Resummed hydrodynamic expansion for an electromagnetic interacting plasma

Outline

- Introduction: hydrodynamic expansions and microscopic theories
- Method of moments for the Boltzmann-Vlasov equation
- Resummed moments expansion
- Numerical results (Bjorken flow)

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Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions 2018

Relativistic hydrodynamics

$$\left.\begin{aligned}\partial_\mu \hat{T}^{\mu\nu} &= 0\\ T^{\mu\nu} &= tr(\hat{\rho}\,\hat{T}^{\mu\nu})\end{aligned}\right\} \rightarrow \partial_\mu T^{\mu\nu} = 0$$

Hydro

$$T^{\mu\nu} = \varepsilon\, u^\mu\, u^\nu \; - p\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

From quantum field theory, but at least ten degrees of freedom and only four equations

Expansions

Gradient expansion

- Well defined in QFT
- Requires small gradients
- Unstable
- Not converging

Method of moments

- Kinetic theory (no fields)
- No small parameter
- Non hydrodynamic modes
- Stable and converging

A Buchel, M P Heller, J Noronha, arXiv:1603.05344

G Denicol, J Noronha, arXiv:1608.07869

Separation between the fluid part and the coherent degrees of freedom

$$\varepsilon\, u^\mu u^\nu \;-p\,\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

$$T^{\mu\nu} = T_f^{\mu\nu} + T_c^{\mu\nu}$$

$$T_c^{\mu\nu} = F^{\mu\alpha} F_\alpha^{\ \nu} + \frac{1}{4}\, g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Particular case:

Magneto-hydrodynamics $E^\mu \simeq 0$

$$F^{\mu\nu} = E^\mu u^\nu - u^\mu E^\nu + \epsilon^{\mu\nu\rho\sigma} u_\rho B_\sigma$$

Moments expansion of the distribution function

Relativistic Boltzmann equation

$$p \cdot \partial f = -\mathcal{C}[f]$$

Expansion of the distribution

$$f = f_{\text{eq.}} \left(1 + \bar{f}_{\text{eq.}} \sum_l \sum_n \mathcal{H}_n^{(l)}(p \cdot u)\, \rho_n^{\mu_1 \cdots \mu_l}\, p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

G S Denicol, H Niemi, E Molnar, D H Rischke, arXiv:1202.4551

Irreducible moments

$$\rho_n^{\mu_1 \cdots \mu_l} = \int_{\mathbf{p}} (p \cdot u)^n\, p^{\langle \mu_1} \cdots p^{\mu_l \rangle} f$$

covariant momentum integral

$$\int_{\mathbf{p}} = \int d^4p\, 2\Theta(p_0)\delta(p^2 - m^2)$$

More convenient basis

$$\mathcal{F}_r^{\mu_1 \cdots \mu_s} = \int_{\mathbf{p}} (p \cdot u)^r\, p^{\mu_1} \cdots p^{\mu_s} f$$

$$\rho_n^{\mu_1 \cdots \mu_l}\, p_{\langle \mu_1} \cdots p_{\mu_l \rangle} = [\mathcal{F}_n^{\mu_1 \cdots \mu_l} - \mathcal{F}_n^{\mu_1 \cdots \mu_l}|_{\text{eq.}}]\, p_{\langle \mu_1} \cdots p_{\mu_l \rangle}$$

Method of moments without external fields

Exact equations for the reducible moments

$$\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} = r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s}$$

$$C_{r-1}^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u)^{r-1} p^{\mu_1}\cdots p^{\mu_s}\mathcal{C}[f]$$

Special case: $T^{\mu\nu}$ (r=0,s=2)

$$\dot{T}^{\mu\nu} + C_{-1}^{\mu\nu} = -\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} - \nabla_\alpha u_\beta\,\mathcal{F}_{-2}^{\alpha\beta\mu\nu}$$

Tower of equations starting from $T^{\mu\nu}$(hydrodynamic degrees of freedom)

Residual moments: $\mathcal{F}_r^{\mu_1\cdots\mu_s} \simeq \mathcal{F}_r^{\mu_1\cdots\mu_s}|_{eq.}$

Particles interacting with external fields

Boltzmann-Vlasov equation

$$p\cdot\partial f + m\partial_\alpha m\,\partial^\alpha_{(p)} f + qF_{\alpha\beta}p^\beta\partial^\alpha_{(p)} f = -\mathcal{C}[f]$$

Immediate (but flawed!) generalization

$$\begin{aligned}\dot{\mathcal{F}}^{\mu_1\cdots\mu_s}_r + C^{\mu_1\cdots\mu_s}_{r-1} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}^{\alpha\mu_1\cdots\mu_s}_{r-1} - \nabla_\alpha\mathcal{F}^{\alpha\mu_1\cdots\mu_s}_{r-1} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}^{\alpha\beta\mu_1\cdots\mu_s}_{r-2}\\ &+m\dot{m}\,(r-1)\,\mathcal{F}^{\mu_1\cdots\mu_s}_{r-2} + s\,m\partial^{(\mu_1}m\,\mathcal{F}^{\mu_2\cdots\mu_s)}_{r-1}\\ &-q(r-1)\,E_\alpha\,\mathcal{F}^{\alpha\mu_1\cdots\mu_s}_{r-2} - q\,s\,g_{\alpha\beta}F^{\alpha(\mu_1}\mathcal{F}^{\mu_2\cdots\mu_s)\beta}_{r-1}\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

Unphysical moments as a source

$$\mathcal{F}_r^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u)^r\, p^{\mu_1}\cdots p^{\mu_s} f$$

$r < -2 - s$, diverging integral in the massless case

Numerical problems for small non-vanishing masses

$$\frac{\mathcal{F}_r^{\mu_1\cdots\mu_s}}{T^{r+s+2}} \propto \left(\frac{m}{T}\right)^{r+s+2}$$

Any non-trivial coupling to an electromagnetic field prevents a systematic expansion in the massless limit!

Moments with large negative r needed, infrared catastrophe!

Resummed expansion

- Resummed moments
- All reducible moments recovered
- Well defined equations

$$\Phi^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u) p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2 (p\cdot u)^2} f$$

$$\mathcal{F}_n^{\mu_1\cdots\mu_l} = \frac{2}{\sqrt{\pi}} \int_0^\infty d\xi \; \Phi^{\mu_1\cdots\mu_l\nu_1\cdots\nu_n} \, u_{\nu_1}\cdots u_{\nu_n}$$

$$\begin{aligned}\dot{\Phi}^{\mu_1\cdots\mu_s} + \delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = \; & \frac{2}{\sqrt{\pi}} \int_\xi^\infty d\zeta \frac{\zeta}{\sqrt{\zeta^2-\xi^2}} \Big\{ \dot{u}_\alpha \, \Phi^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \Phi^{\alpha\mu_1\cdots\mu_s} \\ & + s\left[m\partial^{(\mu_1} m \, \Phi^{\mu_2\cdots\mu_s)} - q\, g_{\alpha\beta} F^{\alpha(\mu_1} \Phi^{\mu_2\cdots\mu_s)\beta} \right] \Big\} \\ & -2\xi^2 \left[\partial_\alpha u_\beta \, \Phi^{\alpha\beta\mu_1\cdots\mu_s} + m\dot{m}\, \Phi^{\mu_1\cdots\mu_s} - qE_\alpha \, \Phi^{\alpha\mu_1\cdots\mu_s} \right]\end{aligned}$$

Contribution from the collisional kernel

$$\delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = \int_{\mathbf{p}} p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2(p\cdot u)^2} \mathcal{C}[f]$$

Exact solutions of the Boltzmann-Vlasov equation

- Maxwell equations, particles as the source: $\partial_\mu F^{\mu\nu} = J^\nu$ $\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$

- Longitudinally boost invariant expansion, and homogeneous in the transverse plane (no parity invariance), RTA

Because of symmetry

$$\partial\tau f(\tau, p_T, p_\eta) = -q\,E_\eta\,\frac{\partial f}{\partial p_\eta} - \frac{1}{\tau_R}(f - f_{eq.})$$

$$\partial\tau \bar{f}(\tau, p_T, p_\eta) = +q\,E_\eta\,\frac{\partial \bar{f}}{\partial p_\eta} - \frac{1}{\tau_R}(\bar{f} - f_{eq.})$$

$$\partial_\tau E_\eta(\tau) = \frac{1}{\tau}E_\eta - J_\eta$$

$$u \cdot J = 0$$

$$f_{eq.} = k\,e^{-\frac{1}{T}(p\cdot u)} \qquad E_\eta = -\tau\,E_L \qquad k = \frac{N_{dof}}{(2\pi)^3} \qquad J_\eta = -\tau\,J_L$$

- Massless particles, $4\pi\,\bar{\eta} = 1$

- Local equilibrium initial conditions, $\tau_0 = 1$ fm/c, $\mathrm{T}_0 = 0.3$ GeV, $\mathrm{E}_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$.

R Ryblewski, W Florkowski, arXiv:1307.0356

Set of independent moments

❖ Linearly independent moments: $\Phi_l^{\pm} = \Phi^{\mu_1 \dots \mu_l} z_{\mu_1} \cdots z_{\mu_l} \pm \bar{\Phi}^{\mu_1 \dots \mu_l} z_{\mu_1} \cdots z_{\mu_l}$

$$[z^{\mu} = (\sinh\eta, 0, 0, \cosh\eta), \quad u^{\mu} = (\cosh\eta, 0, 0, \sinh\eta)]$$

❖ Normalized (dimensionless) moments: $M_l^{\pm} = \dfrac{\Phi_l^{\pm}}{(4\pi k)(l+2)l! T_0^{l+3}}$

In particular

$$(48\,\pi\,k)T^4 = \varepsilon = \frac{2}{\sqrt{\pi}} \int_0^{\infty} d\,\xi \left(-\frac{\partial}{\partial \xi^2} \Phi_0^+ \right) = 16\sqrt{\pi}\,k\,T_0^3 \int_0^{\infty} d\,\xi \left(-\frac{\partial}{\partial \xi^2} M_0^+ \right)$$

$$P_L = \frac{2}{\sqrt{\pi}} \int_0^{\infty} d\,\xi\, \Phi_2^+ = 64\sqrt{\pi}\,k\,T_0^5 \int_0^{\infty} d\,\xi\, M_2^+$$

$$J_L = -q \frac{2}{\sqrt{\pi}} \int_0^{\infty} d\,\xi\, \Phi_1^- = -\,q(48\sqrt{\pi}\,k\,T_0^4) \int_0^{\infty} d\,\xi\, M_1^-$$

Equations to test numerically

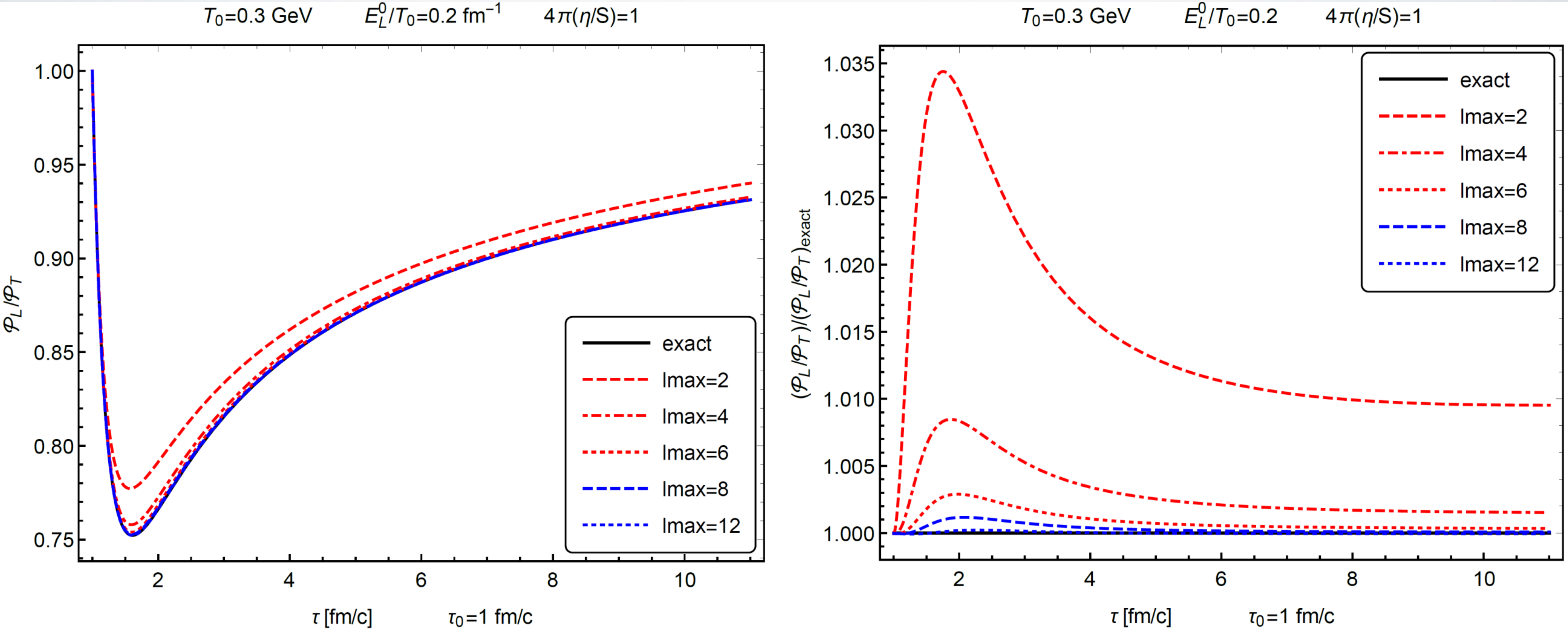
From the resummed moments and Maxwell equations

$$\tau_R \partial_\tau M_l^\pm + \left(M_l^\pm - M_l^\pm \Big|_{eq.} \right) = -\frac{\tau_R}{\tau} \Big[(l+1) M_l^\pm - 2(\xi T_0)^2 (l+4)(l+1) M_{l+2}^\pm + \frac{q E_\eta}{T_0} \left(\frac{l+1}{l+2} M_{l-1}^\mp - 2(\xi T_0)^2 \frac{(l+3)(l+1)}{l+2} M_{l+1}^\mp \right) \Big]$$

$$\frac{\partial_\tau T}{T} = -\frac{1}{4\tau} \left[1 + \frac{4 T_0^5}{3\sqrt{\pi} T^4} \int_0^\infty d\xi \; M_2^+ - q \, E_\eta \frac{T_0^4}{\sqrt{\pi} T^4} \int_0^\infty d\xi \; M_1^- \right]$$

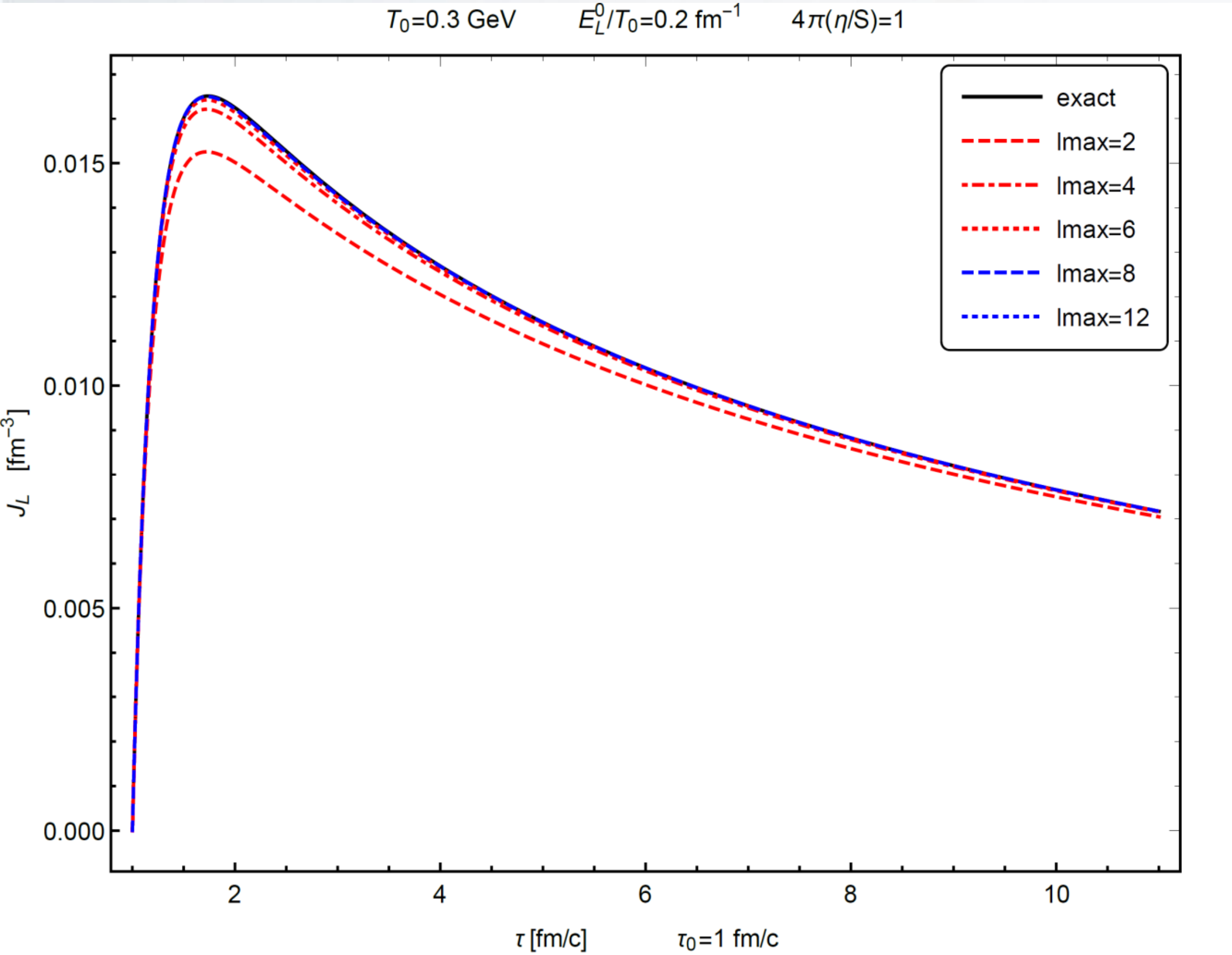
$$\partial_\tau E_\eta = \frac{1}{\tau} E_\eta \; - \tau q (48 \sqrt{\pi} \, k \, T_0^4) \int_0^\infty d \, \xi \; M_1^-$$

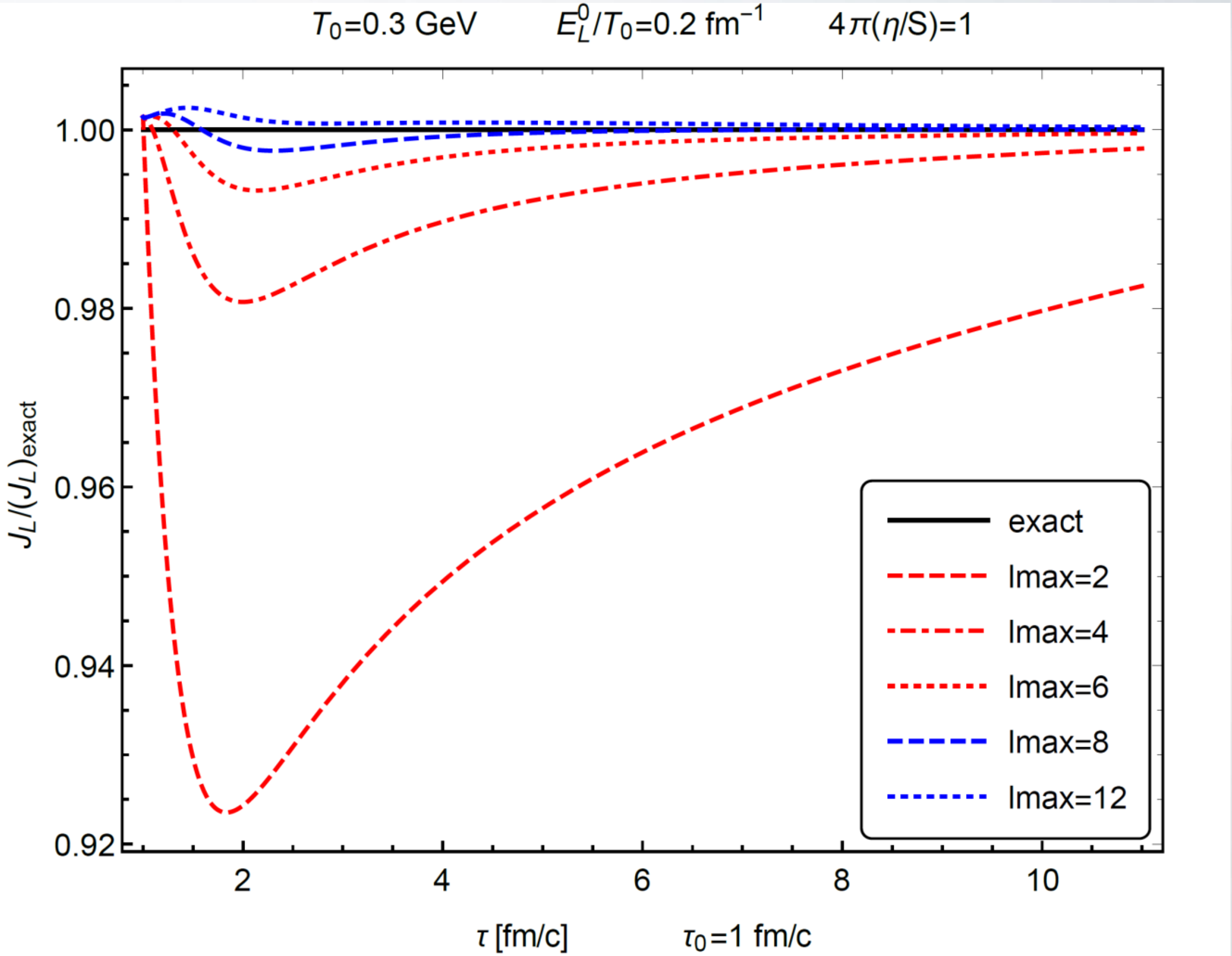
Comparisons: pressure anisotropy



Higher orders ill-defined in the traditional expasion

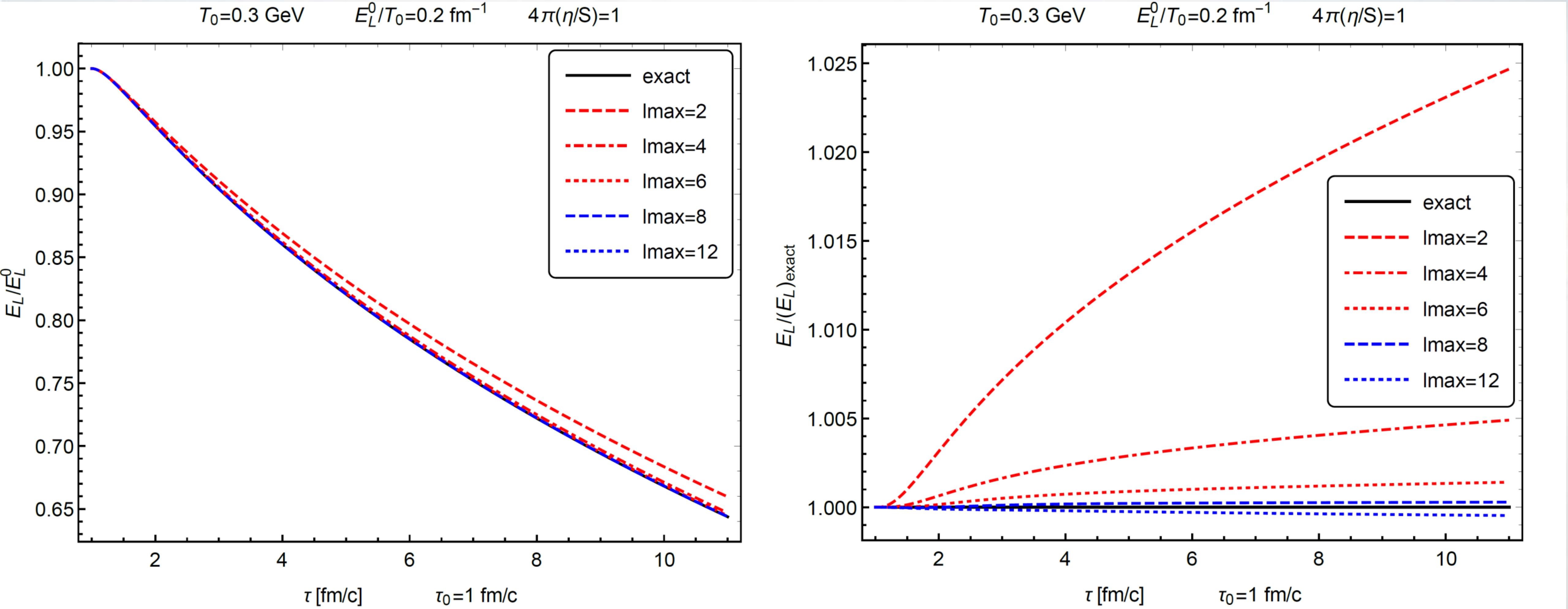
Comparisons:electric current





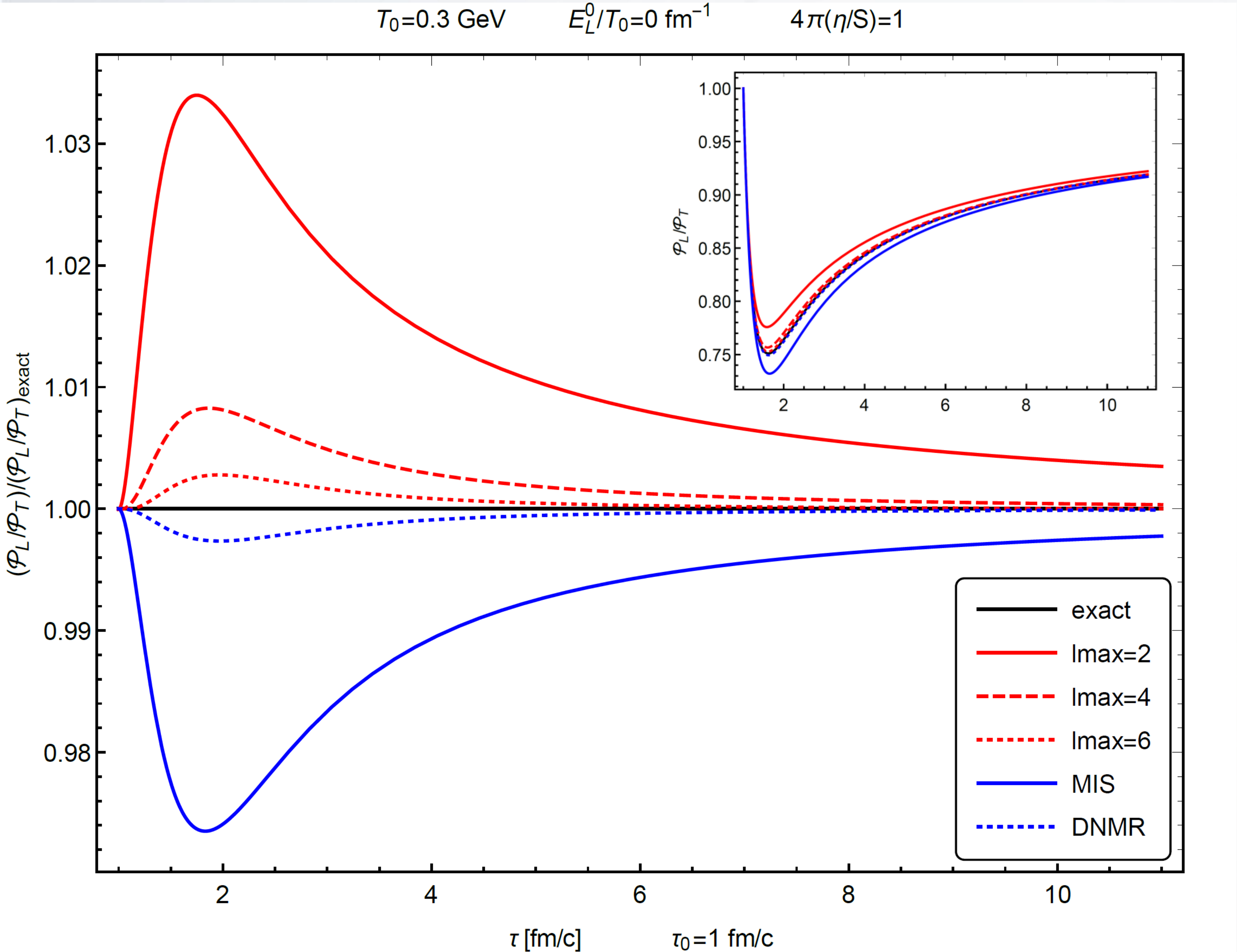
Higher orders ill-defined in the traditional expasion

Comparisons: electric field

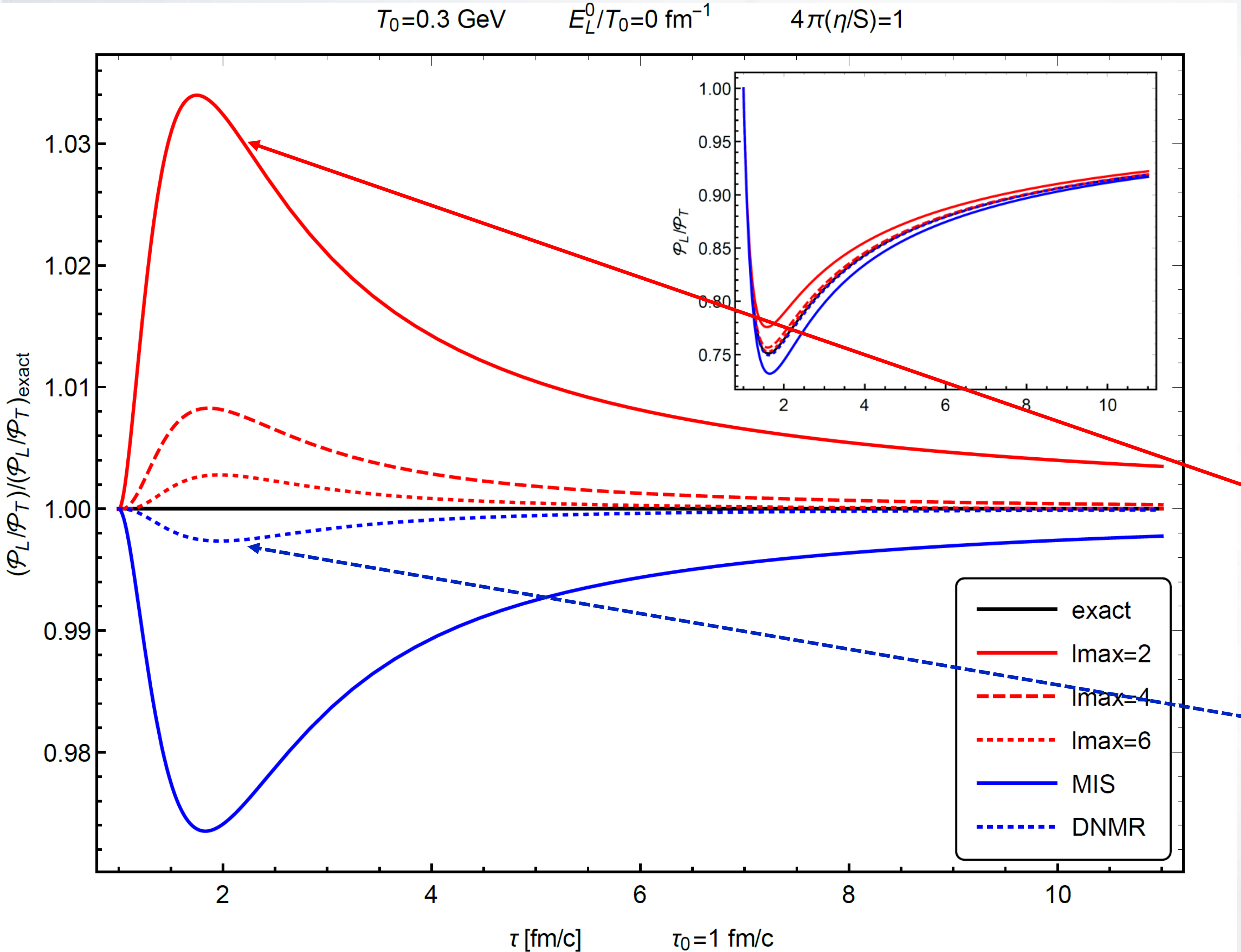


Higher orders ill-defined in the traditional expasion

Vanishing electric fields: resummed moments and viscous hydrodynamics



Vanishing electric fields: resummed moments and viscous hydrodynamics



Both lines from the method of moments, at the same order

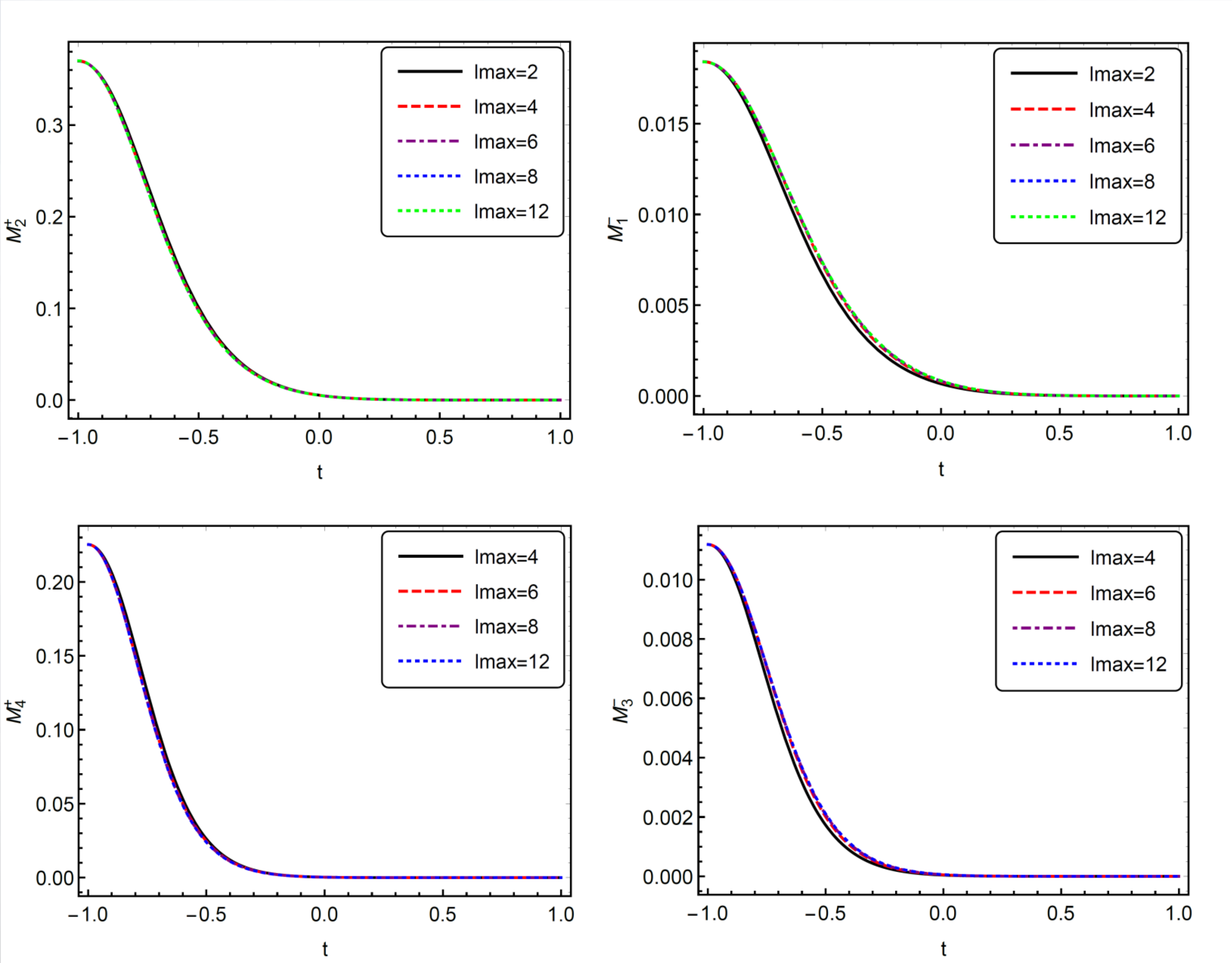
different treatment of the residual moments

$$\mathcal{F}_r^{\mu_1\cdots\mu_s} \to \mathcal{F}_r^{\mu_1\cdots\mu_s}\Big|_{eq.}$$

$$\mathcal{F}_r^{\mu_1\cdots\mu_s} \neq \mathcal{F}_r^{\mu_1\cdots\mu_s}\Big|_{eq.}$$

The shape of the resummed moments

$\tau = 1.6$ fm
$E_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$



$$P_L = 64\sqrt{\pi}\, k\, T_0^5 \int_0^\infty d\,\xi\, M_2^+$$

$$J_L = -48\sqrt{\pi}\, k\, T_0^4 \int_0^\infty d\,\xi\, M_1^-$$

Convenient map $\xi \in [0,\infty) \to t \in [-1,1)$

$$\xi T_0 = \frac{1+t}{1-t} \Rightarrow t = \frac{\xi\, T_0 - 1}{\xi T_0 + 1}$$

Strong suppression for

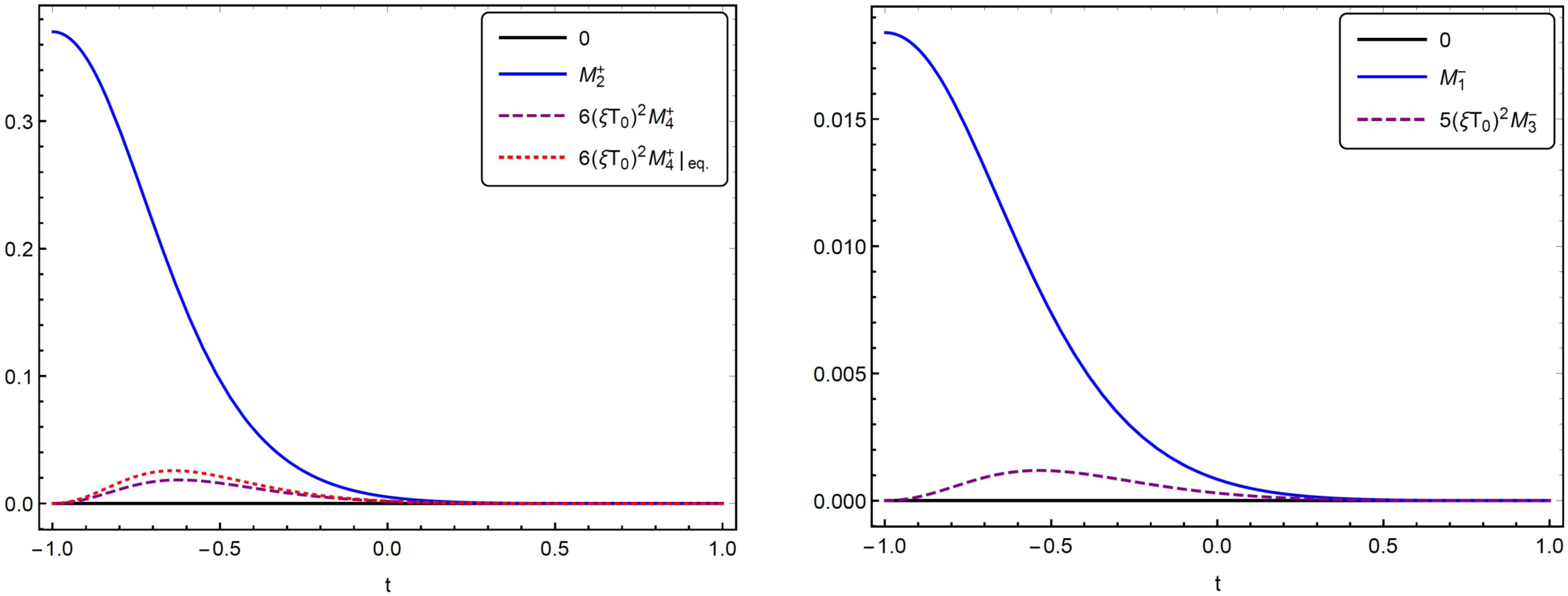
$$\boldsymbol{\xi > T_0^{-1} \qquad (t > 0)}$$

Reason of convergence?

Suppression for small ξ

$$\xi T_0 = \frac{1+t}{1-t} \Rightarrow t = \frac{\xi\, T_0 - 1}{\xi T_0 + 1}$$

$\tau = 1.6$ fm
$E_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$

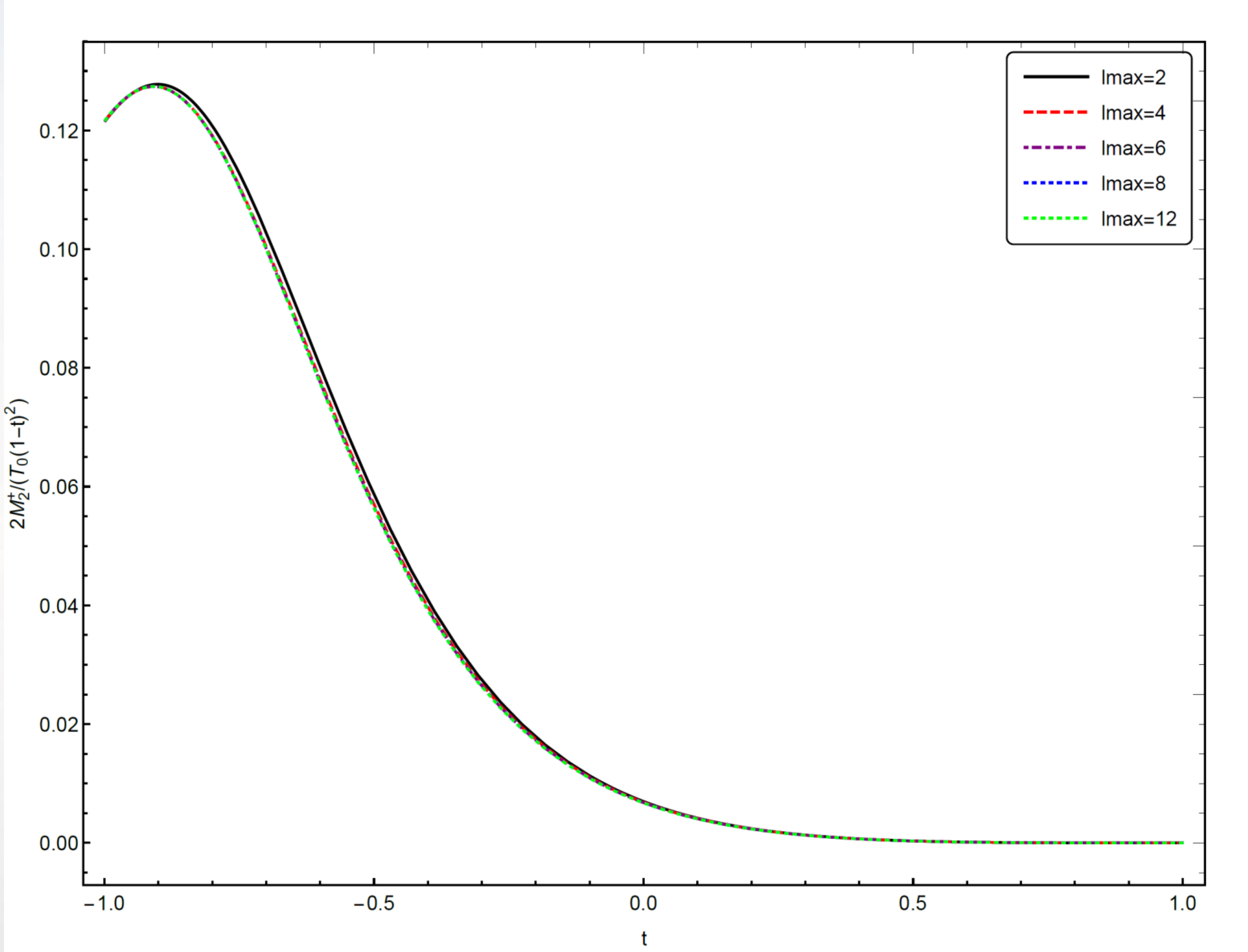


from the EOM: $\tau_R \partial_\tau M_l^{\pm} = -\frac{\tau_R}{\tau}\left[(l+1)M_l^{\pm} - 2(\xi T_0)^2 (l+4)(l+1) M_{l+2}^{\pm} + \cdots\right.$

Conclusions and outlook

- Method of moments ill-defined with external fields
- Resummed moments expansion
- Explanation of the convergence

Back up slides



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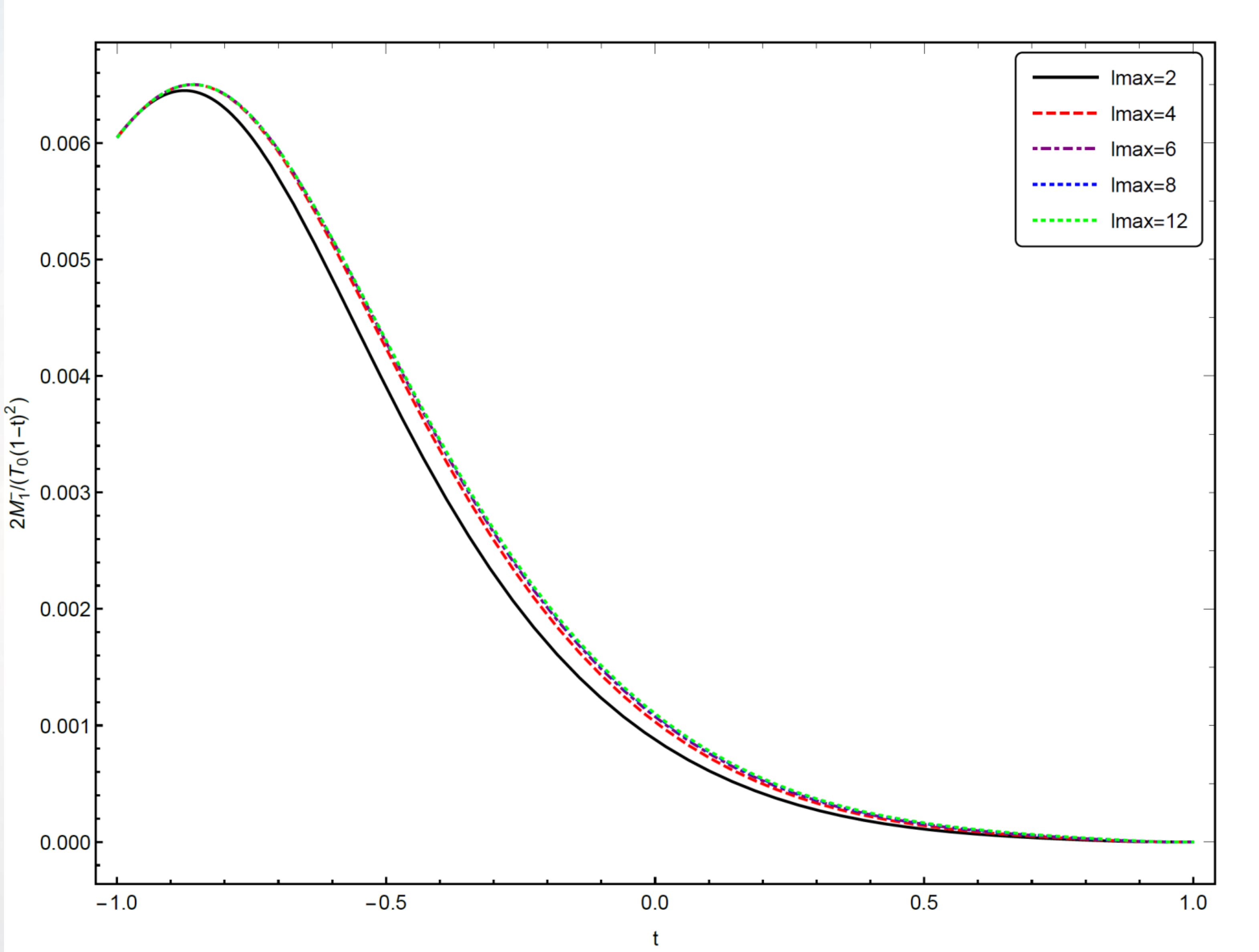
lmax=6
lmax=8
lmax=12
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$2M_2^{+}/(T_0(1-t)^2)$
−1.0
−0.5
0.0
0.5
1.0
t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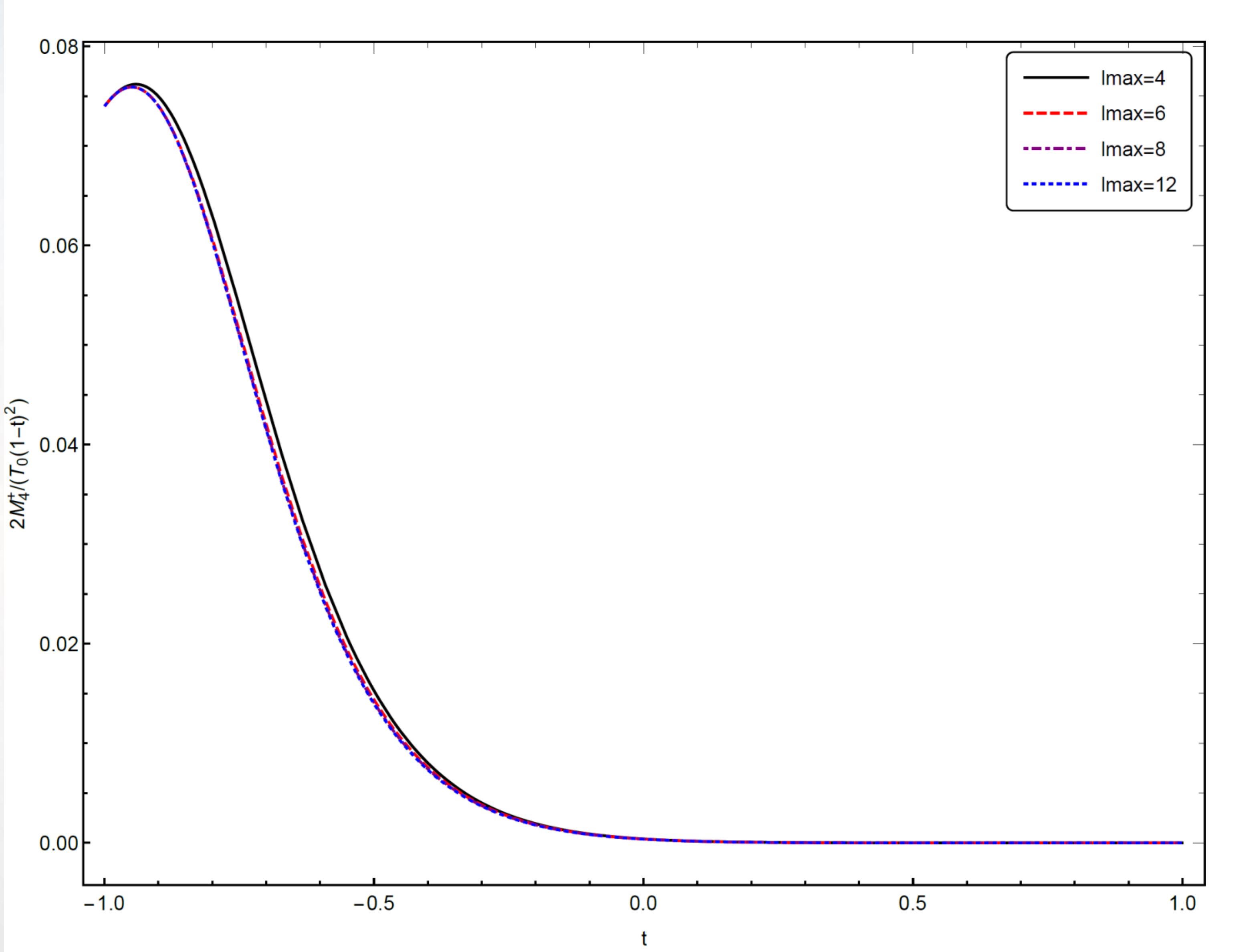
2M̄1/(T0(1−t)^2)
−1.0
−0.5
0.0
0.5
1.0
t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0.5
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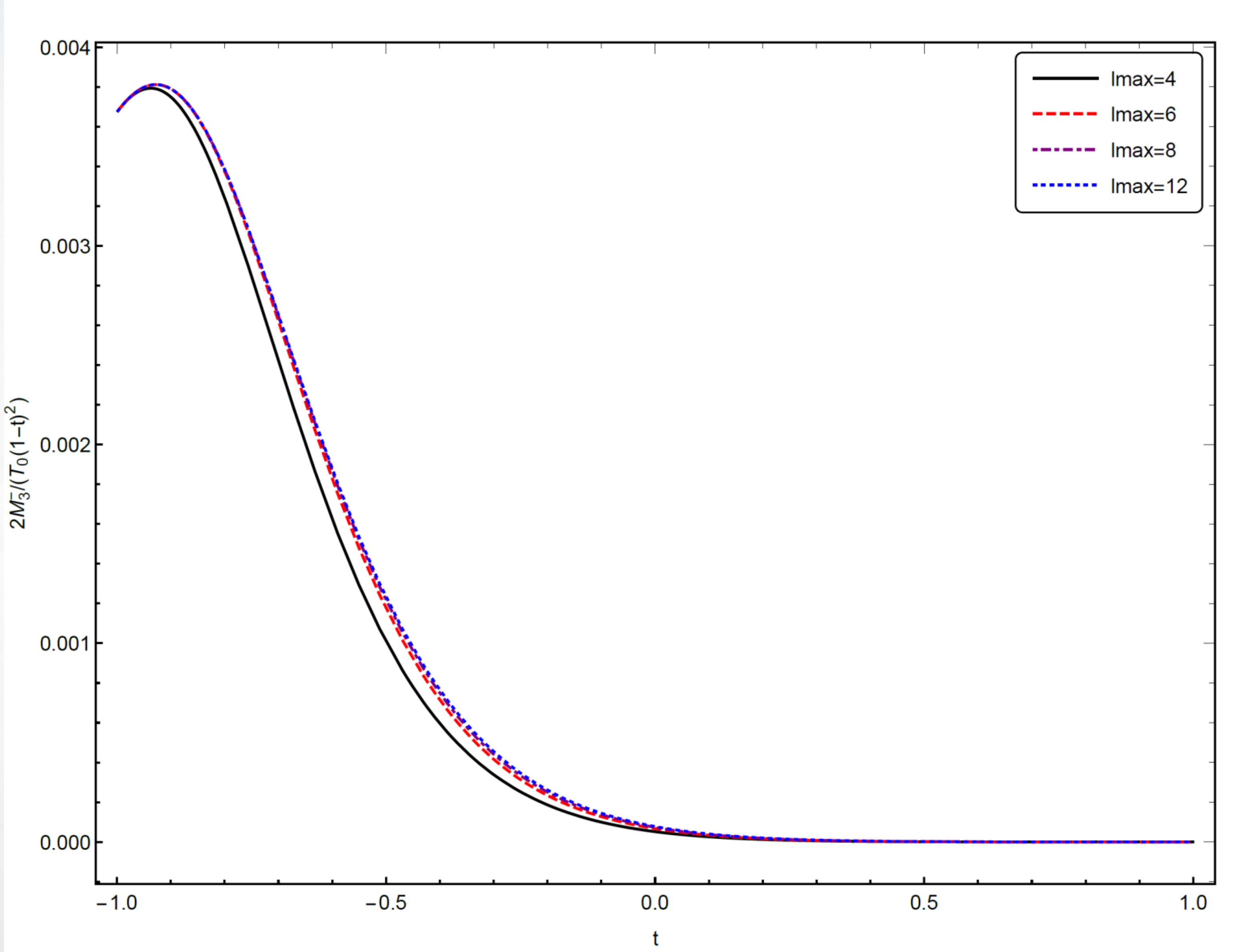
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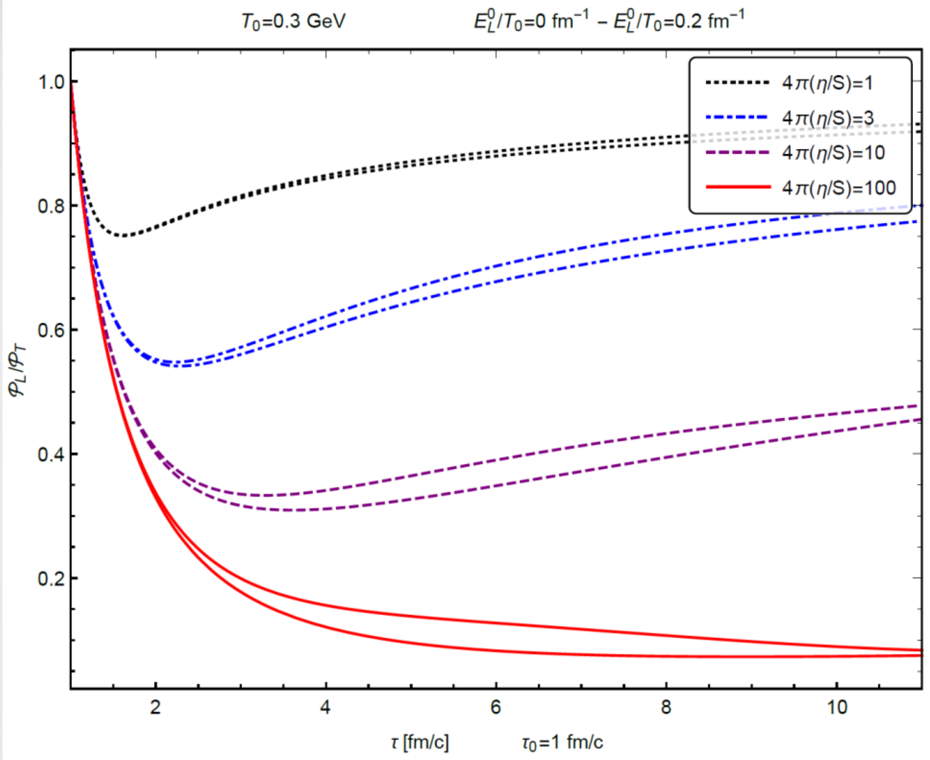
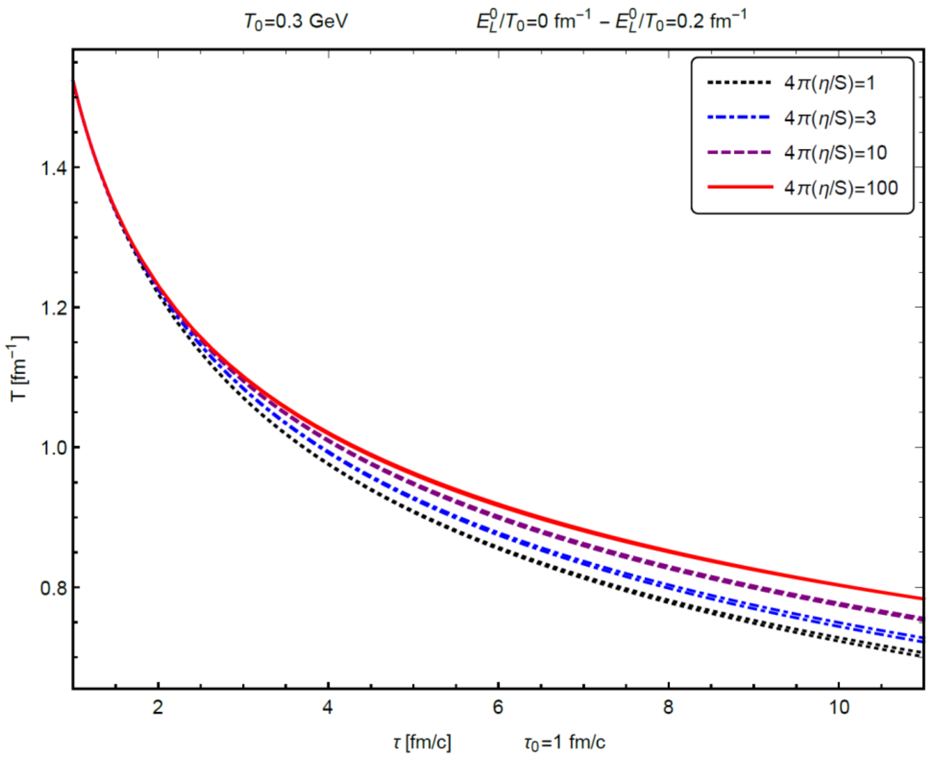
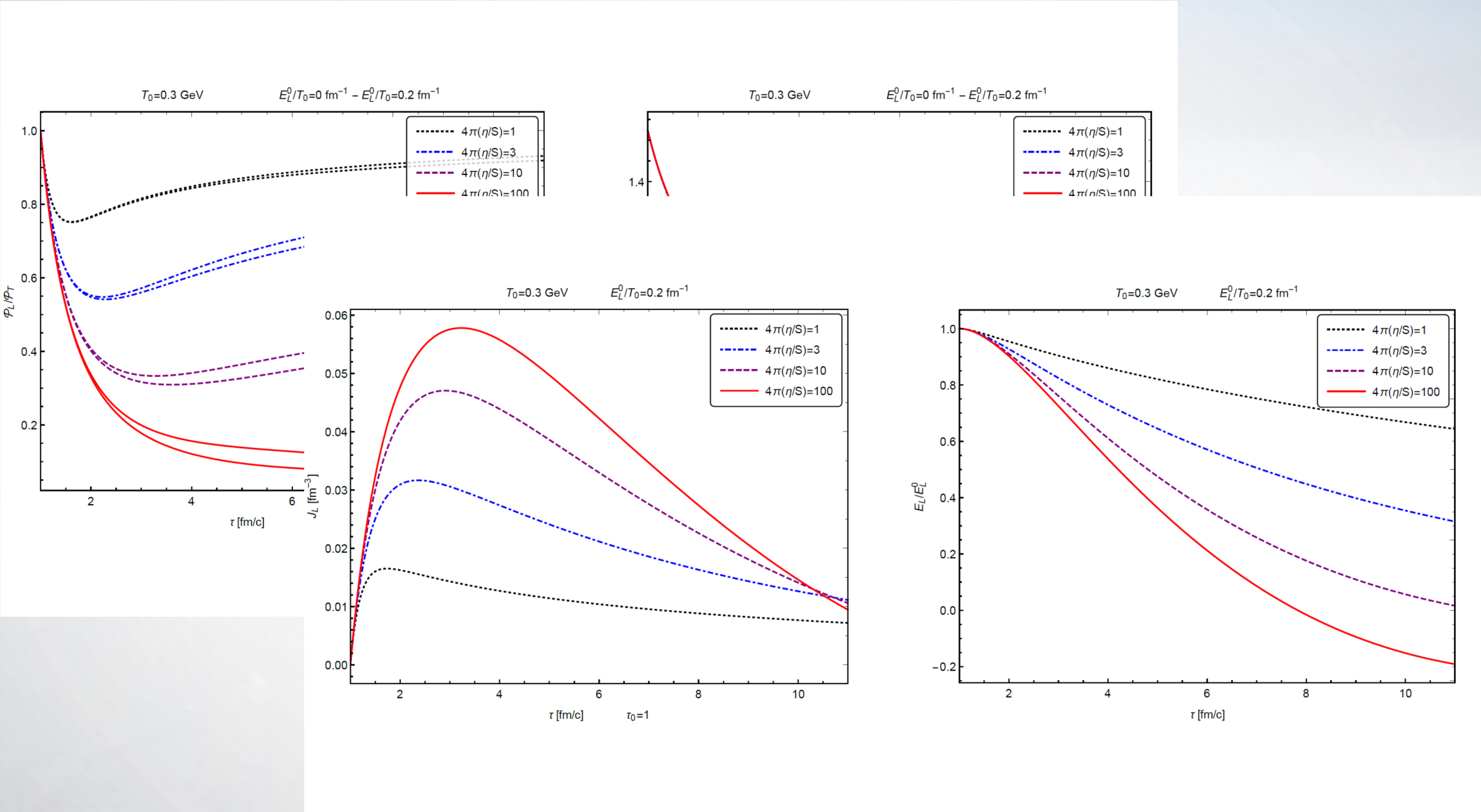
0.5
1.0
t
$2M_{\bar{3}}/(T_0(1-t)^2)$



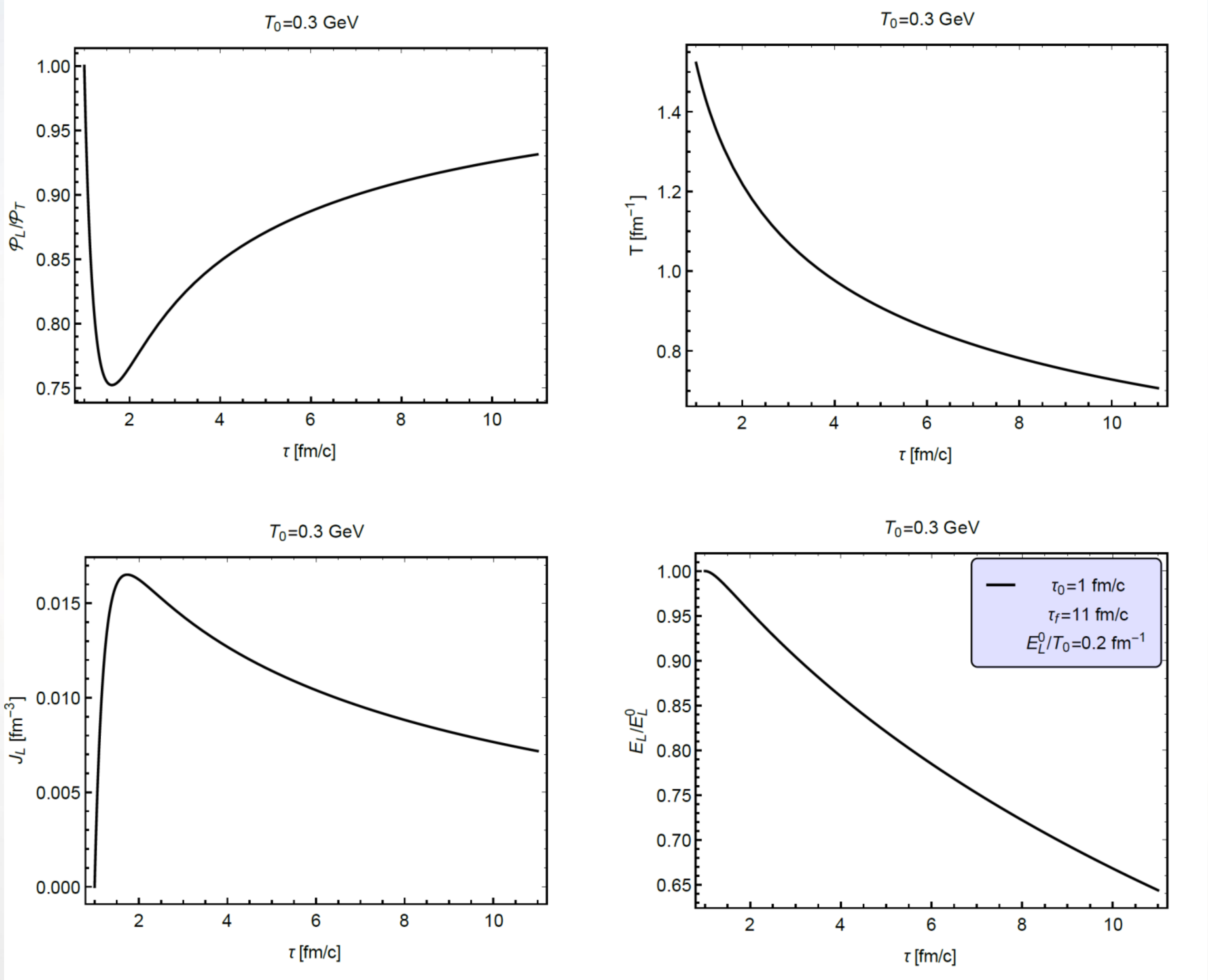
T_0=0.3 GeV
E_L^0/T_0=0 fm^{-1} – E_L^0/T_0=0.2 fm^{-1}
$4\pi(\eta/S)$=1
$4\pi(\eta/S)$=3
$4\pi(\eta/S)$=10
$4\pi(\eta/S)$=100
$\mathcal{P}_L/\mathcal{P}_T$
τ [fm/c]
τ_0=1 fm/c

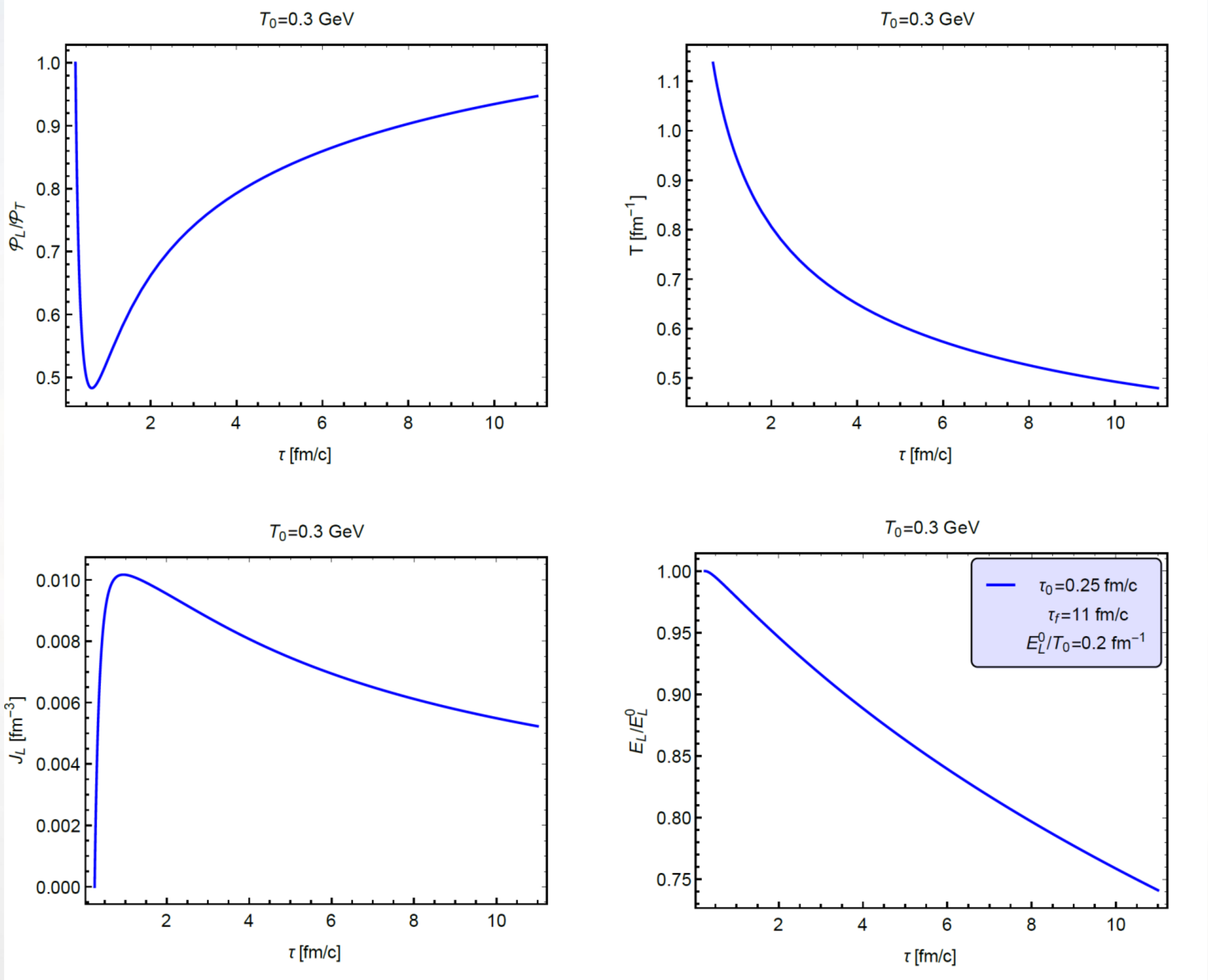


T_0=0.3 GeV
E_L^0/T_0=0 fm^{-1} – E_L^0/T_0=0.2 fm^{-1}
$4\pi(\eta/S)$=1
$4\pi(\eta/S)$=3
$4\pi(\eta/S)$=10
$4\pi(\eta/S)$=100
T [fm^{-1}]
τ [fm/c]
τ_0=1 fm/c

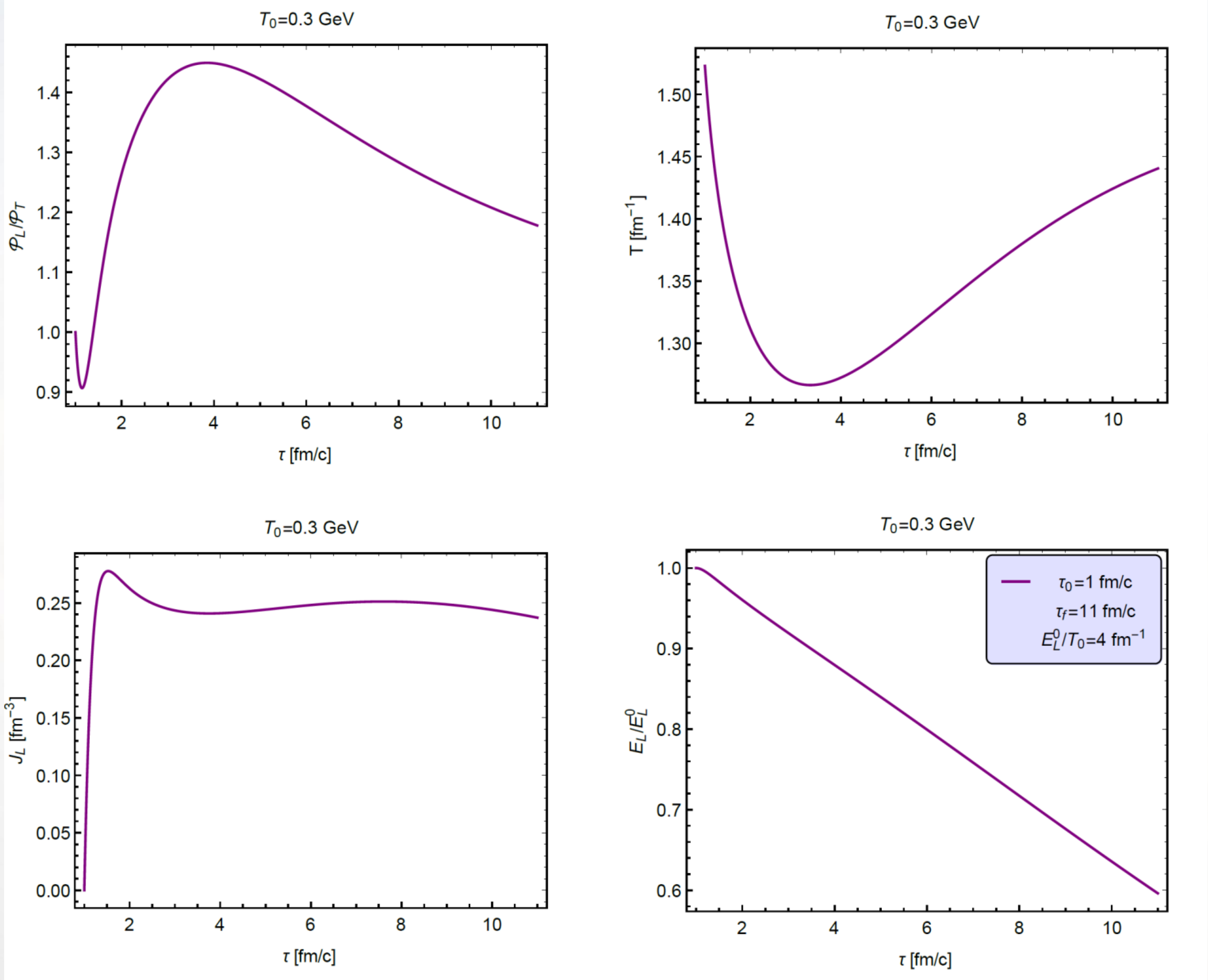


T0=0.3 GeV
EL0/T0=0 fm−1 – EL0/T0=0.2 fm−1
4π(η/S)=1
4π(η/S)=3
4π(η/S)=10
4π(η/S)=100
PL/PT
τ [fm/c]
T0=0.3 GeV
EL0/T0=0 fm−1 – EL0/T0=0.2 fm−1
T0=0.3 GeV
EL0/T0=0.2 fm−1
JL [fm−3]
τ [fm/c]
τ0=1
T0=0.3 GeV
EL0/T0=0.2 fm−1
EL/EL0
τ [fm/c]

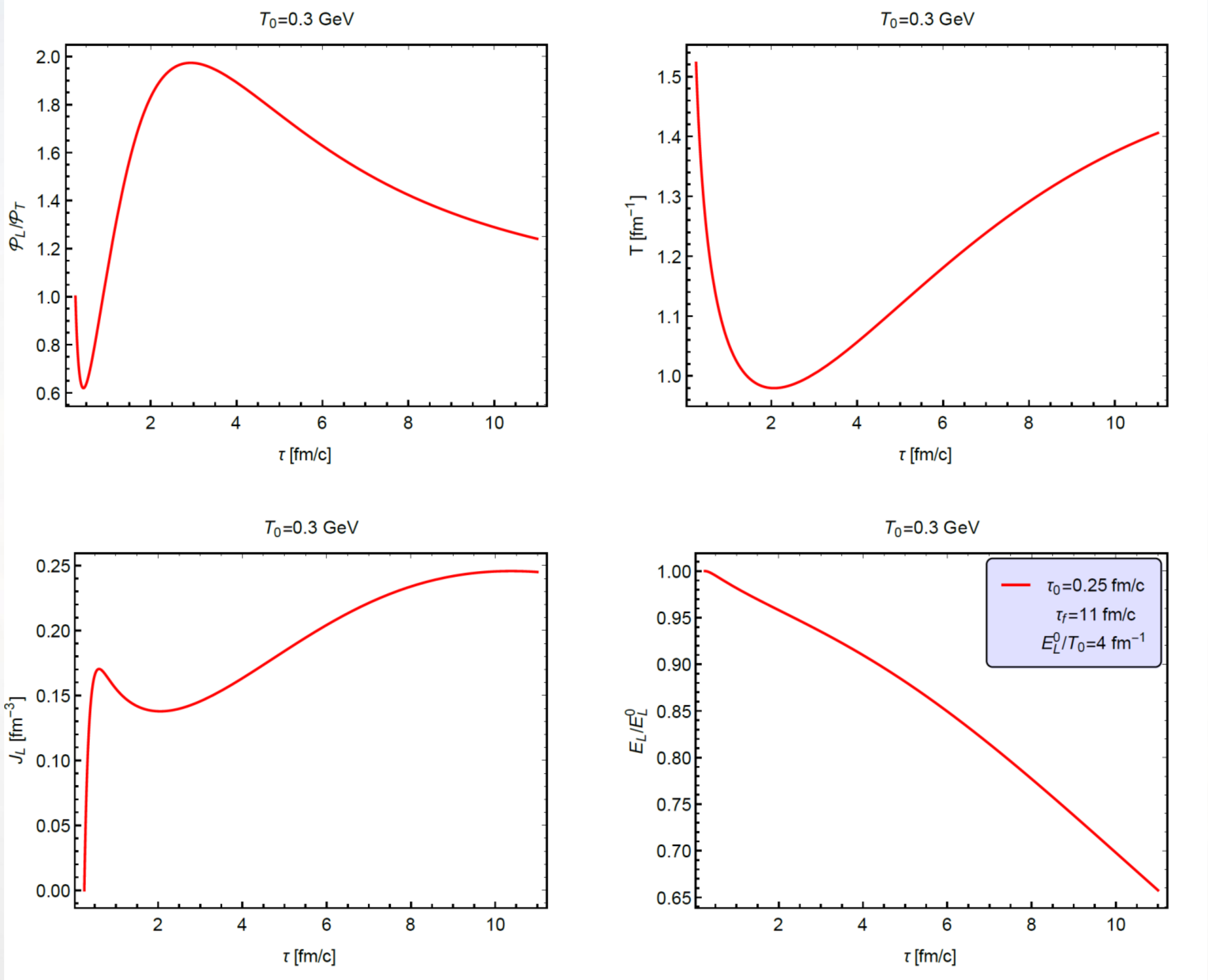

T_0=0.3 GeV
$\mathcal{P}_L/\mathcal{P}_T$
τ [fm/c]
T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
T_0=0.3 GeV
J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
τ [fm/c]
τ_0=1 fm/c
τ_f=11 fm/c
E_L^0/T_0=0.2 fm^{-1}



T_0=0.3 GeV
$\mathcal{P}_L/\mathcal{P}_T$
τ [fm/c]
T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
T_0=0.3 GeV
J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
τ [fm/c]
τ_0=0.25 fm/c
τ_f=11 fm/c
E_L^0/T_0=0.2 fm^{-1}



T_0=0.3 GeV
$\mathcal{P}_L/\mathcal{P}_T$
τ [fm/c]
T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
T_0=0.3 GeV
J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
τ [fm/c]
τ_0=1 fm/c
τ_f=11 fm/c
E_L^0/T_0=4 fm^{-1}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T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
T_0=0.3 GeV
J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
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τ_0=0.25 fm/c
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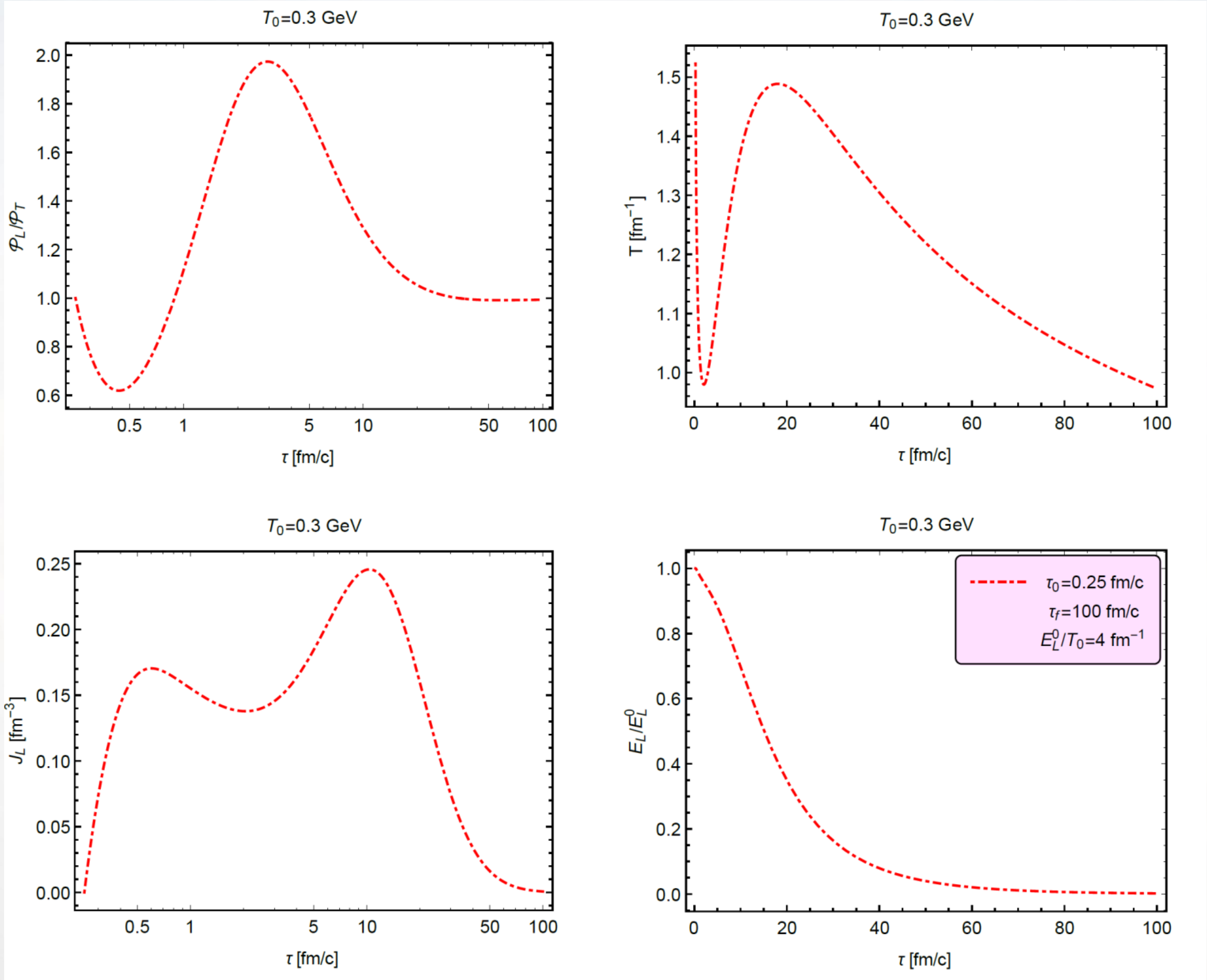
E_L^0/T_0=4 fm^{-1}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τ [fm/c]
T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
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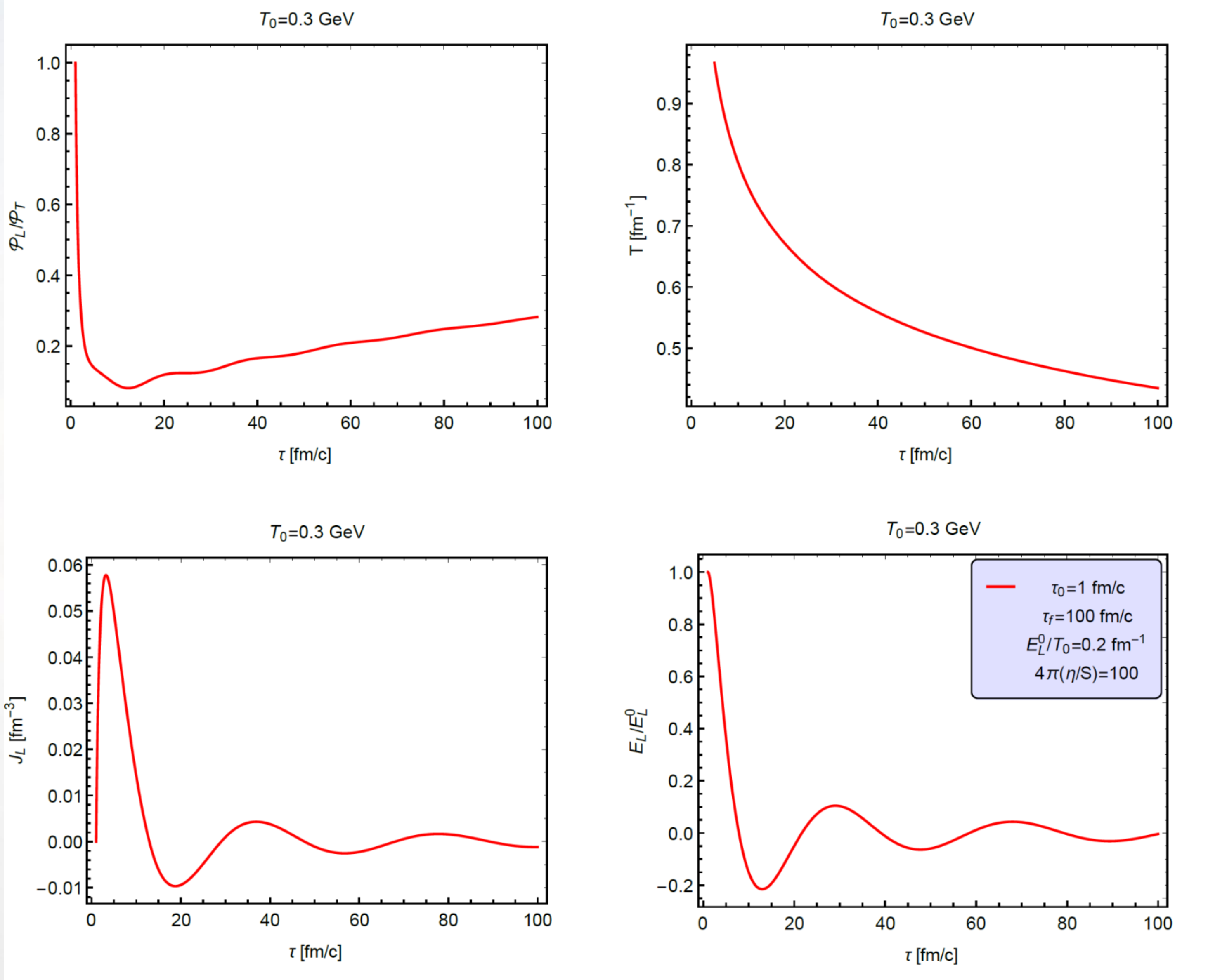
J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
τ [fm/c]
τ_0=0.25 fm/c
τ_f=100 fm/c
E_L^0/T_0=4 fm^{-1}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$\mathcal{P}_L/\mathcal{P}_T$
τ [fm/c]
T_0=0.3 GeV
T [fm^{-1}]
τ [fm/c]
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J_L [fm^{-3}]
τ [fm/c]
T_0=0.3 GeV
E_L/E_L^0
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