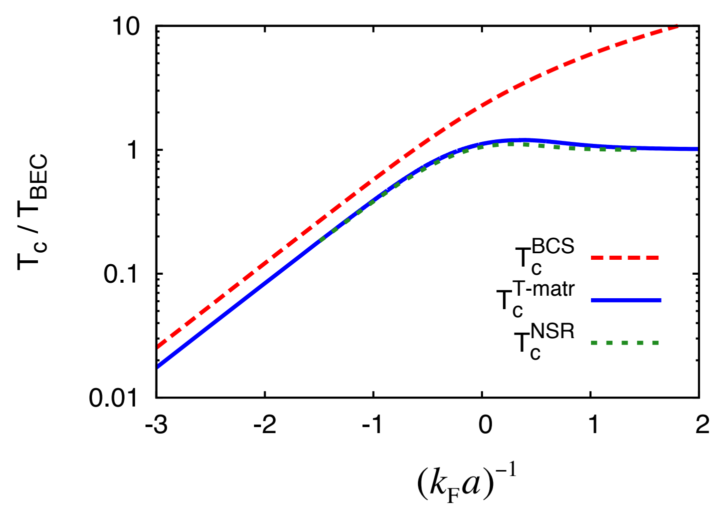
$$\Longrightarrow \quad k_\mu \equiv \sqrt{2m\mu} \cong k_F \left(1 + \frac{2}{3\pi} k_F a\right)$$

Insert it in T_c equation: $T_c = \dfrac{8e^\gamma \mu}{\pi e^2} \exp\left(\dfrac{\pi}{2k_\mu a}\right)$

$$T_c = \frac{8e^\gamma E_F}{\pi e^2} \exp\left(\frac{\pi}{2k_F a_F (1 + 2k_F a / 3\pi)}\right)$$

$$= \frac{8e^\gamma E_F}{\pi e^2} \exp\left(\frac{\pi}{2k_F a_F} (1 - 2k_F a / 3\pi)\right)$$

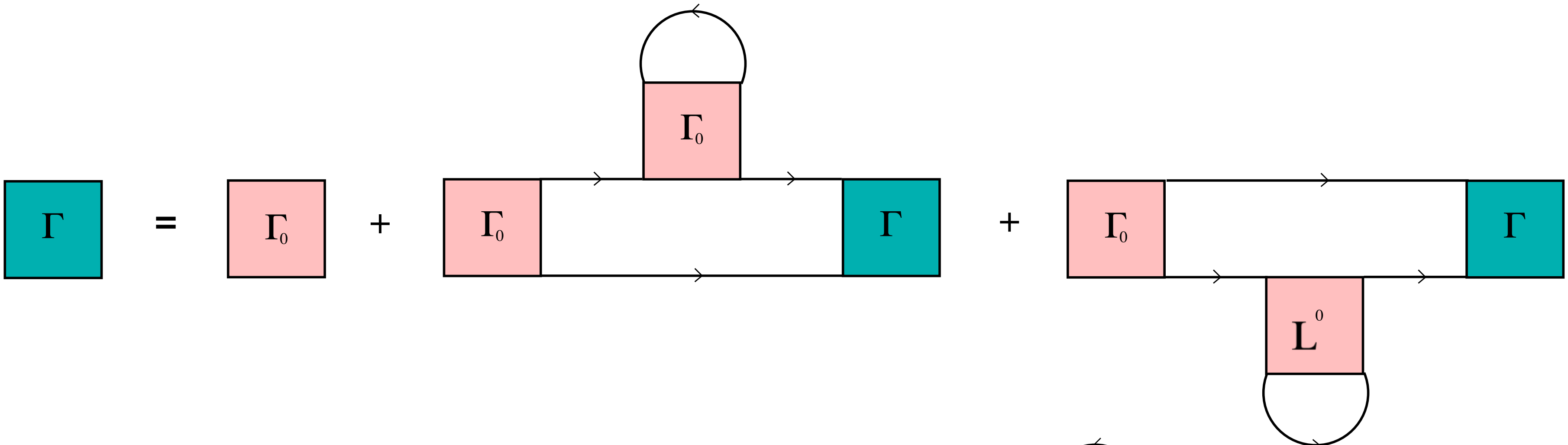
$$= \frac{8e^\gamma E_F}{\pi e^2} \exp\left(\frac{\pi}{2k_F a}\right) \frac{1}{e^{1/3}}$$



Recovering BCS critical temperature in the weak-coupling limit

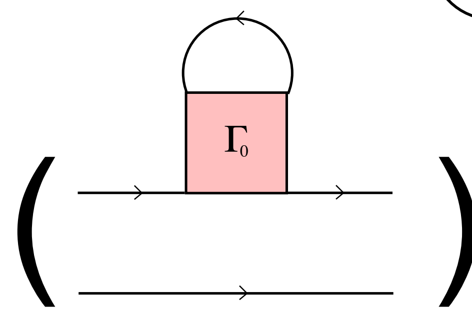
If the fermion propagators within the ladder series are dressed with first-order self-energy $\Sigma_0 = 2\pi na/m$ then $\mu \rightarrow \mu' = \mu - \Sigma_0 = E_F$ and the BCS result for T_c is recovered.

Diagrammatically, to lowest order in weak coupling, it corresponds to **"dressing" the composite-boson propagator** Γ with a "composite-boson self-energy" insertion Σ^{B}.



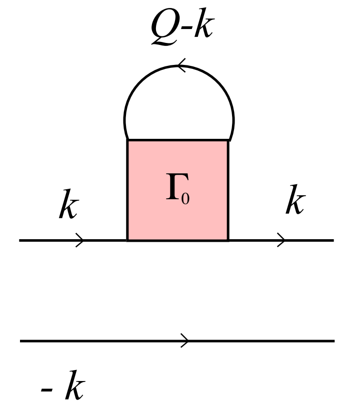
yielding:

$\Gamma^{-1}(Q) = \Gamma_0^{-1}(Q) - \Sigma^{B}_{\text{Pop}}(Q)$ where $\Sigma^{B}_{\text{Pop}} = 2 \times$



“Dressed” Thouless criterion

$$\Gamma^{-1}(Q=0)=0 \quad \Leftrightarrow \frac{m}{4\pi a_F}+\Pi_{\mathrm{pp}}(0)+\Sigma^{\mathrm{B}}_{\mathrm{Pop}}(0)=0$$



$$\Sigma^{\mathrm{B}}_{\mathrm{Pop}}(0)=-2\sum_{k,Q}G_0(k)^2G_0(-k)G_0(Q-k)\Gamma_0(Q)$$

$$\cong -\frac{4\pi a}{m}n_0\frac{m^2}{4\pi^2 k_\mu}\ln\left(\frac{8e^{\gamma-2}\mu}{\pi T}\right) \qquad \cong -\frac{4\pi a}{m}\frac{k_\mu^3}{3\pi^2}\frac{m^2}{4\pi^2 k_\mu}\left(-\frac{\pi}{2k_\mu a}\right) = \frac{mk_\mu}{6\pi^2}$$

For $T \approx T_c$, $\ln(\mu/T)$ originating from $\sum_k G_0(k)^2G_0(-k)$ compensates the small parameter a !

Eq. for T_c becomes $\quad \frac{m}{4\pi a}+\frac{mk_\mu}{2\pi^2}\ln\left(\frac{8e^{\gamma-2}\mu}{\pi T_c}\right)+\frac{mk_\mu}{6\pi^2}=0 \quad \Rightarrow \quad \frac{\pi}{k_\mu a}+\ln\left(\frac{8e^{\gamma-2}\mu}{\pi T_c}\right)+\frac{1}{3}=0$

“Counter-term” 1/3 compensates -1/3 from expansion of k_μ. **BCS result for T_c is recovered.**

The GMB correction in the weak-coupling limit (I)

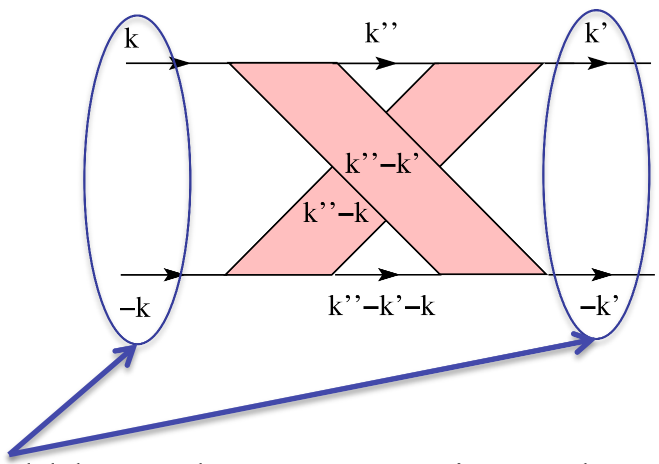
But is BCS expression for T_c really correct in the weak-coupling limit? **No!**

Gorkov and Melik-Barkhudarov (1961): For a low density attractive Fermi gas, the BCS expression **is reduced** by a factor $(4e)^{1/3} \sim 2.2$ in the weak-coupling limit.

Reason? Same mechanism just discussed: **compensation** of small parameter a with large factor $\ln(\mu/T)$ originating from particle-particle bubble $\sum_k G_0(k) G_0(-k)$

Dress Γ with following "self-energy" insertion:

$$\Sigma^{\mathrm{B}}_{\mathrm{GMB}}(0) =$$



which contains two pp bubbles and two $\Gamma_0 \sim a$ in weak coupling.

The GMB correction in the weak-coupling limit (II)

$$\Sigma^{\Gamma}_{\mathrm{GMB}}(0) = \sum_{k,k',k''} G_0(k)G_0(-k)G_0(k')G_0(-k')G_0(k'')G_0(k''-k-k')\Gamma_0(k''-k)\Gamma_0(k''-k')$$

$$\sim \underbrace{\left(-\frac{4\pi a}{m}\sum_k G_0(k)G_0(-k)\right)}_{\substack{=1+O(k_F a)\\ \text{and select } k\cong(\mathbf{k}_\mathrm{F},0)}} \underbrace{\left(-\frac{4\pi a}{m}\sum_{k'} G_0(k')G_0(-k')\right)}_{\substack{=1+O(k_F a)\\ \text{and select } k'\cong(\mathbf{k}'_\mathrm{F},0)}} \underbrace{\sum_{k''}\langle G_0(k'')G_0(k''-k-k')\rangle}_{\substack{=\frac{1}{2}\int d(\hat{\mathbf{k}}_\mathrm{F}\cdot\hat{\mathbf{k}}'_\mathrm{F})\chi_{\mathrm{ph}}(\mathbf{k}_\mathrm{F}+\mathbf{k}'_\mathrm{F},0)\\ =-\frac{mk_\mathrm{F}}{2\pi^2}\ln(4e)^{1/3}}}$$

Equation for T_c: $\quad \dfrac{m}{4\pi a}+\dfrac{mk_\mathrm{F}}{2\pi^2}\ln\left(\dfrac{8e^{\gamma-2}E_\mathrm{F}}{\pi T_c}\right)-\dfrac{mk_\mathrm{F}}{2\pi^2}\ln(4e)^{1/3}=0$

$$T_c = \frac{8e^{\gamma}E_F}{\pi e^2}\exp\left(\frac{\pi}{2k_F a}\right)\frac{1}{(4e)^{1/3}}$$

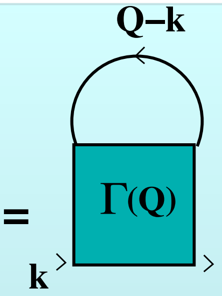
Note: for standard BCS potential, the GMB reduction is instead $\exp(\omega_D/E_F) \sim 1$ since $\omega_D \ll E_F$. GMB correction irrelevant in this case.

GMB (and Popov) correction throughout the BCS-BEC crossover

Coupled equations:

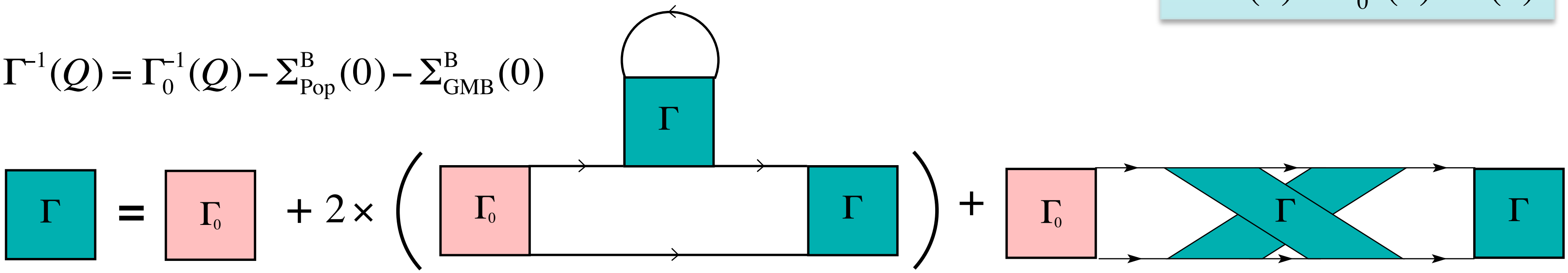
$$\Gamma^{-1}(Q=0)=0 \quad \Leftrightarrow \frac{m}{4\pi a_F}+\Pi_{\mathrm{pp}}(0)+\Sigma^{\mathrm{B}}_{\mathrm{Pop}}(0)+\Sigma^{\mathrm{B}}_{\mathrm{GMB}}(0)=0$$

$$n=\mathrm{Tr}\,\mathrm{G}=2T\sum_n\int\frac{d\mathbf{k}}{(2\pi)^3}G(\mathbf{k},\omega_n)e^{i\omega_n 0^+}$$

$\Sigma(\mathbf{k})=$ 

$$G^{-1}(k)=G_0^{-1}(k)-\Sigma(k)$$

$$\Gamma^{-1}(Q)=\Gamma_0^{-1}(Q)-\Sigma^{\mathrm{B}}_{\mathrm{Pop}}(0)-\Sigma^{\mathrm{B}}_{\mathrm{GMB}}(0)$$



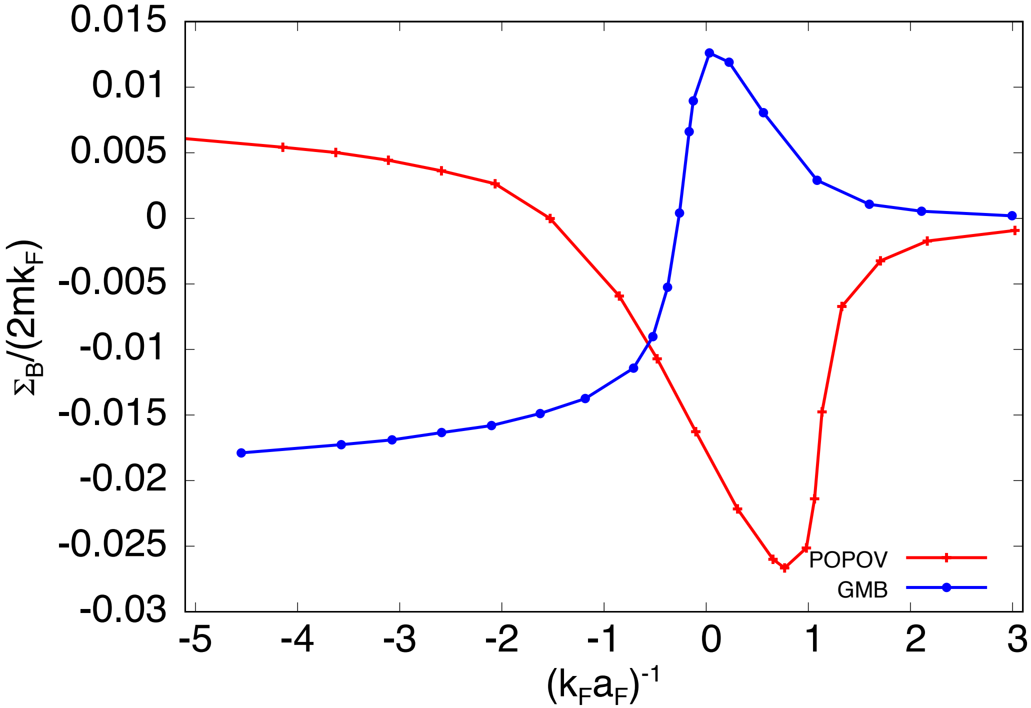
Retain **full momentum and frequency dependence** of Γ and avoid all weak-coupling approximations when calculating $\Sigma^{\mathrm{B}}_{\mathrm{Pop}}$ and $\Sigma^{\mathrm{B}}_{\mathrm{GMB}}$:

$$\Sigma^{\mathrm{B}}_{\mathrm{GMB}}(0)=\sum_{k,k',k''}G_0(k)G_0(-k)G_0(k')G_0(-k')G_0(k'')G_0(k''-k-k')\Gamma(k''-k)\Gamma(k''-k')$$

$$\Sigma^{\mathrm{B}}_{\mathrm{Pop}}(0)=-2\sum_{k,Q}G_0(k)^2G_0(-k)G_0(Q-k)\Gamma(Q)$$

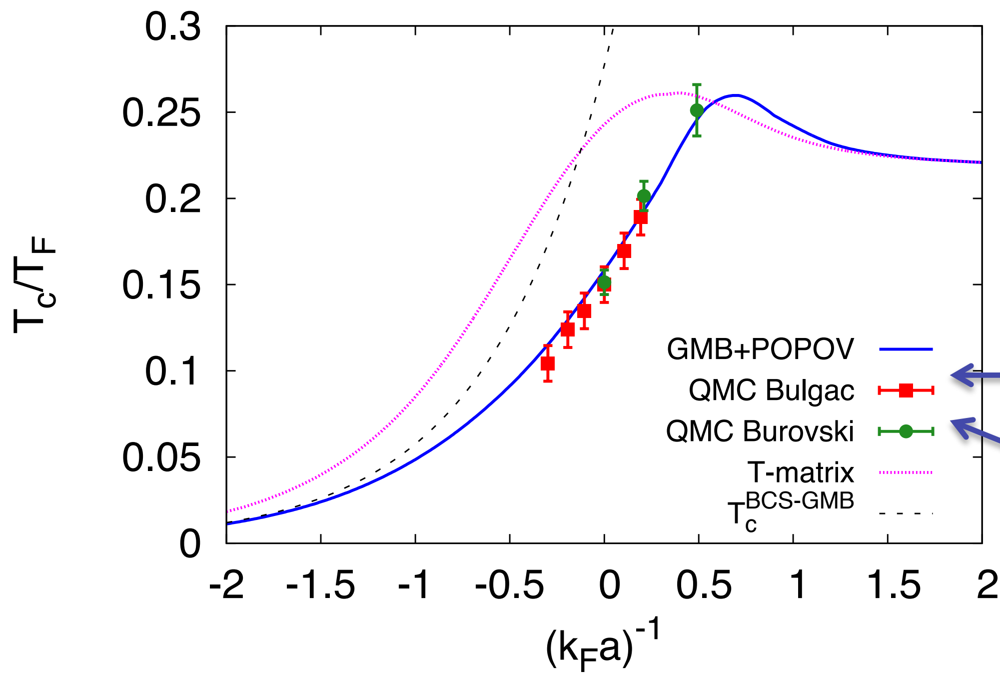
Calculation quite non-trivial!

Numerical results: GMB and Popov self-energies at Tc



Inclusion of both self-energies is important from weak coupling to past unitarity. To some extent, they compensate each other.

Numerical results for the critical temperature



A. Bulgac *et al.*, PRA **78**, 023625 (2008)

E. Burovski *et al.*, PRL **101**, 090402 (2008)

The resulting curve for T_c converges to the **GMB result for weak coupling** and to the **BE critical temperature for strong coupling**. In the intermediate coupling region it **agrees** very well with **diag-QMC and lattice QMC**.

Summary and conclusions

- GMB found that for a dilute attractive Fermi gas the BCS critical temperature is reduced by a numerical factor.
- Their work was formulated in the weak coupling regime of the attraction.
- Standard diagrammatic approximations for the BCS-BEC crossover (NSR, T-matrix) fail to recover GMB result in weak coupling.
- We have reformulated GMB's work in a way which is amenable to extension to the whole BCS-BEC crossover.
- Consistency in the weak-coupling limit required us to also include "Popov" two-particle self-energy.
- The resulting curve for T_c agrees very well with available QMC data.

Thank you!