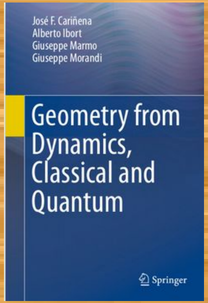
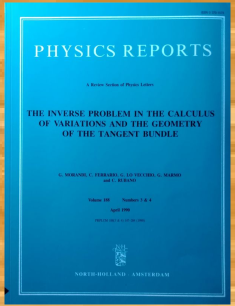
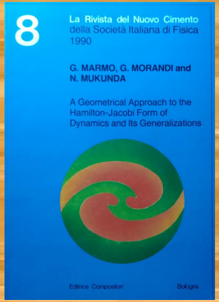
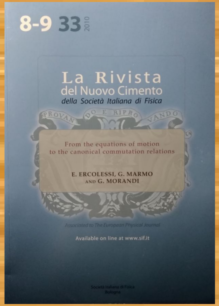
From trajectories to vector fields and commutation relations





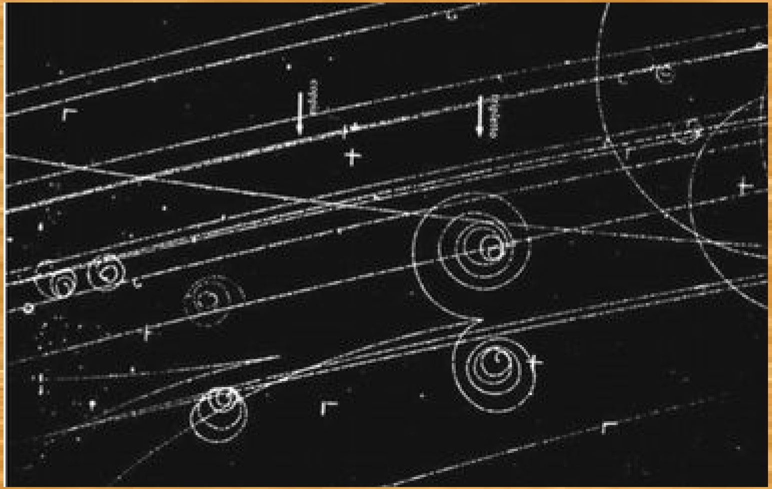




45 years with Giuseppe

From experimental data to differential equations

Implicit versus explicit second order (higher order)



$$S \equiv \{\gamma\colon I \to Q\}\,, \qquad \left(\gamma^j, \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\right) \equiv \chi^j$$

$$\frac{\mathrm{d}\chi^j}{\mathrm{d}t} \to \begin{cases} \dfrac{\mathrm{d}\gamma^j}{\mathrm{d}t} = v^j \\[2ex] \dfrac{\mathrm{d}^2\gamma^j}{\mathrm{d}t^2} = F^j \end{cases}$$

→ Field of forces

→ A vector field on TQ

From forces to Lagrangian description on TQ

Euler-Lagrange equations as an equation for the Lagrangian:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial v^j}-\frac{\partial \mathcal{L}}{\partial x^j}=0 \Longrightarrow \begin{cases} \dfrac{\mathrm{d}x^j}{\mathrm{d}t}=v^j \\ \dfrac{\partial^2 \mathcal{L}}{\partial v^j \partial v^k}\dfrac{\mathrm{d}v^k}{\mathrm{d}t}=\dfrac{\partial \mathcal{L}}{\partial x^j}-\dfrac{\partial^2 \mathcal{L}}{\partial v^j \partial x^k}\dfrac{\mathrm{d}x^k}{\mathrm{d}t} \end{cases}$$

A linear PDE for the Lagrangian

Many fields of forces admit alternative Lagrangian descriptions!

Hamiltonian description on TQ

$$\begin{cases} \dfrac{\mathrm{d}x^j}{\mathrm{d}t} = \{H\,, x^j\}\,, \qquad H = H(\mathbf{x}\,, \mathbf{v}) \\[2ex] \dfrac{\mathrm{d}^2x^j}{\mathrm{d}t^2} = \{H\,, \{H\,, x^j\}\} = F^j(\mathbf{x}\,, \mathbf{v}) \end{cases}$$

Unknown { , } and H, highly nonlinear equation.

Requiring localization $\{x^j, x^k\} = 0$.

The Hamiltonian problem is included in the Lagrangian one.

Alternative descriptions.

Weyl systems for TQ

Assuming $Q = \mathbb{R}^n \,, \quad \det \left\| \frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^k} \right\| \neq 0$

We may use as global coordinates for TQ

$$\left(x^j \,, \frac{\partial \mathcal{L}}{\partial v^j} \right) \equiv (x^j \,, \alpha_j) \;, \quad \omega_{\mathcal{L}} = \mathrm{d}\alpha_j \wedge \mathrm{d}x^j$$

Weyl system $W(x\,,\alpha) \in \mathcal{U}(\mathcal{H})$:

$$W(\mathbf{e}_1) W(\mathbf{e}_2) W^\dagger(\mathbf{e}_1) W^\dagger(\mathbf{e}_2) = \mathrm{e}^{\omega_{\mathcal{L}}(\mathbf{e}_1\,,\mathbf{e}_2)}\,\mathbb{I}$$

Weyl systems for TQ

On a Lagrangian subspace V:

$$\mathcal{H} = L^2(V\,,\mu) \qquad\qquad [W(x\,,\alpha)\psi]\,(y) = \mathrm{e}^{\imath\alpha(y)}\psi(x+y)$$

→ Commutation relations

Alternative Lagrangians, alternative solutions of:

$$\frac{\partial^2\mathcal{L}}{\partial v^j\partial v^k}F^k = \frac{\partial\mathcal{L}}{\partial x^j} - \frac{\partial^2\mathcal{L}}{\partial v^j\partial x^k}v^k$$

→ Alternative linear structures

→ Alternative commutation relations

Weyl systems for H

Replacing TQ with the Hilbert space H we may consider:

$$W(\psi_1), W(\psi_2)\, W^\dagger(\psi_1)\, W^\dagger(\psi_2) = \mathrm{e}^{-\imath\langle\psi_1|\psi_2\rangle}\, \mathbb{I}$$

Along the way:

- Integrability;
- Geometrization of algebraic structures;
- Physical aspects of topology (monopoles, Berry phase, QHE).