Poincaré-Covariant Analysis: Physics Case

Within quantum field theories, the Bethe–Salpeter framework, underpinned by the Dyson–Schwinger equation controlling the dressed quark propagator, enables the Poincaré-covariant description of quark–antiquark bound states. This quark Dyson–Schwinger equation is part of the infinite tower of coupled Dyson–Schwinger equations, which requires to truncate this tower to a finite set of coupled relations. The merits of such a covariant approach are evident:

- Quark models constitute a convenient framework for the comprehensive investigation of hadron states by comparatively simple technical means.
- Technical/computational constraints limit nonperturbative approaches.
- The covariant analyses use QCD input and modelling to bridge this gap.

Understandably, the first target of the covariant approach usually is the case of quarkonia, bound states of a quark and its antiquark, and thus flavourless. In order to gain a comprehensive picture, we complete this kind of studies by applying a single common framework to all conceivable flavour combinations and fathom its implications for the predicted meson masses, decay constants and in-meson condensates by comparison with experiment or other findings:

- Covariant studies of open-flavour mesons have been and remain limited.
- Nonetheless, a covariant study yields utmost extensive sets of results [1].
Dyson–Schwinger–Bethe–Salpeter Liaison

The Bethe–Salpeter framework represents a bound state of total momentum $P$, composed of quark and antiquark of relative momentum $p$, by such state’s Bethe–Salpeter amplitude $\Gamma(p; P)$ or Bethe–Salpeter wave function $\chi(p; P)$, related by the dressed propagators of the two bound-state constituents, $S_{1,2}$:

$$\chi(p; P) \equiv S_1(p + \eta P) \Gamma(p; P) S_2(p - (1 - \eta) P) , \quad \eta \in \mathbb{R} .$$

These propagators may be obtained as the solutions of the Dyson–Schwinger equation for the quark two-point function, in rainbow truncation of the form

$$S^{-1}(p) = Z_2 (i \gamma \cdot p + m_b) + \frac{4}{3} Z_2^2 \int_q \mathcal{G}((p - q)^2) T_{\mu\nu}(p - q) \gamma_\mu S(q) \gamma_\nu ,$$

adopting current-quark wave-function renormalization constant $Z_2$ and bare mass $m_b$, the free Landau-gauge gluon-propagator transverse-projector part

$$T_{\mu\nu}(k) \equiv \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} ,$$

translationally invariant integration measure $\int_q^\Lambda$ Pauli–Villars regularized at scale $\Lambda$, and an effective coupling $k^2 \mathcal{G}(k^2)$ mimicking the effects of full gluon propagator and full quark–gluon vertex; the mass renormalization factor $Z_m$ relates bare quark mass $m_b$ and renormalized quark mass $m_q(\mu)$ at a scale $\mu$:

$$m_b = Z_m m_q(\mu) .$$

A bound state’s Bethe–Salpeter amplitude or wave function is governed by a homogeneous Bethe–Salpeter equation, which in ladder truncation (in order to satisfy the QCD axial-vector Ward–Takahashi identity) reads, for mesons,

$$\Gamma(p; P) = -\frac{4}{3} Z_2^2 \int_q^\Lambda \mathcal{G}((p - q)^2) T_{\mu\nu}(p - q) \gamma_\mu \chi(q; P) \gamma_\nu .$$

Expansion of $\Gamma(p; P)$ in Lorentz covariants recasts this bound-state equation into a system of four (for bound states of spin zero) or eight (for bound states of non-zero spin) coupled equations. An estimate of (some of) the systematic uncertainties inherent to such treatment can be acquired by adopting for the effective couplings $k^2 \mathcal{G}(k^2)$ at least two different (rather popular) models [2].

From the solution for a bound state’s Bethe–Salpeter amplitude $\Gamma(p; P)$ and mass $M$, we find its decay constant $f$ and in-hadron condensate $|\langle \bar{q} q \rangle|^{1/3}$ [3], the hadron-to-vacuum matrix element of the relevant quark-bilinear density.
It’s a Long Way to Tipperary Bound States

Within a covariant approach, the classification of predicted states in terms of quantum numbers is not as straightforward as in nonrelativistic frameworks.

★ In quark models: construction of a quark-bilinear bound state with total spin $s = 0, 1$ and relative orbital angular momentum $\ell$ of its ingredients.

★ Permitted bound-state spectrum identified by total angular momentum $J$, parity $P = (-1)^{\ell+1}$, and charge-conjugation parity $C = (-1)^{\ell+s}$ (for states with well-defined $C$), constrained by $|\ell - s| \leq J \leq |\ell + s|$ to some assignment $J^{PC} \in \{0^{++}, 0^{--}, 1^{++}, 1^{--}, 2^{++}, 2^{--}, 3^{++}, \ldots\}$.

★ States with $J^{PC} \in \{0^{--}, 0^{+-}, 1^{+-}, 2^{+-}, 3^{+-}, \ldots\}$ are viewed as exotic.

★ So far, just hints of isovector $1^{+-}$ states have been found experimentally.

★ This situation carries over to open-flavour mesons where one encounters quasi-exotic mesons, mirroring exotic mesons in the equal-mass case [4].

★ Bethe–Salpeter amplitude: more complex than solution of quark model.

★ So, a covariant approach predicts more meson states than quark models.

★ Orbital angular momentum $\ell$ can be identified also in the covariant case, e.g., for the $\rho$ meson and its excitations [5]. This can be visualized via the contributions to the norm of $\chi(p; P)$ of the various Lorentz covariants of different values of $\ell$ attributable to them in the given meson’s rest frame.

★ For ground and first excited state of the $\rho$, one finds [5, Figs. 7 and 12], for the $8 \times 8$ combinations of Lorentz covariants in the vector-meson case.
Meson Mass, Decay Constant, Condensate

For a meson composed of antiquark $\bar{q}$ and quark $q'$, we depict our predictions for the masses $M_{\bar{q}q'}$ and leptonic decay constants $f_{\bar{q}q'}$ of its ground state and lowest radial excitations in single shared plots, as exemplified by two mesons:

- strange meson involving massless ($\chi$) or light ($q$) and $s$ quark [1, Fig. 14]:

- charmed, strange heavy meson, formed by an $s$ and a $c$ quark [1, Fig. 17]:

![Diagram showing meson mass and decay constant predictions for different configurations.](image-url)
For comparison with experiment, we combine our results found from the effective-interaction models under study [2] for two fits [1] of the involved quark masses to single predictions (given by boxes), as illustrated for the strange [1, Fig. 24] (top) or charmed, strange [1, Fig. 25] (bottom) states:

It goes without saying that the example findings presented above should merely serve as a teaser: the complete sets of our results may be found in Ref. [1]. A first idea of the size of the systematic uncertainties inherent to the employed approach can be inferred by variation of the model details.
Finally, we find the in-hadron condensates $\langle \bar{q} q \rangle$ of all mesons [1, Fig. 21]:

![Graph showing in-hadron condensates](image)

**Summary: Findings, Conclusions, Outlook**

- The Dyson–Schwinger–Bethe–Salpeter-rooted covariant framework has both qualitative and quantitative features inherited directly from QCD.
- Therein, model studies are comparatively cheap, as far as the computing power required is concerned, and may be implemented comprehensively.
- With technical issues overcome, a new class of QCD models will emerge.
- The models will have scope and comprehension akin to the quark model.


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