Why does relativistic quantum mechanics need complex Hilbert spaces?

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References

- V.M. and M.Oppio: *Quantum theory in quaternionic Hilbert space: How Poincaré symmetry reduces the theory to the standard complex one*, submitted, arXiv:1709.09246 (71 pages)
- V.M. and M.Oppio: *Quantum theory in real Hilbert space: How the complex Hilbert space structure emerges from Poincaré symmetry*
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spectral theory, **operator algebra theory**, and **functional analysis** results in **quaternionic** Hilbert spaces.

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- R.Ghiloni, V.M. and A.Perotti: *Continuous slice functional calculus in quaternionic Hilbert spaces*
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1. General structure of Quantum Theories

1.1. Abstract Orthomodular Lattice of Propositions

- QM in a $\mathbb{C}$ Hilbert space H can be formulated using **elementary observables** with possibles outcomes 1 (TRUE) or 0 (FALSE) represented by *orthogonal projectors* $P \in \mathfrak{L}(\mathsf{H})$.
- General **obeservables** $A = A^\dagger$ give rise to elementary observables $P_E^{(A)}$ =the outcome of A stays in $E \subset \sigma(A)$.
- $P_E^{(A)} \in \mathfrak{L}(H)$ defines the **spectral measure** of A such that $A = \int_{\sigma(A)} \lambda dP^{(A)}(\lambda)$.
- $\mathfrak{L}(\mathsf{H})$ is a lattice with respect to the operations $P \vee P'$ (projector onto $\overline{P(\mathsf{H}) + P'(\mathsf{H})}$) and $P \wedge P'$ (projector onto $P(\mathsf{H}) \cap P'(\mathsf{H})$) with orthocomplement $P^\perp$ (projector onto $P(\mathsf{H})^\perp$).
- $(\mathfrak{L}(\mathsf{H}), \vee, \wedge, ^\perp)$ enjoys formal properties similar to those of the classical logic intepreting $\vee = OR$, $\wedge = AND$, $^\perp = NOT$, provided the involved projectors **commute**, i.e. are quantistically compatible.

1.2. The coordinatization problem

- $(\mathcal{L}(\mathsf{H}), \vee, \wedge, ^{\perp})$ is not a classical logic (a Boolean lattice) due to **incompatible elements**. It includes minimal elements (projectors onto one-dimensional subspaces): atoms.
- $(\mathcal{L}(\mathsf{H}), \vee, \wedge, ^{\perp})$ has the abstract structure of an orthomodular, σ-complete, irreducible, atomic, lattice. Structure defined **without** referring to a Hilbert space.
- Every property of $(\mathcal{L}(\mathsf{H}), \vee, \wedge^{\perp})$ has an operationistic interpretation.
- Coordinatization problem (since von Neumann $\sim$ 1939): Is every orthomodular, σ -complete, irreducible, atomic lattice isomorphic to the concrete lattice $\mathcal{L}(\mathsf{H})$ over a complex Hilbert space?
- Several attempts, especially by Piron $\sim$ 1960
- **Solér's theorem (1995)**: Every such lattice with infinite orthogonal atoms is isomorphic to a concrete lattice of orthogonal projectors of a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ (algebra of quaternions: $a + bi + cj + dk$).

1.3.1. A common formulation

Let H be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ indifferently.

- quantum states are σ-**additive probability measures**

  $$\mu : \mathcal{L}(\mathsf{H}) \ni P \mapsto \mu(P) \in [0, 1]$$

  Gleason-Varadarajan's theorem: if $\dim \mathsf{H} \neq 2$ such measures are **density matrices** $\rho : \mathsf{H} \to \mathsf{H}$ and

  $$\mu_\rho(P) = tr(P\rho) \quad \text{for } P \in \mathcal{L}(\mathsf{H}) \qquad \square$$

- extremal states called **pure states** are of the form $\rho = |\psi\rangle\langle\psi|$ with $||\psi|| = 1$ determined up to signs ($\mathbb{R}$, $\mathbb{H}$) or phases ($\mathbb{C}$).
- observables are collections of $P_E \in \mathcal{L}(\mathsf{H})$, with $E \subset \mathbb{R}$ Borel set, enjoying properties of Projection-Valued-Mesures over $\mathbb{R}$, e.g. $P_E P_F = P_{E \cap F}$.
- The **spectral theorem** holds and $A = \int_{\mathbb{R}} \lambda dP(\lambda)$ is a **self-adjoint operator** with $\sigma(A) = supp(P) \subset \mathbb{R}$. Observables and self-adjoint operators are one-to-one.

1.3.2. A common formulation

- **Lüders-von Neuamnn reduction postulate** is valid for the three cases $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$:

  If P has outcome YES in the state ρ, after measurement
  $$\rho \to \frac{P\rho P}{tr(P\rho)}.$$
- **Wigner-Kadison-Segal theorems** hold: **symmetries** are always determined by **unitary** ($\mathbb{R}$, $\mathbb{H}$) or also **anti-unitary** ($\mathbb{C}$) operators $U : \mathsf{H} \to \mathsf{H}$, up to signs ($\mathbb{R}$, $\mathbb{H}$) or phases ($\mathbb{C}$).
- **Continuous symmetries** are strongly continuous one-paramiter groups of unitaries $\mathbb{R} \ni s \mapsto U_s$

REMARK: For a σ-complete **sublattice** $\mathcal{L} \subset \mathcal{L}(\mathsf{H})$ (e.g., presence of **superselection rules**) everything holds but
(1) correspondence **states-density matrices** not one-to-one
(2) arbitrary **signs/phases** of unitaries representing symmetries replaced by **operators**.

1.4. Problems

- **Composite systems** $\Leftrightarrow$ **tensor product**.
  PROBLEM: $\otimes$ does not exist in quaternionic Hilbert spaces.
  (candidate solutions exist)
- **Continuous dynamical symmetries** $U_s \Leftrightarrow$ **generators** A **constant of motion** using **Stone theorem** $U_s = e^{sA_0}$ with $A_0^\dagger = -A_0$ and $A := -iA_0$ is an observable.
  PROBLEM: Stone theorem holds in real and quaternionic spaces but i does not exist in $\mathbb{R}$ Hilbert spaces and, in a sense, there are no or too many i in $\mathbb{H}$ Hilbert spaces. (candidate solutions exist)
- $\mathbb{C}$-**Hilbert space formulation** revealed to be the **fundamental description** of quantum physics (at elementary level at least)
  PROBLEM: Though theoretically permitted, there is no physical evidence of elementary quantum systems described in $\mathbb{R}$- or $\mathbb{H}$-Hilbert space.

2. Elementary (Relativistic) Quantum Systems

2.1. von Neumann Algebras and Schur's Lemma

- **Unbounded observables** of a quantum system S are (strong) limits of **bounded observables**. **Bounded observables** are the self-adjoint elements of a *-algebra R_S of operators called the **von Neumann algebra** of the quantum system enjoying important algebraic and topological properties.
- R_S includes other relevant operators like those representing symmetries etc. R_S includes the full information on the system.
- R_S is **irreducible** if there are no non-trivial orthogonal projectors $P \in \mathfrak{L}(\mathsf{H})$ such that $AP = PA$ for every $A \in R_S$
- The **commutant** R'_S is the set of bounded operators commuting with every operator in R_S. It holds (**von Neumann double commutant theorem**): $R_S = R''_S$

Complex Schur's lemma: R vN algebra over a $\mathbb{C}$-Hilbert space. R irriducible $\Leftrightarrow$ $R' = \{cI\}_{c\in\mathbb{C}}$. Therefore $R = R'' = \mathbb{C}I'' = B(\mathsf{H})$. $\square$

Real and Quaternionic Schur's lemma: R vN algebra over an either $\mathbb{R}$- or $\mathbb{H}$-Hilbert space.
R irreducible $\Leftrightarrow$ one of the three possibilities occurs for R':

- (real type) $R' = \{rI\}_{r\in\mathbb{R}}$;
- (complex type) $R' = \{aI + bJ\}_{a,b\in\mathbb{R}}$, where $JJ = -I$ and $J^* = -J$ is fixed up to sign and $J \in R \cap R'$.
- (quaternionic type) $R' = \{aI + bJ_1 + cJ_2 + dJ_3\}_{a,b,c,d\in\mathbb{R}}$, where $J_rJ_r = -I$, $J_r^* = -J_r$ and $J_rJ_s = -J_sJ_r$.

The **center** $Z_R := R \cap R'$ is

- $Z_R = \{rI\}_{r\in\mathbb{R}}$ in the real and quaternionc case,
- $Z_R = \{aI + bJ\}_{a,b\in\mathbb{R}}$ in the complex case. $\square$

2.2. Elementary Quantum Systems

Assume the **lattice of elementary observables** $\mathcal{L}_S$ of a quantum system S is the set of orthogonal projectors of a certain vN algebra R_S, nomatter if H_S is over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

DEF. S is **elementary** if R_S is irreducible.

MOTIVATION

- The Hilbert space H_S describes only S: R'_S cannot be intepreted as the vN algebra of a **complementary independent** system S' since R'_S does not includes non-trivial observables.
- In $\mathbb{C}$-Hilbert spaces H_S, **Wigner's notion of elementary particle** and the standard **non-relativistic particle with spin** are **elementary systems**: $R_S = B(\mathsf{H}_S)$
- R_S is irreducible under standard conditions (1)+(2):
  (1) absence of **superselection rules** (center of $\mathcal{L}_S$ trivial)
  (2) existence of a **maximal set of commuting observables**.

2.2. Elementary Relativistic Quantum Systems

DEF. S is **relativistic** if it supports a **(locally faithful) representation**

$$SL(2,\mathbb{C}) \ltimes \mathbb{R}^4 \ni g \mapsto h_g^{(S)}$$

in terms of **automorphism** $h_g^{(S)} : \mathcal{L}_S \to \mathcal{L}_S$ of the lattice $\mathcal{L}_S \subset R_S$ of elementary observables and $g \mapsto \mu(h_g^{(S)}(P))$ is **continuous** for every state μ and $P \in \mathcal{L}_S$.

THM. If S is **elementary** and $h^{(S)}$ is **irreducible** (no invariant P), then there is a strongly-continuous unitary representation

$$SL(2,\mathbb{C}) \ltimes \mathbb{R}^4 \ni g \mapsto U_g^{(S)}$$

such that (*) $h_g(P) = U_g^{(S)} P U_g^{(S)\dagger}$ for $P \in \mathcal{L}_S$, $g \in SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$. $U_g^{(S)}$ is determined by (*) up to elements of the center Z_{R_S}.

2.2. Elementary Relativistic Quantum Systems

An **elementary relativistic quantum system**;

- is elementary as a quantum system,
- admits a continuous action of Poincaré group
- no non-trivial elementary observables is fixed under the representation;
- its vN algebra is completely determined by its symmetry group.

DEF. S is an **elementary relativistic quantum system** if is elementary, relativistic, the representation h is irreducible, and its unitary implementation U generates the vN algebra R_S of the system ($R_S = U^{(S)} \vee Z_{R_S}$)

NB: If H_S is complex, S is an elementary quantum system if and only if is an elementary perticle according to **Wigner's definition**.

3. Equivalence to the Complex Hilbert Space Case

3.1. Main Technical Result

THEOREM. Let S be a **elementary relativistic quantum system** real, complex or quaternionic.
If the anti-selfadjoint generators of **spacetime displacements** P_μ of the Poincaré unitary representation U satisfy

$$P_0 P_0 - \sum_{k=1}^{3} P_k P_k \geq 0$$

on **Gårding's domain**. The following facts hold

(a) the commutant R'_S of the vN algebra R_S of the system

- **is trivial** $\{cI \mid c \in \mathbb{C}\}$ for **complex** H_S,
- is of **complex type** $R'_S = \{aI + bJ\}_{a,b\in\mathbb{R}}$, where $JJ = -I$ and $J^* = -J$ is fixed up to sign and $J \in R_S \cap R'_S$ for **real** and **quaternionic** H_S.

(b) $J|P_0| = P_0$ is the **polar decomposition** of P_0.

(c) J is **Poincaré invariant**: $U_g^{(S)} J U_g^{(S)\dagger} = J$.

(d) The representation $U^{(S)}$ alone generates the full vN algebra R_S.

3.2.1. Equivalence with the Complex Hilbert Space Case

Suppose H_S is **real**, define the **complex** Hilbert space H_{SJ}defining

$$(a+ib)\psi := a\psi + bJ\psi, \quad a,b \in \mathbb{R},\ \psi \in \mathsf{H}_S = \mathsf{H}_{SJ}$$

$$\langle\psi|\psi'\rangle_J := \langle\psi|\psi'\rangle - i\langle\psi|J\psi'\rangle\ , \quad \psi,\psi' \in \mathsf{H}_S = \mathsf{H}_{SJ}$$

- Only the operators in $R_S \subsetneq B(\mathsf{H}_S)$ are $\mathbb{C}$-**linear** (commute with J) and can be viewed as operators in H_{SJ}.
- $R_S \ni A \mapsto A \in B(\mathsf{H}_{SJ})$ and $\mathcal{L}_{R_S} \ni P \mapsto P \in \mathcal{L}(\mathsf{H}_{SJ})$ **continuous *-isomorphism** of real (vN) algebras and **isomorphism** of orthocomplemented lattices.
- $B(\mathsf{H}_{SJ})$ includes exactly the relevant operators describing S.
- **Quantum states** (probability measures) on $\mathcal{L}_{R_S}$ are exaclty **density matrices** on $\mathcal{L}(\mathsf{H}_{SJ})$, in particular pure states $\equiv$ unit vectors up to phases.
- The representation U_S viewed in H_{SJ} is a standard **Wigner elementary particle** (with $m^2 \geq 0$).

3.2.2. Equivalence with the Complex Hilbert Space Case

Suppose H_S is **quaternionic**, define the **complex** Hilbert space

$$\mathsf{H}_{SJ} := \{\psi \in \mathsf{H}_S \mid J\psi = i\psi\}$$

that is a **complex subspace** but **not** a **quaternionic subspace**,

$$\langle\psi|\psi'\rangle_J := \langle\psi|\psi'\rangle\ , \quad \psi, \psi' \in \mathsf{H}_{SJ} \subset \mathsf{H}_S$$

- Only the operators in $R_S \subsetneq B(\mathsf{H}_S)$ **leave** H_{SJ} **invariant** (commute with J) and can be viewed as operators in H_{SJ}.
- $R_S \ni A \mapsto A|_{\mathsf{H}_{SJ}} \in B(\mathsf{H}_{SJ})$ and $\mathcal{L}_{R_S} \ni P \mapsto P|_{\mathsf{H}_{SJ}} \in \mathcal{L}(\mathsf{H}_{SJ})$ **continuous *-isomorphism** of real (vN) algebras and **isomorphism** of orthocomplemented lattices.
- $B(\mathsf{H}_{SJ})$ includes exactly the relevant operators describing S.
- **Quantum states** (probability measures) on $\mathcal{L}_{R_S}$ are exaclty **density matrices** on $\mathcal{L}(\mathsf{H}_{SJ})$, in particular pure states $\equiv$ unit vectors up to phases.
- The representation U_S restricted to H_{SJ} is a standard **Wigner elementary particle** (with $m^2 \geq 0$).

3.3.1 Conclusions: does some way out remain?

A **relativistic elementary system** (absence of superselection rules, maximal set of commuting observables, irreducible representation of Poincaré group determining the algebra of observables) always admits a faithful **complex-Hilbert space description** even if starting form **real** or **quaternionic** Hilbert spaces.

- The arising complex description is of Wigner type.
- The Poincaré invariant **complex structure** J responsible for the complex description arises form the symmetry group: it is suggested by physics.
- The **complex** description is non-redundant with respect to the **real** and **quaternionic** one as all self-adjoint operators have a physical role of **observables**.

3.3.2 Conclusions: does some way out remain?

QUESTION. Do real or quaternionic theories make sense?

There are very important **physical systems** whose description in complex Hilbert spaces is given in terms of von Neumann algebras that are **not** isomorfic to some $B(\mathsf{H})$ differently from the studied cases.

- **extended thermodynamical systems** (*type-III* vN algebras)
- **local algebras of quantum fields** (*type-III* vN algebras))
- elementary quantum systems with internal **nonAbelian gauge group** like **quarks** (non-trivial R' in complex Hilbert space)

Are there some corresponding descriptions in real or quaternionic Hilbert spaces perhaps involving new physics?

Thank you very much for your attention!