

Simplified Bethe–Salpeter Description of Basic Pseudoscalar-Meson Features

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Pseudoscalar Mesons of Goldstone Nature

The Goldstone theorem necessitates the presence of a massless boson in the physical particle spectrum for every spontaneously broken chiral symmetry of quantum chromodynamics (QCD); these Goldstone bosons are identified with the ground-state pseudoscalar mesons (pions, kaons, and η), with their finite (but comparatively small) masses attributed to the additional explicit breakdown of the chiral symmetries induced by nonvanishing quark masses.

We analyze the Goldstone-boson nature of the lightest pseudoscalar mesons within a formalism [1,2] positioned somewhere between the fully relativistic Bethe–Salpeter approach to bound states [3], with several yet to be resolved inherent obstacles, and the latter’s extreme instantaneous limit represented by its three-dimensional reduction devised by Salpeter [4]. A very promising tool to judge the merits of this kind of intermediate framework proves to be, among others, the fulfilment of generalized Gell-Mann–Oakes–Renner-type relations [5] by the characteristic properties of light pseudoscalar mesons [6].

Quark–Antiquark Bound State Formalism

The homogeneous Bethe–Salpeter equation describes in Poincaré-covariant manner a bound state $|B(P)\rangle$ of mass \widehat{M} and momentum P built up by two particles of relative momentum p by its Bethe–Salpeter amplitude $\Phi(p, P)$. One ingredient are the bound-state constituents’ full propagators, given, for spin- $\frac{1}{2}$ fermions, by mass $M_i(p^2)$ and wave-function renormalization $Z_i(p^2)$:

$$S_i(p) = \frac{i Z_i(p^2)}{\not{p} - M_i(p^2) + i\varepsilon}, \quad \not{p} \equiv p^\mu \gamma_\mu, \quad \varepsilon \downarrow 0, \quad i = 1, 2.$$

The interactions responsible for the formation of bound states are the other ingredient. Ignoring their dependence on time components of momenta and both propagators' dependence on the momentum zero components allowed us to devise a bound-state equation [1] for the Salpeter amplitude [4]

$$\phi(\mathbf{p}) \propto \int dp_0 \Phi(p, P) ,$$

where the (by assumption instantaneous) effective interactions experienced by the bound-state constituents are captured by an integral kernel $K(\mathbf{p}, \mathbf{q})$.

Our equation reads, in terms of free energies and apposite energy projectors

$$E_i(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + M_i^2(\mathbf{p}^2)} , \quad \Lambda_i^\pm(\mathbf{p}) \equiv \frac{E_i(\mathbf{p}) \pm \gamma_0 [\boldsymbol{\gamma} \cdot \mathbf{p} + M_i(\mathbf{p}^2)]}{2 E_i(\mathbf{p})} ,$$

for fermion–antifermion bound states in the center-of-momentum frame [1],

$$\phi(\mathbf{p}) = Z_1(\mathbf{p}^2) Z_2(\mathbf{p}^2) \int \frac{d^3q}{(2\pi)^3} \left(\frac{\Lambda_1^+(\mathbf{p}) \gamma_0 [K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q})] \Lambda_2^-(\mathbf{p}) \gamma_0}{\widehat{M} - E_1(\mathbf{p}) - E_2(\mathbf{p})} - \frac{\Lambda_1^-(\mathbf{p}) \gamma_0 [K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q})] \Lambda_2^+(\mathbf{p}) \gamma_0}{\widehat{M} + E_1(\mathbf{p}) + E_2(\mathbf{p})} \right) .$$

For one-particle states $|B(P)\rangle$ normalized Lorentz-invariantly according to

$$\langle B(P)|B(P')\rangle = (2\pi)^3 2 P_0 \delta^{(3)}(\mathbf{P} - \mathbf{P}') ,$$

the normalization condition of the corresponding Salpeter amplitudes reads

$$\int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[\phi^\dagger(\mathbf{p}) \frac{\gamma_0 [\boldsymbol{\gamma} \cdot \mathbf{p} + M_1(\mathbf{p}^2)]}{E_1(\mathbf{p})} \phi(\mathbf{p}) \right] = 2 P_0 .$$

Assuming Generalized Flavour Symmetry

Things simplify considerably if the propagator functions of antifermion and fermion happen to be identical. The Salpeter amplitude of any pseudoscalar bound state is fully defined by just two Lorentz-scalar components, $\varphi_{1,2}(\mathbf{p})$:

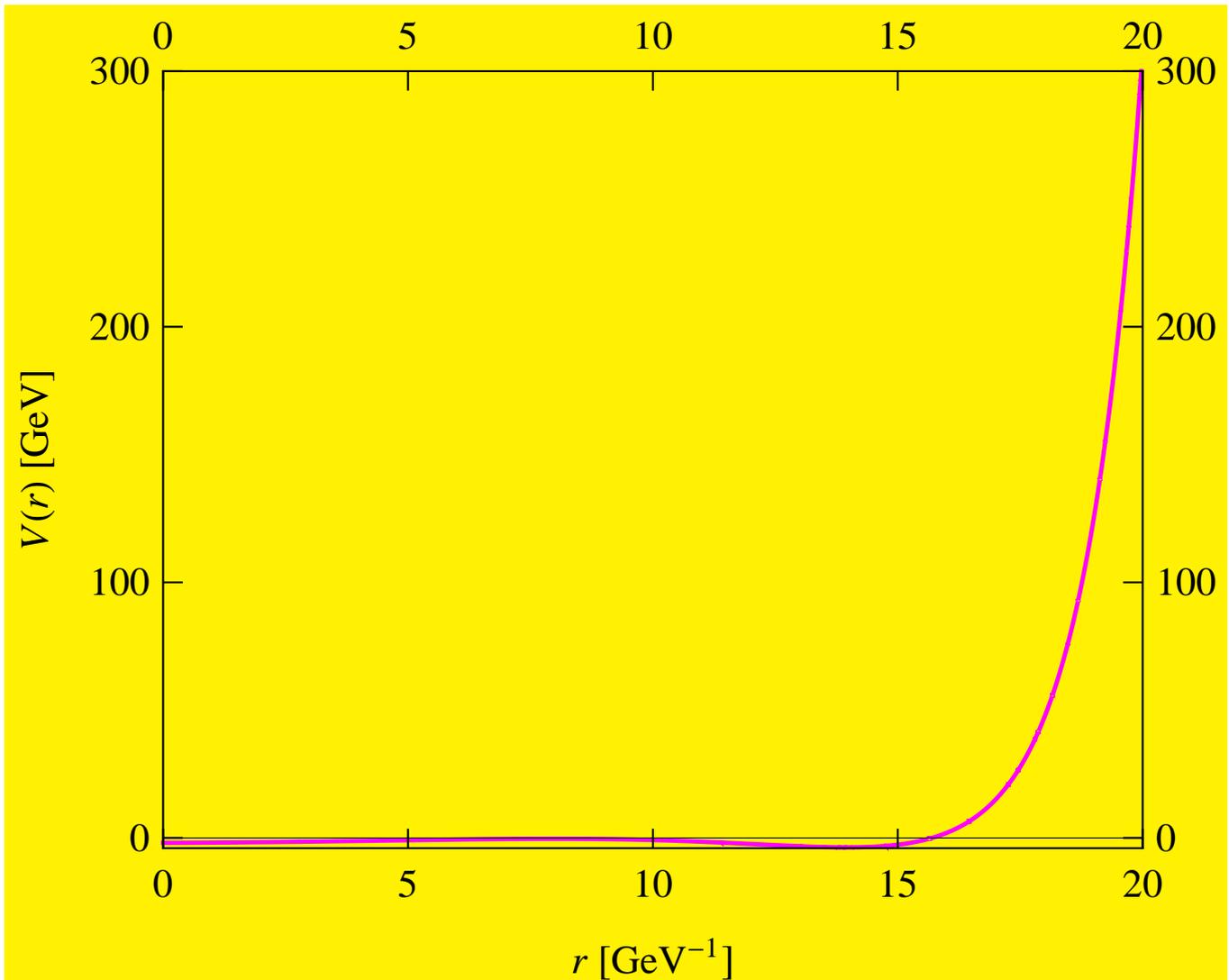
$$\phi(\mathbf{p}) = \frac{1}{\sqrt{3}} \left[\varphi_1(\mathbf{p}) \frac{\gamma_0 [\boldsymbol{\gamma} \cdot \mathbf{p} + M(\mathbf{p}^2)]}{E(\mathbf{p})} + \varphi_2(\mathbf{p}) \right] \gamma_5 .$$

If $K(\mathbf{p}, \mathbf{q})$ is compatible with spherical and specific Fierz symmetries of the effective interactions, our bound-state equation governing $\phi(\mathbf{p})$ collapses to an eigenvalue problem [7] fixing the radial parts $\varphi_{1,2}(p)$, $p \equiv |\mathbf{p}|$, of $\varphi_{1,2}(\mathbf{p})$, with the effective interactions between bound-state constituents encoded in a single spherically symmetric configuration-space potential $V(r)$, $r \equiv |\mathbf{x}|$:

$$E(p) \varphi_2(p) + \frac{2 Z^2(p^2)}{\pi p} \int_0^\infty dq q dr \sin(p r) \sin(q r) V(r) \varphi_2(q) = \frac{\widehat{M}}{2} \varphi_1(p) ,$$

$$E(p) \varphi_1(p) = \frac{\widehat{M}}{2} \varphi_2(p) .$$

The **interaction potential** $V(r)$ was determined pointwise [2] by inversion [8] of our Bethe–Salpeter-inspired bound-state formalism [1], starting from the Salpeter amplitude $\phi(\mathbf{p})$ for massless pseudoscalar mesons derived from the chiral-quark propagator [9] plus some QCD Ward–Takahashi identity [5,10]. It rises confiningly from a slightly negative value at $r = 0$ steeply to infinity:



Pseudoscalar-Meson Properties Revisited

In order to analyze pseudoscalar quark–antiquark bound states for physical (or non-chiral) quark masses, we have to find the corresponding solutions to our system of coupled equations for the radial Salpeter components $\varphi_{1,2}(p)$. That task is facilitated by an obvious move, enabled by the purely algebraic nature of one of these relations: Inserting any of these into the other leads to single explicit eigenvalue problems for either $\varphi_1(p)$ or $\varphi_2(p)$ with eigenvalue \widehat{M}^2 [2, 7]; conversion to equivalent matrix eigenvalue problems by expansion over a basis in function space is one of the standard solution procedures [11]. With the solutions at hand, we may then create trust in the reliability of our approach by assessing or scrutinizing its predictions for hadron observables.

The spatial extension of the pion deduced from the ground-state solution to our bound-state formalism in form of the pion’s average interquark distance $\langle r \rangle = 0.478$ fm or root-mean-square radius $\sqrt{\langle r^2 \rangle} = 0.529$ fm fits nicely to its observed electromagnetic charge radius $\sqrt{\langle r_\pi^2 \rangle} = (0.672 \pm 0.008)$ fm [12]. However, this agreement cannot qualify as confirmation of the credibility of our framework since the Salpeter amplitude for chiral quarks served already as input to the inversion procedure yielding the shape of the potential $V(r)$, and the experimental u/d quark masses are pretty close to their chiral limit.

Equating the residues of pseudoscalar-meson pole terms in axial-vector and pseudoscalar vertex functions, entering in the axial-vector Ward–Takahashi identity of QCD, leads to a generalization of the Gell-Mann–Oakes–Renner relation [13]; this innovation relates, for a pseudoscalar bound state $|B(P)\rangle$, its **decay constant**, f_B , defined in terms of the axial-vector quark current by

$$\langle 0 | : \bar{\psi}_1(0) \gamma_\mu \gamma_5 \psi_2(0) : | B(P) \rangle = i f_B P_\mu \quad \rightsquigarrow \quad f_B \propto \int d^3 p \operatorname{Tr}[\gamma_0 \gamma_5 \phi(\mathbf{p})] ,$$

and its (vacuum-quark-condensate universalizing) **in-hadron condensate** [5]

$$\mathbb{C}_B \equiv \langle 0 | : \bar{\psi}_1(0) \gamma_5 \psi_2(0) : | B(P) \rangle \propto \int d^3 p \operatorname{Tr}[\gamma_5 \phi(\mathbf{p})]$$

to that bound state’s mass \widehat{M}_B and the quark mass parameters in the QCD Lagrangian [5]. For the case of equal quark masses m , this relationship reads

$$f_B \widehat{M}_B^2 = 2 m \mathbb{C}_B .$$

Compatibility with this relation may be inspected by solving our formalism with the once specified potential $V(r)$ for bound states of chiral, u/d , and s quarks, taking advantage of the appropriate model propagator functions [9]. Comparison of the quark masses m , fixed by the thereby predicted values of \widehat{M}_B , f_B , and \mathbb{C}_B , with the current-quark masses $\overline{m}(\mu)$ in modified minimal subtraction at scale μ proves that our m outcomes are in the right ballpark:

Constituents	\widehat{M}_B [MeV]	f_B [MeV]	\mathbb{C}_B [GeV ²]	m [MeV]	$\overline{m}(2 \text{ GeV})$ [MeV] [12]
chiral quarks	6.8	151	0.585	0.0059	—
u/d quarks	148.6	155	0.598	2.85	$3.5^{+0.7}_{-0.3}$
s quarks	620.7	211	0.799	51.0	96^{+8}_{-4}

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