Spin-dependent distribution functions for relativistic hydrodynamics of spin-1/2 particles

Enrico Speranza

with W. Florkowski, B. Friman, A. Jaiswal, and R. Ryblewski
arXiv:1705.00587, arXiv:1712.07676

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC-TR 211

The 4th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions
Galileo Galilei Institute

Florence, March 19-22, 2018

Introduction

- Relativistic hydrodynamics with spin based on distribution functions
  see W. Florkowski's talk, and R. Ryblewski's poster

- Study formal aspects of distribution functions

- Connection between polarization 3-vector, spin tensor
  and Pauli-Lubański 4-vector

Spin-density matrix

- **Pure state**: $|\psi\rangle = \sum_\lambda c_\lambda |\lambda\rangle$
  Expectation value of an operator $\langle O\rangle = \langle\psi|O|\psi\rangle$
- **Mixed state**: incoherent mixture of $|\psi_i\rangle$ with statistical weight a_i

$$f = \sum_i a_i |\psi_i\rangle\langle\psi_i| = \sum_{\lambda,\lambda'} f_{\lambda\lambda'} |\lambda\rangle\langle\lambda'|$$

$f_{\lambda\lambda'} = \sum_i a_i\, c_\lambda^{(i)} c_{\lambda'}^{(i)*}$. Expectation value: $\langle O\rangle = \mathrm{Tr}(f\, O)$

Spin-1/2 particle (2×2 hermitian matrix):

$$f = \frac{1}{2}(1 + \boldsymbol{\mathcal{P}}\cdot\boldsymbol{\sigma})$$

- Polarization 3-vector: $\boldsymbol{\mathcal{P}} = \langle\boldsymbol{\sigma}\rangle = \mathrm{Tr}(f\,\boldsymbol{\sigma})$

$\lvert\boldsymbol{\mathcal{P}}\rvert = 1$	Pure state
$0 < \lvert\boldsymbol{\mathcal{P}}\rvert < 1$	Mixed state
$\lvert\boldsymbol{\mathcal{P}}\rvert = 0$	Completely unpolarized mixed state

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Local distribution functions for spin-1/2 particles

- Starting point (Becattini et al., Annals. Phys. 338 32):

$$f^+_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+u_s(p), \qquad f^-_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^-v_r(p)$$

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right] M^\pm$$

$$M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]$$

with $\beta^\mu = u^\mu/T$, $\xi = \mu/T$, $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$

- $\omega_{\mu\nu}$ analogue to EM field-strength tensor $F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma}u^\beta B^\gamma$

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma}u^\beta\omega^\gamma,$$

- Polarization tensor expressed like EM field-strength tensor

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

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Spin matrices $M^{\pm}$

- General expression

$$M^{\pm} = 1_{4\times 4}\left[\Re(\cosh z) \pm \Re\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right] + i\gamma_5\left[\Im(\cosh z) \pm \Im\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right]$$

with $z = \frac{1}{2\sqrt{2}}\sqrt{\omega_{\mu\nu}\omega^{\mu\nu} + i\omega_{\mu\nu}\tilde{\omega}^{\mu\nu}} = \frac{1}{2}\sqrt{k\cdot k - \omega\cdot\omega + 2ik\cdot\omega}$

- Assumptions: $k\cdot\omega = 0$, and $k\cdot k - \omega\cdot\omega \geq 0$ (ζ is real)

$$M^{\pm} = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\omega_{\mu\nu}\Sigma^{\mu\nu}, \qquad \zeta \equiv \frac{1}{2}\sqrt{k\cdot k - \omega\cdot\omega}$$

- ζ imaginary – F. Becattini, W. Florkowski, E.S. (to be published)

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Polarization 3-vector $\mathcal{P}$

- Expansion in terms of Pauli matrices

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]$$

$$\boldsymbol{P} = \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] = \boldsymbol{b}_*$$

$*$ denotes the PARTICLE REST FRAME

- Average polarization vector

$$\mathcal{P} = \frac{1}{2}\frac{\mathrm{tr}_2\left[(f^+ + f^-)\sigma\right]}{\mathrm{tr}_2\left[f^+ + f^-\right]} = -\frac{1}{2}\tanh(\zeta)\frac{\boldsymbol{P}}{2\zeta}$$

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- **CAUTION**: Valid for small $\omega_{\mu\nu}$ and particle momenta p

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Spin tensor $S^{\lambda,\mu\nu}$

- Energy-momentum tensor $T^{\mu\nu}$
  Total angular momentum tensor ("orbital"+"spin"):

$$J^{\lambda,\mu\nu} = x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu} + S^{\lambda,\mu\nu}$$

  $S^{\lambda,\mu\nu}$ = Spin tensor

- **Conservation of energy, momentum and total angular momentum**

$$\partial_{\mu} T^{\mu\nu} = 0, \qquad \partial_{\lambda} J^{\lambda,\mu\nu} = 0 \Longrightarrow \partial_{\lambda} S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Total energy-momentum and angular momentum must be fixed

$$P^{\mu} = \int d^3\Sigma_{\lambda}\, T^{\lambda\mu} \qquad J^{\mu\nu} = \int d^3\Sigma_{\lambda}\, J^{\lambda,\mu\nu}$$

- Densities are defined up to divergences
  $\Longrightarrow$ **Pseudo-gauge transformations:**

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_{\lambda}(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda})$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_{\alpha} Z^{\alpha\lambda,\mu\nu}$$

Leave P^{μ} and $J^{\mu\nu}$ invariant

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Leave P^{μ} and $J^{\mu\nu}$ invariant

Pauli-Lubański four-vector (I)

- Phase-space density of total angular momentum of particle with momentum p

$$E_p \frac{dJ^{\lambda,\mu\nu}(x,p)}{d^3p}$$

- Pauli-Lubański four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}$$

- **Which spin tensor do we use?**

(Becattini, Tinti, Annals Phys. 325, 1566)
used in W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, E.S. arXiv:1705.00587

$$S^{\lambda,\mu\nu} = \kappa\int\frac{d^3p}{2E_p}\,p^\lambda\,\mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] = \frac{wu^\lambda}{4\zeta}\omega^{\mu\nu}$$

See W. Florkowski's talk

- Total angular momentum density becomes

$$E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}\,p^\lambda\left(x^\nu p^\alpha - x^\alpha p^\nu\right)\mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2}\,p^\lambda\mathrm{tr}_4\left[\left(X^+ - X^-\right)\Sigma^{\nu\alpha}\right]$$

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Pauli-Lubański four-vector (II)

- Particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2}\,\Delta\Sigma \cdot p\,\mathrm{tr}_4\left(X^+ + X^-\right)$$

- PL per particle

$$\pi_\mu(x,p) = \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)}$$

- PL four-vector in the PRF agrees with the Polarization vector (!)

$$\pi^0_* = 0, \qquad \boldsymbol{\pi}_* = \boldsymbol{\mathcal{P}}$$

- **What about other forms for spin tensor?**

Different form for spin tensor

- PL four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}$$

- Canonical spin tensor

$$S^{\lambda,\mu\nu}_{\rm can} = \kappa\int\frac{d^3p}{2E_p}\Big(p^\lambda {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] - p^\mu {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\nu}\right] + p^\nu {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\mu}\right]\Big)$$

- de Groot, van Leeuwen, van Weert spin tensor

$$S^{\lambda,\mu\nu}_{\rm GLW} = \kappa\int\frac{d^3p}{2E_p}p^\lambda\left({\rm tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] + \frac{i}{2m^2}{\rm tr}_4\left[(X^+ - X^-)p_\alpha\gamma^\alpha\left(\gamma^\mu p^\nu - \gamma^\nu p^\mu\right)\right]\right)$$

PL is identical for all forms of spin tensor considered!

Different form for spin tensor

- PL four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}$$

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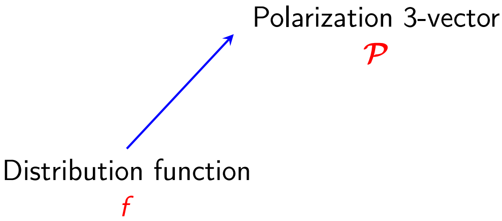
Summary

Distribution function

f

Summary



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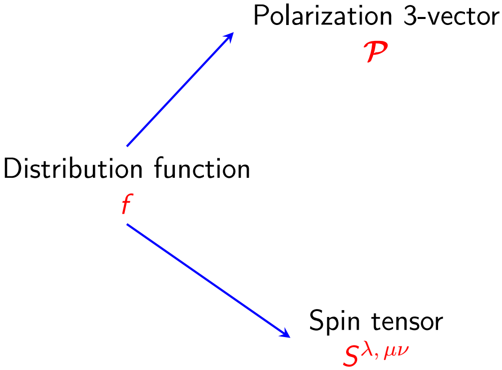
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$\mathcal{P}$
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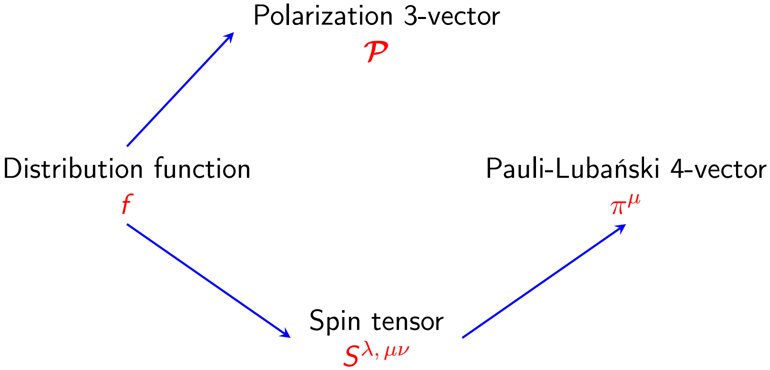
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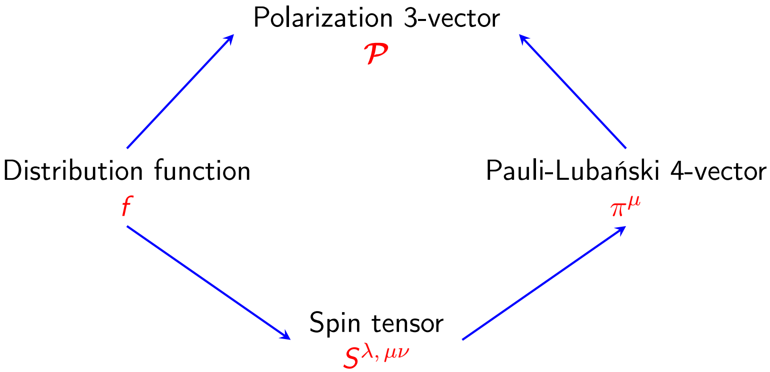

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