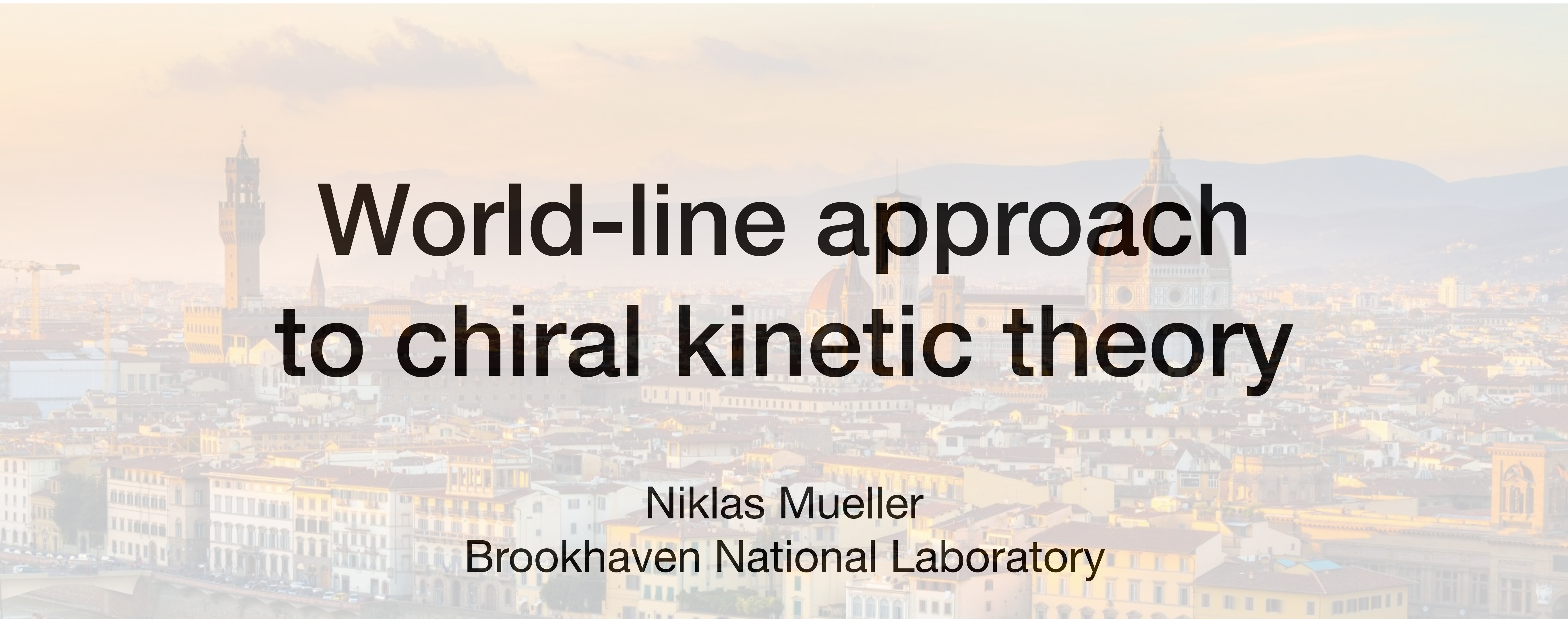


World-line approach to chiral kinetic theory

Niklas Mueller
Brookhaven National Laboratory

with Raju Venugopalan and Yi Yin

arXiv:1701.03331, 1702.01233, 1712.04057

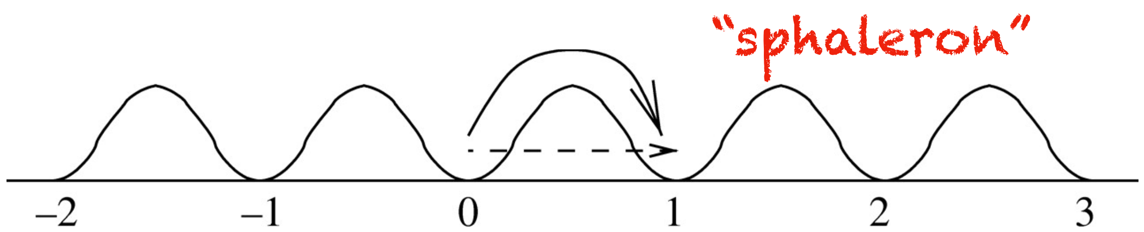
Workshop on Chirality, Vorticity and Magnetic Fields in Heavy Ion Collisions

Galileo Galilei Institute, Florence, Italy

20 March 2018

Probing topological properties of QCD

- **P and CP *odd* field configurations are connected to the topological structure of QCD**



$$\frac{dN_{CS}}{dt} = \frac{g^2}{8\pi^2} \int d^3x \; E_i^a(\mathbf{x}) B_i^a(\mathbf{x})$$

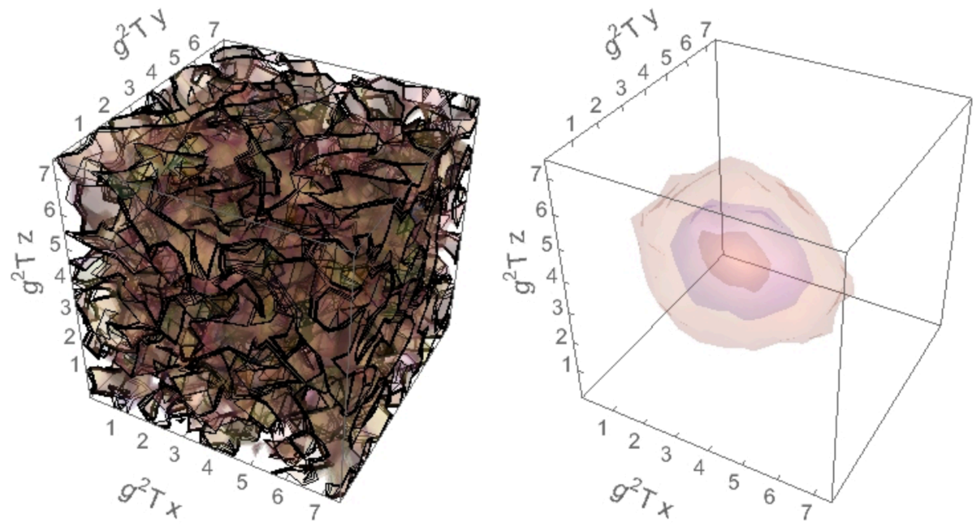
- **In the EW sector, topological transitions might be responsible for the observed baryon asymmetry of the Universe.**

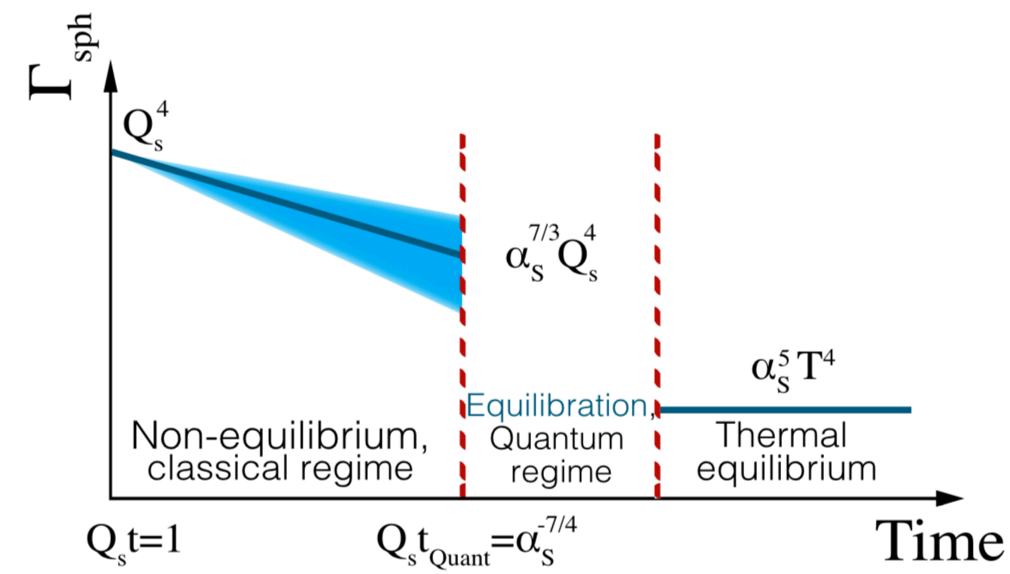
- **Puzzling properties:**

QCD is P and CP even.
Axial currents are anomalous.

Sphalerons in the Glasma

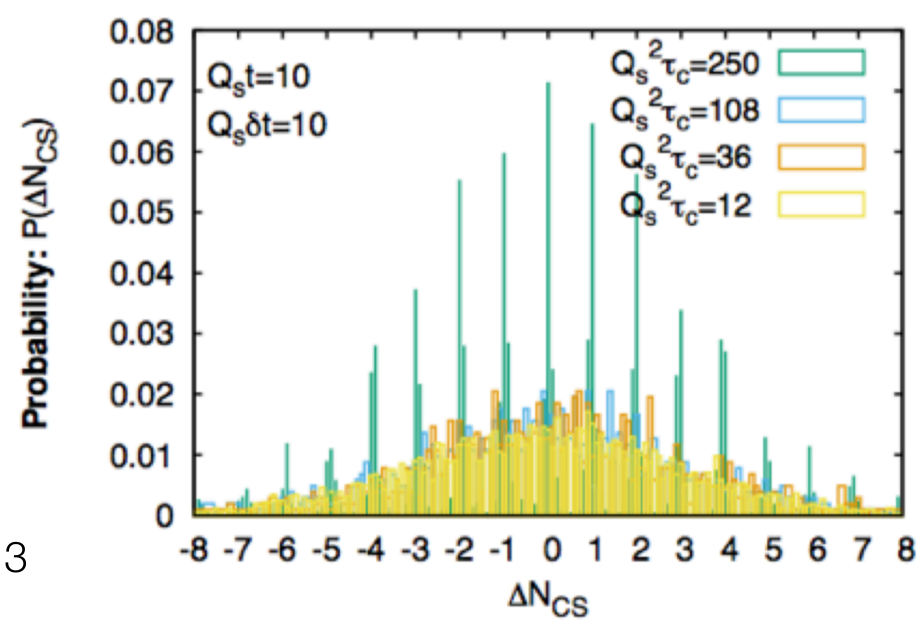
- **In the matter created in the earliest moments of a heavy ion collision, *(the 'Glasma')* sphaleron transitions are abundant!**





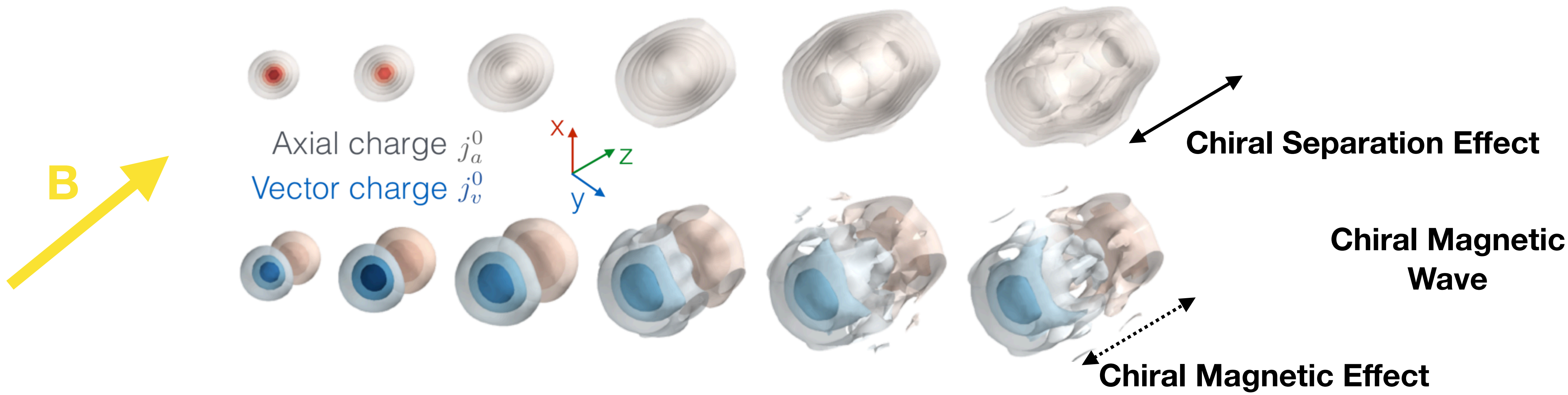
- **Real-time lattice simulations:**
  Significant sphaleron transitions on time scales of the order 1/Q_s

$$C(t,\delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t)\right)^2 \right\rangle$$



Real-time lattice simulations
Mace, Schlichting, Venugopalan PRD93 (2016) no.7, 074036

Anomalous fermion production



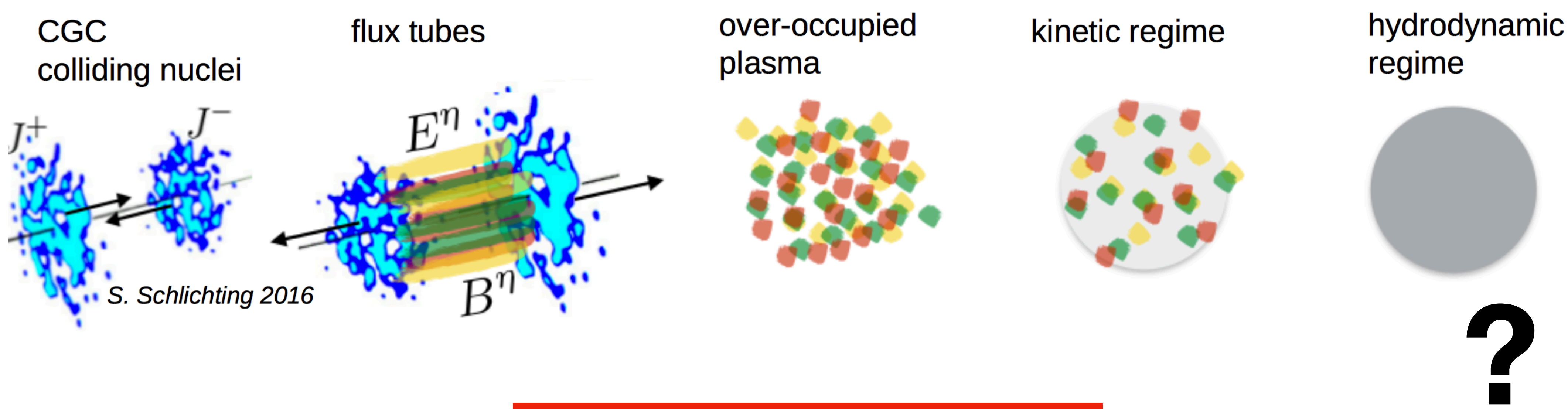
Classical-statistical lattice simulations

- **Dynamics of anomalous fermion production at earliest times**
- **Onset and properties of of anomalous transport**

See also talk by N. Tanji

M. Mace, NM, S. Schlichting, S. Sharma: PRL 117, 061601, PRD 95, 036023, NPA 967, 752

Pre-equilibrium dynamics of the CME



Abundance of topological transitions

Large Magnetic Fields

Fermion production

'Anomalous transport'

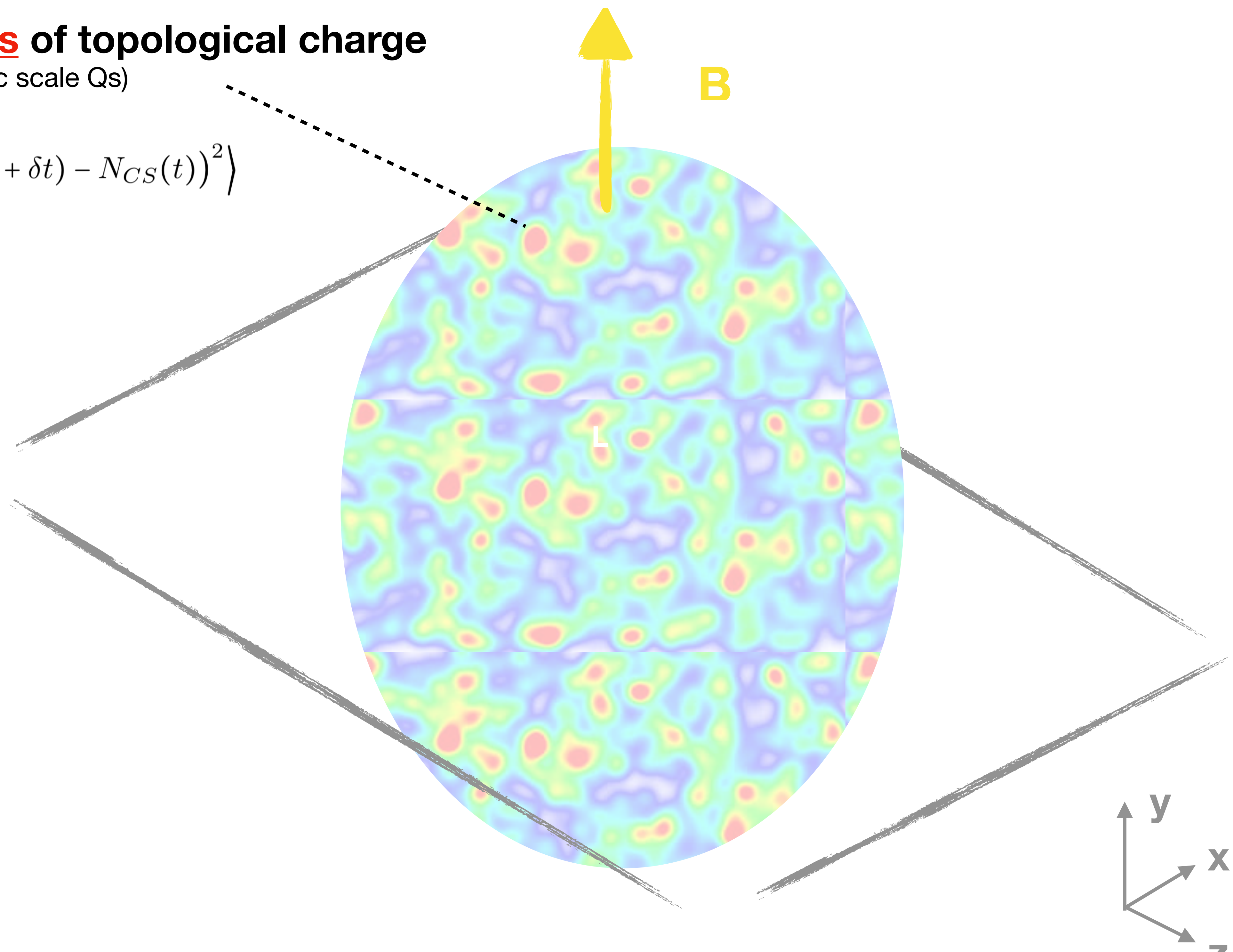
Understand how messengers of topological transitions are produced

Understand how they persist, interact with medium and turn into measurable quantities

Anomalous Transport

local <u>fluctuations</u> of topological charge
(dynamical, characteristic scale Qs)

$$C(t,\delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t) \right)^2 \right\rangle$$

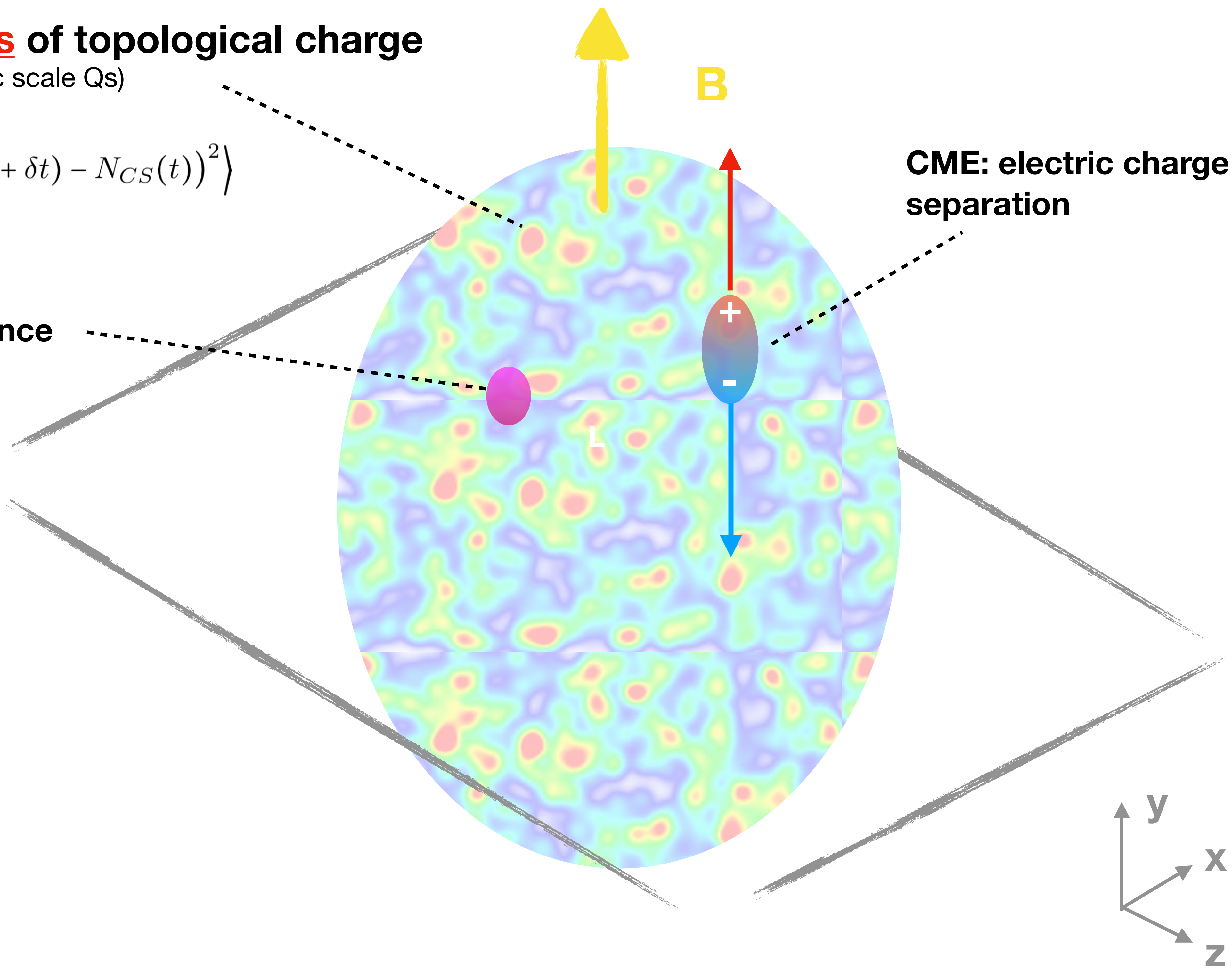


Anomalous Transport

local fluctuations of topological charge
(dynamical, characteristic scale Qs)

$$C(t, \delta t) = \frac{1}{V}\Big\langle\Big(N_{CS}(t+\delta t) - N_{CS}(t)\Big)^2\Big\rangle$$

production of initial
chiral charge imbalance

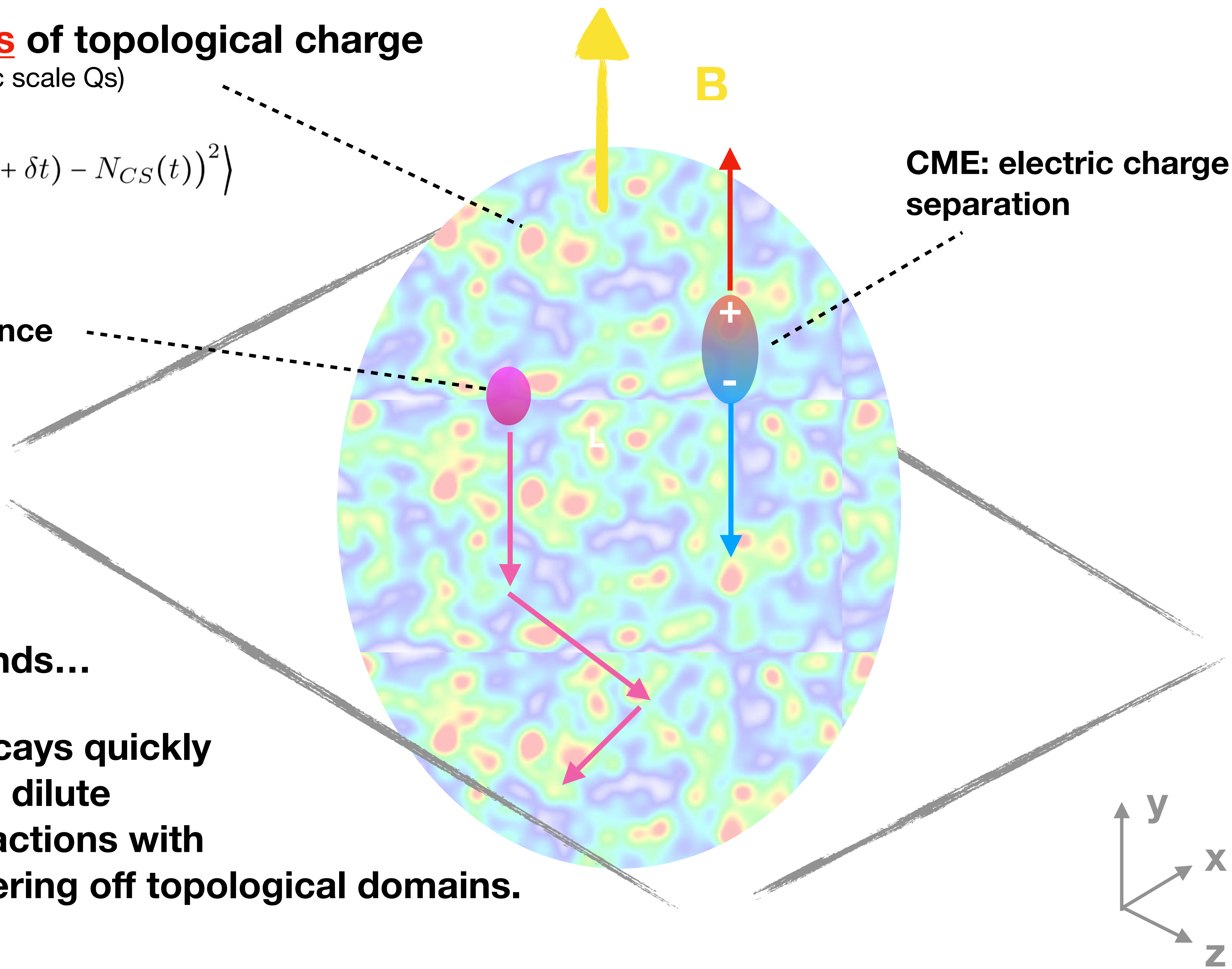


Anomalous Transport

local <u>fluctuations</u> of topological charge
(dynamical, characteristic scale Qs)

$$C(t, \delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t) \right)^2 \right\rangle$$

production of initial
<u>chiral</u> charge imbalance



CME: electric charge separation

As the fireball expands…

- **Magnetic field decays quickly**
- **medium becomes dilute**
- **Subsequent interactions with the fireball, scattering off topological domains.**

Chiral Kinetic Theory

Conceptual Challenges

- ☐ **Lorentz covariant description of chiral fermions, spin and helicity in "point-particle" picture.**
  Some very non-intuitive aspects: scattering and Lorentz covariance, "side-jumps", covariance of distribution functions?

- ☐ **Dynamics of the chiral anomaly consistently included.**
  Well known in QFT, but point-particle picture requires fundamental insights.

- ☐ **Interactions (=physics of anomalous transport)**
  Interactions with topological and non-topological fluctuations in the fireball. Scales and mechanisms? What can we learn for anomalous hydro?

Chiral Kinetic Theory from
the World-line approach to quantum field theory

1. World-line approach to QFT

- **One-loop effective action (euclidean for now)**

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right)$$

$$\mathcal{L} = \Phi^\dagger D^2 \Phi$$

Feynman, Schwinger 50's; Polyakov, 80's

1. World-line approach to QFT

- **One-loop effective action (euclidean for now)**

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right)$$

$$\mathcal{L} = \Phi^\dagger D^2 \Phi$$

- **Integral representation of log (heat-kernel)**

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t}\left(e^{-\sigma t} - e^{-t}\right)$$

Feynman, Schwinger 50's; Polyakov, 80's

1. World-line approach to QFT

- **One-loop effective action (euclidean for now)**

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right)$$

$$\mathcal{L} = \Phi^\dagger D^2 \Phi$$

- **Integral representation of log (heat-kernel)**

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t}\left(e^{-\sigma t} - e^{-t}\right)$$

- **Effective action: QM path integral of particle on circle (Strassler, 1992)**

$$\Gamma[A] \;=\; \int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\,\mathcal{P}\exp\left[-\underline{\int_0^T d\tau\,\left(\frac{1}{2\varepsilon}\dot{x}^2 + igA[x(\tau)]\cdot\dot{x}\right)}\right]$$

relativistic point-particle action

Feynman, Schwinger 50's; Polyakov, 80's

1. World-line approach to QFT

- **‘SUSY spinning particle models’ via anti-commuting variables**
  Berezin & Marinov, Barducci, Balachandran, Casalbuoni, Brink, Howe, DiVecchia (70s-80s)

$$\mathcal{L} = \frac{\dot{x}^2}{2\mathcal{E}} + \dot{x}_\mu A^\mu(x) + \underbrace{\frac{i}{2}\psi^\mu \dot{\psi}_\mu - \frac{i\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu}_{} + \ldots$$

Grassmanian spin variables

$$\psi_\mu \to \sqrt{\frac{\hbar}{2}}\gamma_5\gamma_\mu$$

Particle Spin Dynamics as the Grassmann Variant of Classical Mechanic

F. A. Berezin

Department of Mathematics, Moscow State University, Moscow, U.S.S.R.

AND

M. S. Marinov

Institute for Theoretical and Experimental Physics, Moscow 117259, U.S.S.R.

Received August 9, 1976

A generalization of classical mechanics is presented. The dynamical variables (functions on the phase space) are assumed to be elements of an algebra with anticommuting generators (the Grassmann algebra). The action functional and the Poisson brackets are defined. The equations of motion are deduced from the variational principle. The dynamics are also

1. World-line approach to QFT

- **'SUSY spinning particle models' via anti-commuting variables**
  Berezin & Marinov, Barducci, Balachandran, Casalbuoni, Brink, Howe, DiVecchia (70s-80s)

$$\mathcal{L} = \frac{\dot{x}^2}{2\mathcal{E}} + \dot{x}_\mu A^\mu(x) + \underline{\frac{i}{2}\psi^\mu \dot{\psi}_\mu - \frac{i\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu} + \ldots$$

Grassmanian spin variables

$$\psi_\mu \to \sqrt{\frac{\hbar}{2}}\gamma_5\gamma_\mu$$

Particle Spin Dynamics as the Grassmann Variant of Classical Mechanic

F. A. Berezin

Department of Mathematics, Moscow State University, Moscow, U.S.S.R.

AND

M. S. Marinov

Institute for Theoretical and Experimental Physics, Moscow 117259, U.S.S.R.

Received August 9, 1976

A generalization of classical mechanics is presented. The dynamical variables (functions on the phase space) are assumed to be elements of an algebra with anticommuting generators (the Grassmann algebra). The action functional and the Poisson brackets are defined. The equations of motion are deduced from the variational principle. The dynamics are also

- **Related to world-line representation for fermions (QED/QCD)**

$$W[A,B] = \log\det(i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/)$$

$$W[A,B] = W_{\mathbb{R}}[A,B] + iW_{\mathbb{I}}[A,B]$$

1. World-line approach to QFT

- **'SUSY spinning particle models' via anti-commuting variables**
  Berezin & Marinov, Barducci, Balachandran, Casalbuoni, Brink, Howe, DiVecchia (70s-80s)

$$\mathcal{L} = \frac{\dot{x}^2}{2\mathcal{E}} + \dot{x}_\mu A^\mu(x) + \frac{i}{2}\psi^\mu \dot{\psi}_\mu - \frac{i\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu \; + \ldots$$

Grassmanian spin variables

$$\psi_\mu \to \sqrt{\frac{\hbar}{2}}\gamma_5\gamma_\mu$$

Particle Spin Dynamics as the Grassmann Variant of Classical Mechanic

F. A. Berezin

Department of Mathematics, Moscow State University, Moscow, U.S.S.R.

AND

M. S. Marinov

Institute for Theoretical and Experimental Physics, Moscow 117259, U.S.S.R.

Received August 9, 1976

A generalization of classical mechanics is presented. The dynamical variables (functions on the phase space) are assumed to be elements of an algebra with anticommuting generators (the Grassmann algebra). The action functional and the Poisson brackets are defined. The equations of motion are deduced from the variational principle. The dynamics are also

- **Related to world-line representation for fermions (QED/QCD)**

$$W[A,B] = \log\det(\, i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/\,) \qquad\qquad W[A,B] = W_{\mathbb{R}}[A,B] + iW_{\mathbb{I}}[A,B]$$

… which has that exact same "point-particle" Lagrangian (no approximations!)

$$W_{\mathbb{R}} = \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathcal{N}\int_P \mathcal{D}x \int_{AP}\mathcal{D}\psi \,\mathrm{tr}\exp\left\{-\int_0^T d\tau\, \mathcal{L}(\tau)\right\}$$

2. Origin of the anomaly

2. Origin of the anomaly

- **Origin of anomaly from fermionic effective action well known**

$$W[A,B] = W_{\mathbb{R}}[A,B] + \underline{iW_{\mathbb{I}}[A,B]}$$

" $: \log\det(\, i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/\,)$ "

(euclidean)

- **Phase of the determinant / <u>imaginary part</u> of effective action is ill-defined, <u>origin of the anomaly</u>** (Alvarez-Gaume, Witten 1984)

2. Origin of the anomaly

- **Origin of anomaly from fermionic effective action well known**

$$W[A,B] = W_{\mathbb{R}}[A,B] + iW_{\mathbb{I}}[A,B]$$

" : $\log\det(i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/)$ "
(euclidean)

- **Phase of the determinant / <u>imaginary part</u> of effective action is ill-defined, <u>origin of the anomaly</u>** (Alvarez-Gaume, Witten 1984)

- **A world-line representation of the imaginary part can be found, but only if we give up chiral symmetry** (D'Hoker and Gagne)

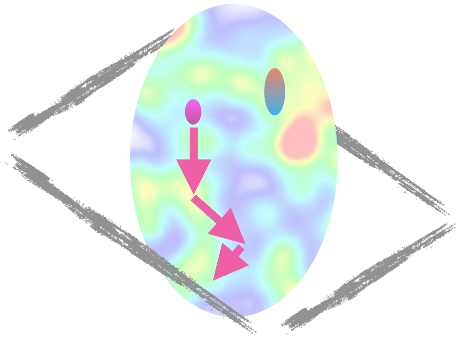
$$W_{\mathbb{I}} = \frac{i\mathcal{E}}{64}\int_{-1}^{1} d\alpha \int_{0}^{\infty} dT\ \mathrm{Tr}\left\{\hat{M} e^{-\frac{\mathcal{E}}{2}T\tilde{\Sigma}^2_{(\alpha)}}\right\}$$

- **Anomaly related to Grassmann zero modes on the world line** (see also Polyakov's book 80's)

$$\partial_\mu \langle j^5_\mu(y)\rangle \equiv \partial_\mu \frac{i\delta W_{\mathbb{I}}}{\delta B_\mu(y)}\Big|_{B=0} = -\frac{1}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(y)F_{\rho\sigma}(y)$$

Constructing a quantum kinetic theory for chiral fermions

3. Chiral Kinetic Theory



- **World-line approach naturally continued to Schwinger-Keldysh (SK) non-equilibrium path integral**

$$Z = \int [d\xi]\ \exp\left(-G[\xi]\right) \int_{\mathcal{C}} [dA]\ \exp\left(iS_{\mathrm{eff}}\right)$$

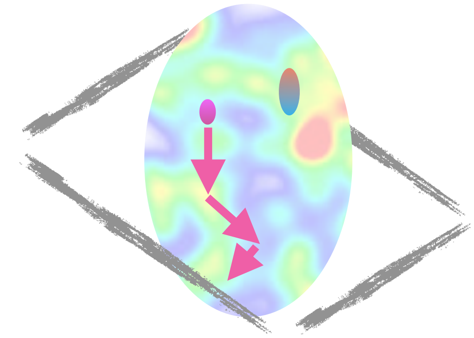
$$S_{\mathrm{eff}}[A,\xi] \ = \ -\tfrac{1}{4}\int_{\mathcal{C}} d^4x\, F_{\mu\nu}F^{\mu\nu} + \underline{W[A,\xi]}$$

(SK) world-line effective action,

can derive (!) that from original QFT

Barducci 1984

3. Chiral Kinetic Theory



- **World-line approach naturally continued to Schwinger-Keldysh (SK) non-equilibrium path integral**

$$Z = \int [d\xi]\, \exp\left(-G[\xi]\right) \int_{\mathcal{C}} [dA]\, \exp\left(iS_{\rm eff}\right)$$

$$S_{\rm eff}[A,\xi] \;=\; -\tfrac{1}{4}\int_{\mathcal{C}} d^4x\, F_{\mu\nu}F^{\mu\nu} + \underline{W[A,\xi]}$$

(SK) world-line effective action,

can derive (!) that from original QFT

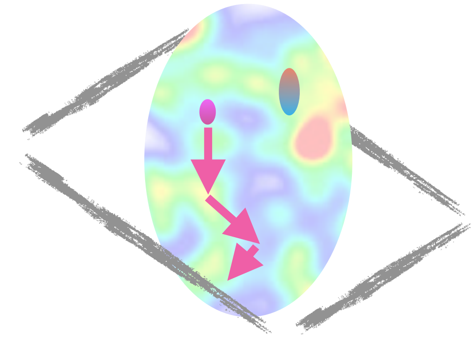
- **In saddle-point limit (Truncated Wigner approximation) the central object is a Wigner distribution**

  - dynamics is determined by a Liouville equation

$$\{f,H\} = f\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\dot{x}^\mu + \frac{\overleftarrow{\partial}}{\partial P^\mu}\dot{P}^\mu + \frac{\overleftarrow{\partial}}{\partial \psi^\mu}\dot{\psi}^\mu\right) = 0$$

Barducci 1984

3. Chiral Kinetic Theory



- **World-line approach naturally continued to Schwinger-Keldysh (SK) non-equilibrium path integral**

$$Z = \int [d\xi]\, \exp\left(-G[\xi]\right) \int_{\mathcal{C}} [dA]\, \exp\left(iS_{\text{eff}}\right)$$

$$S_{\text{eff}}[A,\xi] \;=\; -\tfrac{1}{4}\int_{\mathcal{C}} d^4x\, F_{\mu\nu}F^{\mu\nu} + \underline{W[A,\xi]}$$

(SK) world-line effective action,

can derive (!) that from original QFT

- **In saddle-point limit (Truncated Wigner approximation) the central object is a Wigner distribution**

  - dynamics is determined by a Liouville equation

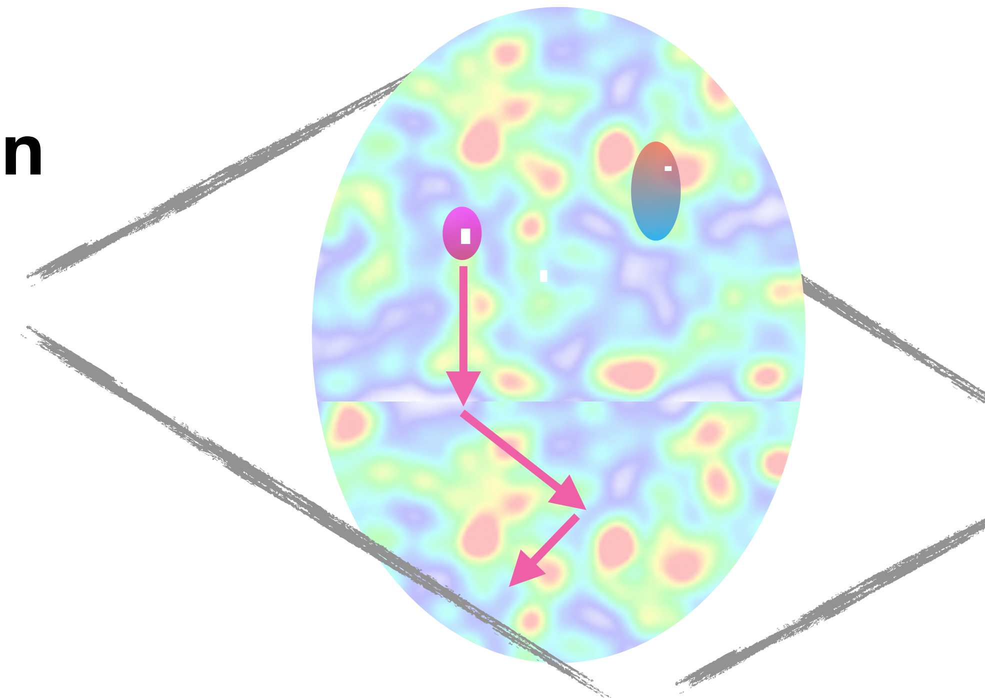
$$\{f,H\} = f\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\dot{x}^\mu + \frac{\overleftarrow{\partial}}{\partial P^\mu}\dot{P}^\mu + \frac{\overleftarrow{\partial}}{\partial \psi^\mu}\dot{\psi}^\mu\right) = 0$$

- governed by chiral fermion Hamiltonian **(not a model, derived from QFT!)**

$$H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_- \qquad c_\pm \;\equiv\; \tfrac{1}{2}\left(\pm P_\mu\psi^\mu + \tfrac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right)$$

Barducci 1984

3. Chiral Kinetic Theory



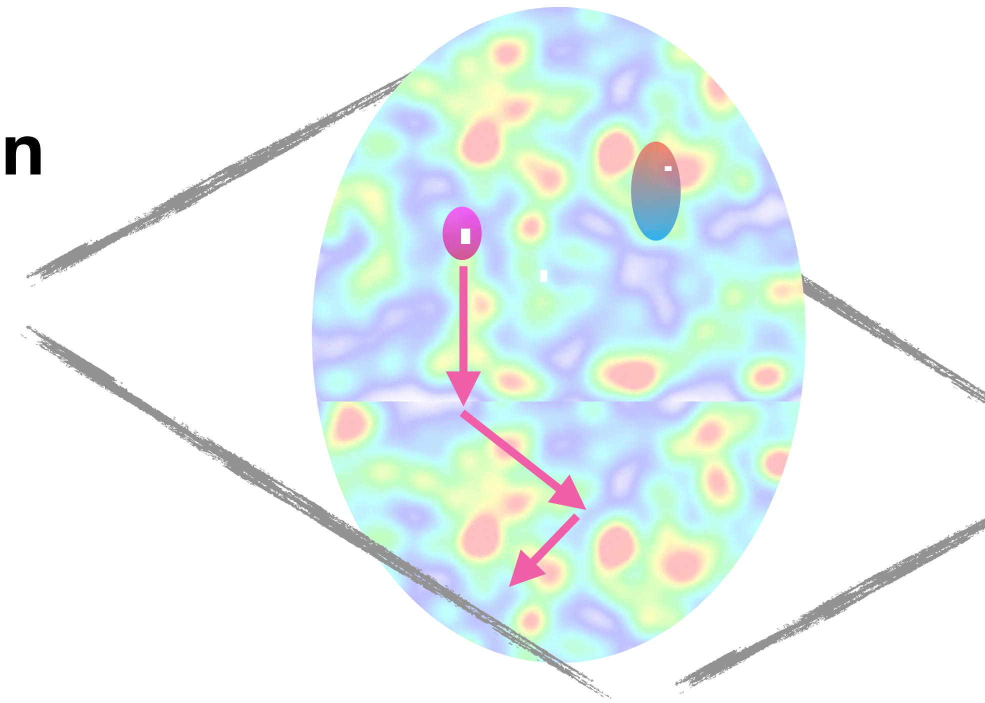
Quantum Kinetic Theory from Liouville equation

- **need to understand interactions of anomalous messengers with local topological fluctuations**

$$C(t,\delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t) \right)^2 \right\rangle$$

3. Chiral Kinetic Theory

Quantum Kinetic Theory from Liouville equation



- need to understand interactions of anomalous messengers with local topological fluctuations

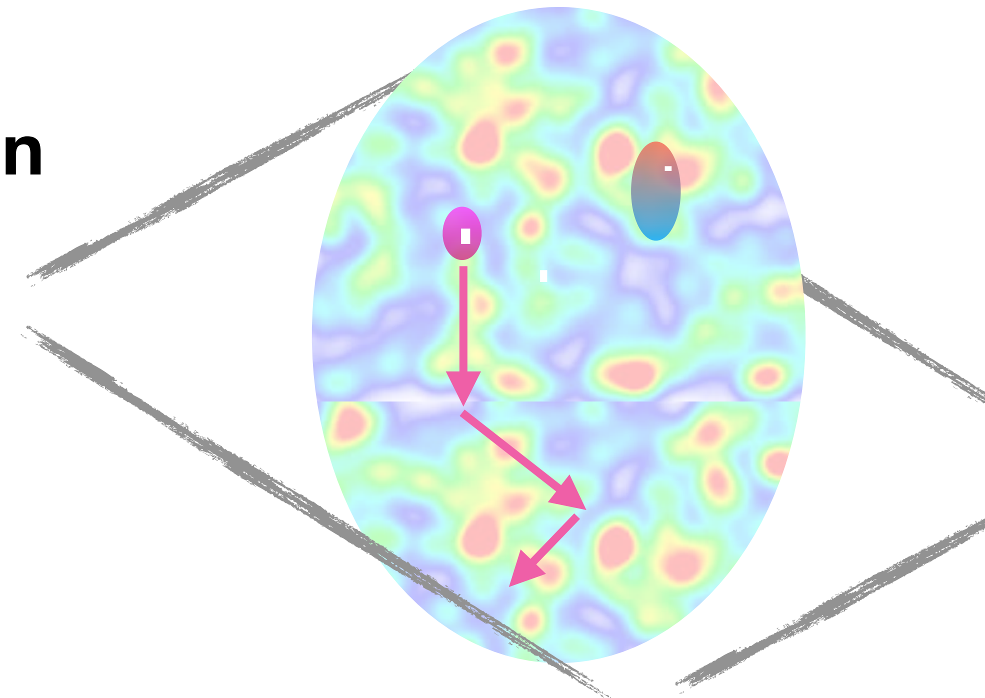
$$C(t,\delta t) = \frac{1}{V}\left\langle\left(N_{CS}(t+\delta t) - N_{CS}(t)\right)^2\right\rangle$$

- SK path integral specifies (quantum and statistical) ensemble, through initial density matrix / Wigner distribution. Fluctuations understood naturally.

$$f \equiv \bar{f} + \delta f$$ 1-particle distribution function

3. Chiral Kinetic Theory

Quantum Kinetic Theory from Liouville equation



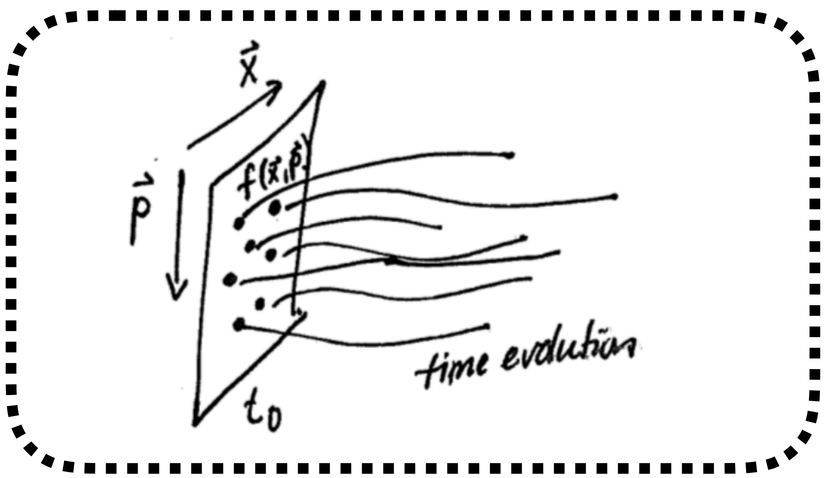
- need to understand interactions of anomalous messengers with local topological fluctuations

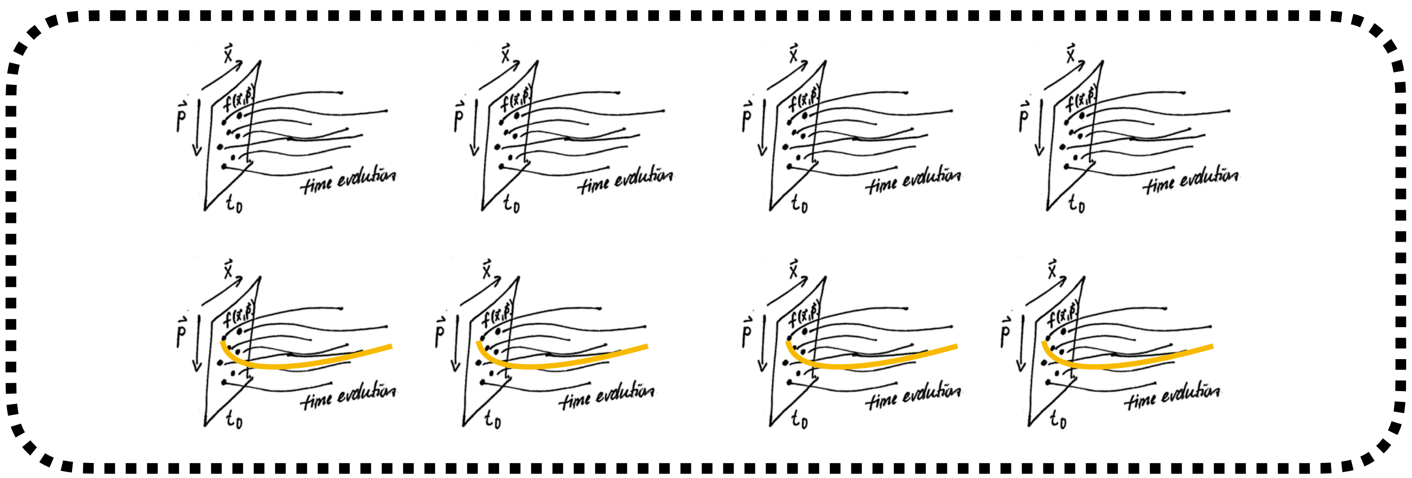
$$C(t,\delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t) \right)^2 \right\rangle$$

- SK path integral specifies (quantum and statistical) ensemble, through initial density matrix / Wigner distribution. Fluctuations understood naturally.

$$f \equiv \bar{f} + \delta f$$

1-particle distribution function





Quantum and statistical ensemble

3. Chiral Kinetic Theory

Quantum Kinetic Theory

$$\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{\epsilon^{\mu\nu\alpha\beta}}{6}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]\Big)$$

$$+ \frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta + \frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] = C[\delta f, \delta F]$$

c.f. BBGKY hierarchy, Landau Lifshitz, "Statistical Physics" and "Physical Kinetics"

3. Chiral Kinetic Theory

Quantum Kinetic Theory

$$\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{\epsilon^{\mu\nu\alpha\beta}}{6}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]\Big)$$

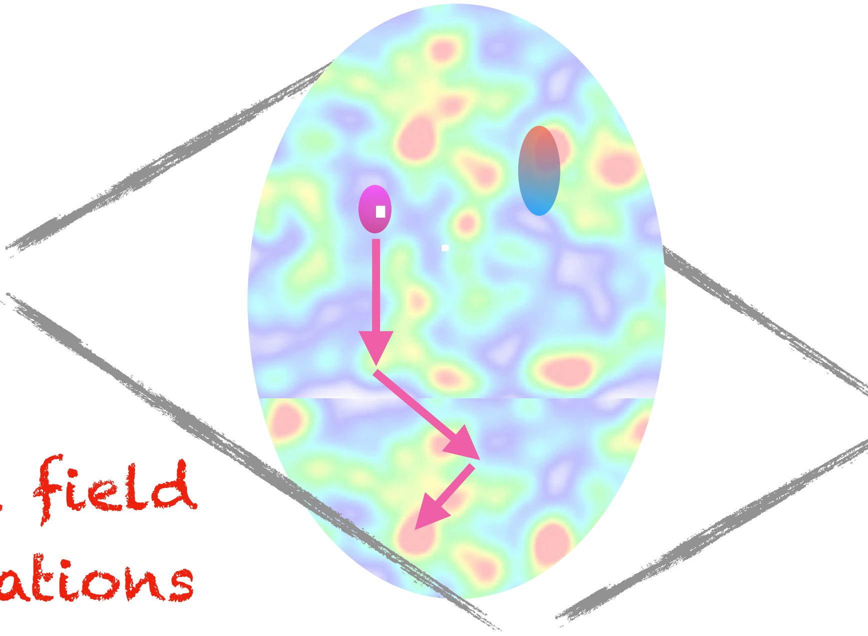
$$+\frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta + \frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] = C[\delta f, \delta F]$$

- **Collision terms from fluctuations**

$$C[\delta f, \delta F] \equiv -\varepsilon\langle\delta f\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\underline{\delta F^{\mu\nu}}\rangle\psi_\nu + \frac{i\varepsilon}{2}\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\partial^\mu\underline{\delta F_{\alpha\beta}}\rangle\psi^\alpha\psi^\beta$$

$$-\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\underline{\delta F^{\mu\alpha}}\rangle\Big(\varepsilon P_\alpha + \frac{i}{2}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big)$$

Gauge field Fluctuations



c.f. BBGKY hierarchy, Landau Lifshitz, "Statistical Physics" and "Physical Kinetics"

3. Chiral Kinetic Theory

Quantum Kinetic Theory

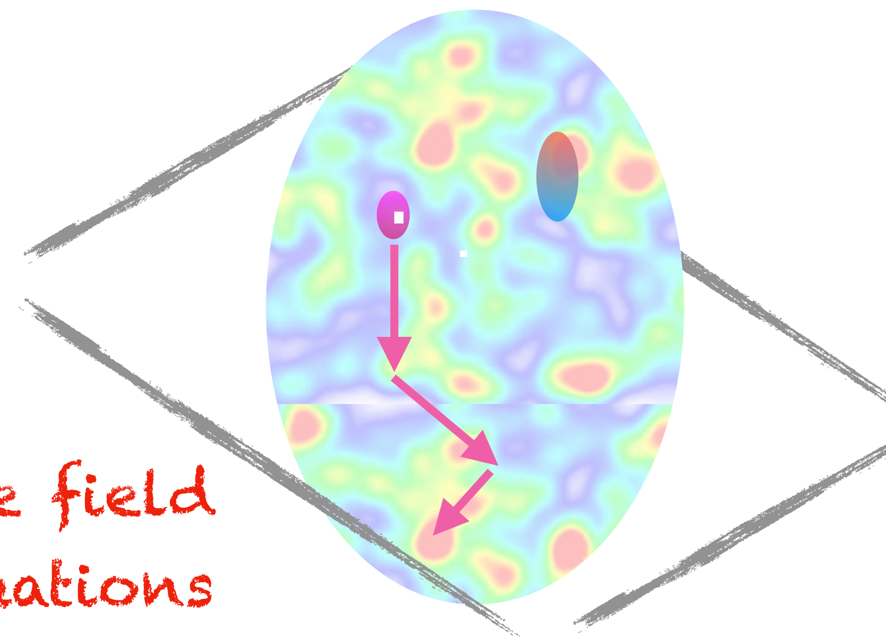
$$\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{\epsilon^{\mu\nu\alpha\beta}}{6}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]\Big)$$

$$+\frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta + \frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] = C[\delta f, \delta F]$$

- **Collision terms from fluctuations**

$$C[\delta f,\delta F] \equiv -\varepsilon\langle\delta f\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\underline{\delta F^{\mu\nu}}\rangle\psi_\nu + \frac{i\varepsilon}{2}\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\partial^\mu\underline{\delta F_{\alpha\beta}}\rangle\psi^\alpha\psi^\beta$$

$$-\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\underline{\delta F^{\mu\alpha}}\rangle\Big(\varepsilon P_\alpha + \frac{i}{2}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big)$$



Gauge field Fluctuations

- **Using Maxwell / Yang-Mills equations —> hierarchy of equations**

$$\langle\delta f\,\delta f\rangle,\quad \langle\delta F^{\alpha\beta}\,\delta F^{\mu\nu}\rangle,\quad \underline{\langle\delta F\,\delta F\,\delta F\,\delta F}\,\rangle,\ldots \longleftrightarrow C(t,\delta t) = \frac{1}{V}\Big\langle\underline{\big(N_{CS}(t+\delta t) - N_{CS}(t)\big)^2}\Big\rangle$$

c.f. BBGKY hierarchy, Landau Lifshitz, "Statistical Physics" and "Physical Kinetics"

3. Bonus: Side-jumps

An issue of Lorentz covariance and spin: side jumps

Lorentz Invariance in Chiral Kinetic Theory
Jing-Yuan Chen, Dam T. Son (Chicago U.), Mikhail A. Stephanov (Illinois U., Chicago & Chicago U.), Ho-Ung Yee (Illinois U., Chicago & RIKEN BNL), Yi Yin (Illinois U., Chicago).
Published in **Phys.Rev.Lett. 113 (2014) no.18, 182302**

Collisions in Chiral Kinetic Theory
Jing-Yuan Chen, Dam T. Son (Chicago U.), Mikhail A. Stephanov (Illinois U., Chicago). Feb 24, 2015. 5 pp.
Published in **Phys.Rev.Lett. 115 (2015) no.2, 021601**

Seems puzzling, but is not

$$H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_- \qquad c_\pm \equiv \frac{1}{2}\left(\pm P_\mu\psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right)$$

mass-shell constraint

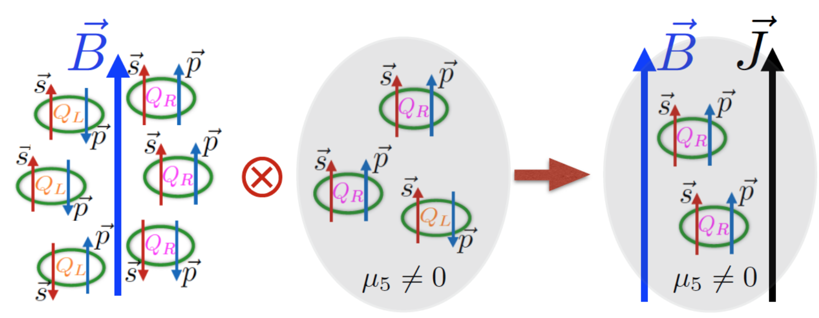
helicity constraint

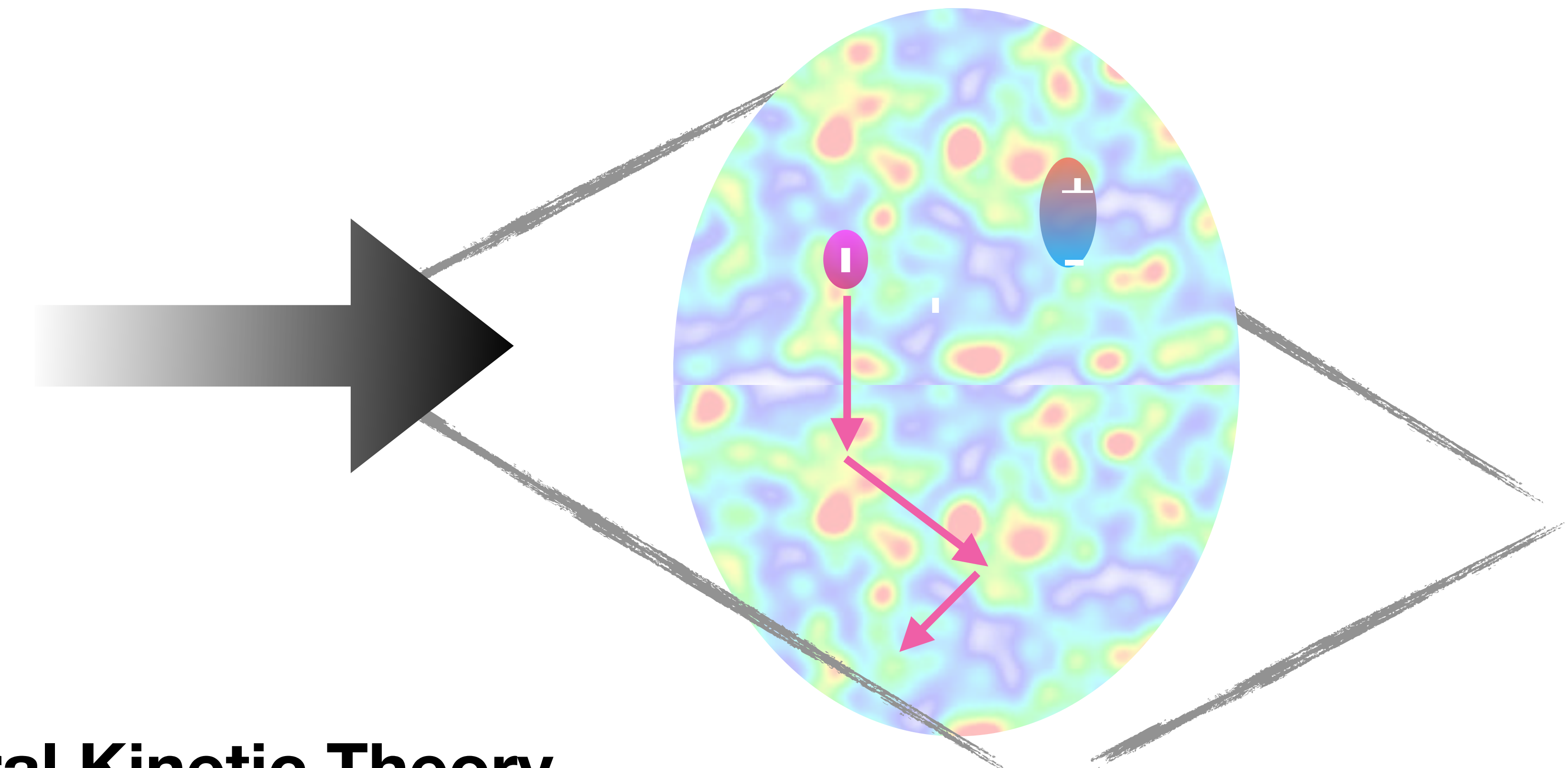
gauge parameters (Lagrange multiplier), 1st class constraints

- **'Position' $x^\mu(\tau)$ at given 'time' is not gauge invariant, because τ is not (reparametrization invariance of world line of relativistic particle).**
- **Similarly defining the local spin frame is somewhat ambiguous.**

Summary

Fluctuations matter!



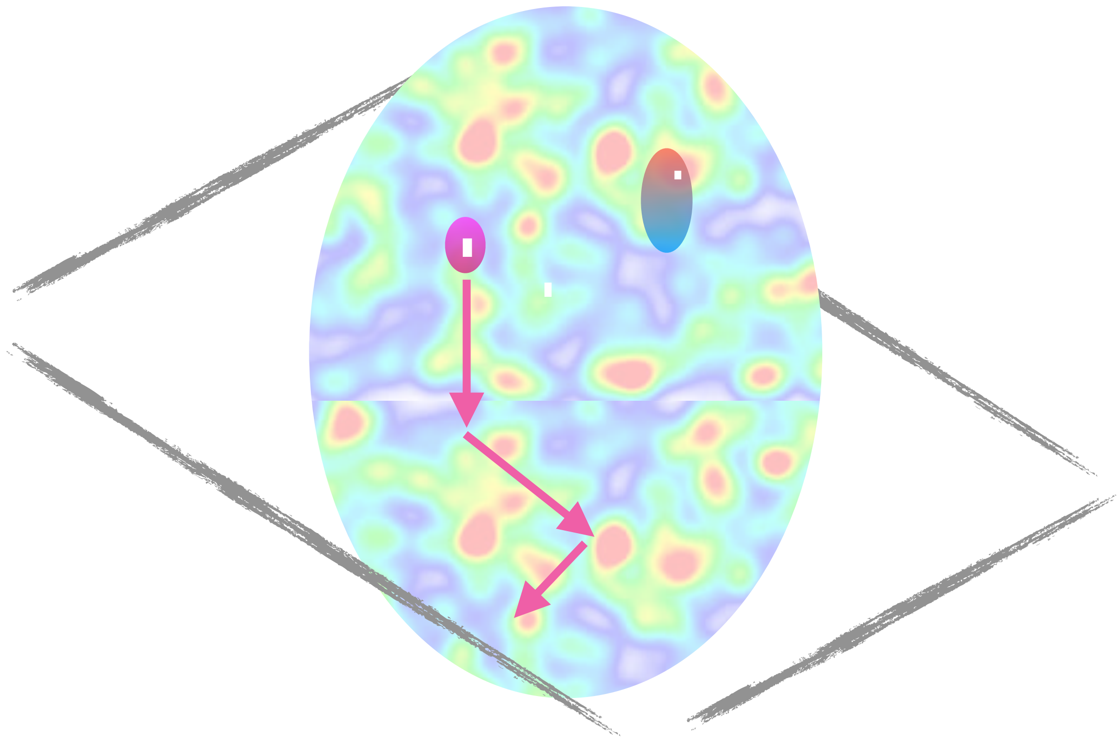


- **Derived a consistent Chiral Kinetic Theory using the world-line approach to QFT**

- **Issues of Lorentz covariance and chirality/spin/helicity naturally understood**

- **Non-equilibrium many-body (SK) formulation: quantum kinetic theory in saddle point limit (Truncated Wigner approx.)**

What next?



A lot more work ahead!

- compute collision terms, identify mechanisms for anomalous transport
- Connect to lattice simulations at early times and anomalous hydro at late times

- better insight into anomalous hydro/hydrodynamics of chiral fluids, deriving it from the kinetic theory

- Applications to other fields, many-body systems where the dynamics is nearly chiral and relativistic (neutron-stars, supernovae etc.)

Back-up:
Origin of the anomaly and Berry phase

The non-relativistic, $H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S} \cdot \left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right] \times \boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B} \cdot \boldsymbol{S}}{m}$ $\quad S^i = -\frac{i}{2}\epsilon^{ijk}\psi^j\psi^k$

and adiabatic limit of the world-line representation contains a Berry phase

$$W_{\mathbb{R}} = \int \mathscr{D}x \mathscr{D}p \, \exp\left(i \int dt \left[\dot{\mathbf{x}} \cdot \mathbf{p} - \tilde{H}\right]\right)$$

$$\tilde{H} = mc^2 + \frac{(\boldsymbol{p} - \boldsymbol{A}/c)^2}{2m} + A^0(x) \underline{- \dot{\mathbf{p}} \cdot \mathscr{A}(\boldsymbol{p})} \qquad \mathscr{A}(\boldsymbol{p}) \equiv -i\langle \psi^+(\boldsymbol{p}) | \boldsymbol{\nabla}_p | \psi^+(\boldsymbol{p}) \rangle$$

Interesting related development: K. Fujikawa 2005 and 2017

Back-up:
Origin of the anomaly and Berry phase

The non-relativistic, $H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S}\cdot\left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right]\times\boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B}\cdot\boldsymbol{S}}{m}$ $\qquad S^i = -\frac{i}{2}\epsilon^{ijk}\psi^j\psi^k$

and adiabatic limit of the world-line representation contains a Berry phase

$$W_{\mathbb{R}} = \int \mathscr{D}x\mathscr{D}p\, \exp\left(i\int dt\left[\dot{\mathbf{x}}\cdot\mathbf{p} - \tilde{H}\right]\right)$$

$$\tilde{H} = mc^2 + \frac{(\boldsymbol{p}-\mathbf{A}/c)^2}{2m} + A^0(x) - \underline{\dot{\mathbf{p}}\cdot\mathscr{A}(\boldsymbol{p})} \qquad \mathscr{A}(\boldsymbol{p}) \equiv -i\langle\psi^+(\boldsymbol{p})|\boldsymbol{\nabla}_p|\psi^+(\boldsymbol{p})\rangle$$

In the adiabatic limit, for large chemical potential and zero mass we reproduce the results of Son & Yamamoto, Stephanov and Yin …

… from the real part, which we have shown to be independent of the anomaly

Interesting related development: K. Fujikawa 2005 and 2017

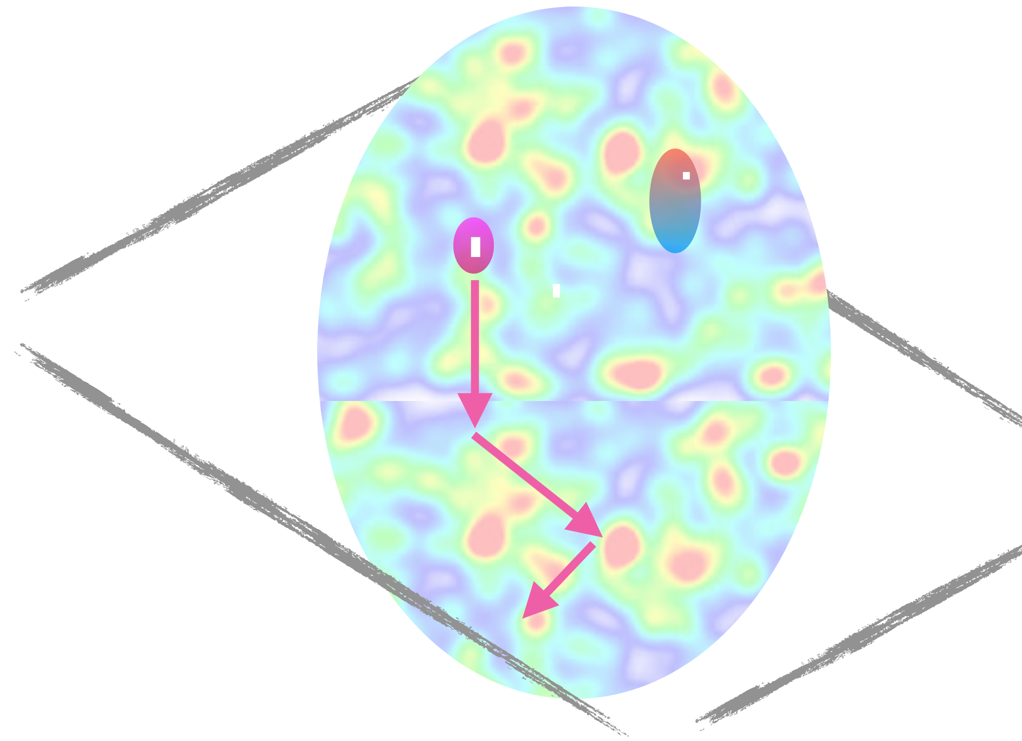
Fujikawa 2005

The notion of Berry's phase is known to be useful in various physical contexts [17]-[18], and the topological considerations are often crucial to obtain a qualitative understanding of what is going on. Our analysis however shows that the topological interpretation of Berry's phase associated with level crossing generally fails in practical physical settings with any finite T. The notion of "approximate topology" has no rigorous meaning, and it is important to keep this approximate topological property of geometric phases associated with level crossing in mind when one applies the notion of geometric phases to concrete physical processes. This approximate topological property is in sharp contrast to the Aharonov-Bohm phase [8] which is induced by the time-independent gauge potential and topologically exact for any finite time interval T. The similarity and difference between the geometric phase and the Aharonov-Bohm phase have been recognized in the early literature [1, 8], but our second quantized formulation, in which the analysis of the geometric phase is reduced to a diagonalization of the effective Hamiltonian, allowed us to analyze the topological properties precisely in the infinitesimal neighborhood of level crossing.

What we have shown in the present paper is that this expectation is not realized, and the similarity between the two is superficial.

7

Backup: Chiral Kinetic Theory



- **Collision terms generated by (topological) fluctuations in the medium**

- **Similar equations exist for the fluctuations, which in the dilute limit can be solved in closed form (c.f. BBGKY hierarchy)**

$$
\begin{aligned}
\delta f\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}P_\alpha \\
- \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta + \frac{i}{2}\bar{F}^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\bar{F}^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] \\
+ \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i\epsilon^{\mu\nu\alpha\beta}}{4}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big] = K[\delta F]\,,
\end{aligned}
$$

$$
\begin{aligned}
K[\delta F] \equiv -\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial P^\mu}\Big[\varepsilon\delta F^{\mu\alpha}P_\alpha - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu\delta F_{\alpha\beta}\psi^\beta \\
+ \frac{i}{2}\delta F^{\mu\alpha}\psi_\alpha\bar{\chi} - \frac{\epsilon_{\alpha\beta\lambda\sigma}}{12}\delta F^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\delta F^{\mu\alpha}\psi_\alpha\Big]\Big)
\end{aligned}
$$

Backup: Grassmann extended phase space

Poisson/Dirac Brakets

$$\{A,B\}_{\mathrm{D}} = A\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\vec{\partial}}{\partial p^\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\vec{\partial}}{\partial x^\mu} + \frac{1}{2}\Big[\frac{\overleftarrow{\partial}}{\partial \psi^\mu}\frac{\vec{\partial}}{\partial p^\mu_\psi} + \frac{\overleftarrow{\partial}}{\partial p^\mu_\psi}\frac{\vec{\partial}}{\partial \psi^\mu}\Big] + \frac{1}{2}\left[\frac{\overleftarrow{\partial}}{\partial \psi_5}\frac{\vec{\partial}}{\partial p_5} + \frac{\overleftarrow{\partial}}{\partial p_5}\frac{\vec{\partial}}{\partial \psi_5}\right]\Big)B\,. \tag{6}$$

The Dirac brackets between any two elements of the extended phase space are given by

$$\{x^\mu, p^\nu\} = \{x^\mu, P^\nu\} = g^{\mu\nu}, \tag{7}$$
$$\{P^\mu, P^\nu\} = F^{\mu\nu}\,, \tag{8}$$
$$\{P^\alpha, F^{\mu\nu}\} = -\partial^\alpha F^{\mu\nu}\,, \tag{9}$$
$$\{p_{\psi,\mu}, \psi_\nu\} = \frac{g_{\mu\nu}}{2}\,, \tag{10}$$
$$\{p_5, \psi_5\} = \frac{1}{2}\,, \tag{11}$$

while all other brackets vanish. Further, using Eq.(4) and Eq.(5), we obtain

$$\{\psi_\mu, \psi_\nu\} = -ig_{\mu\nu}\,, \tag{12}$$
$$\{\psi_5, \psi_5\} = -i\,. \tag{13}$$

The conjugate variables for x^μ are

$$p^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = P^\mu + A^\mu\,,$$

where

$$P^\mu \equiv \frac{\dot{x}^\mu}{\varepsilon} - \frac{i}{4\varepsilon}\dot{x}_\mu\left(\psi^\mu + \frac{i\epsilon^{\mu\nu\alpha\beta}}{3}\psi_\nu\psi_\alpha\beta\right)\chi_+ + \frac{i}{4\varepsilon}\dot{x}_\mu\left(\psi^\mu - \frac{i\epsilon^{\mu\nu\alpha\beta}}{3}\psi_\nu\psi_\alpha\beta\right)\chi_-\,.$$

Backup: Chiral phase space

Weyl equation $$\frac{1}{2}(\gamma \cdot p)(1 \pm \gamma^5)\Psi = 0\,.$$

Weyl Hamiltonian $$H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_-\,,$$

$$c_\pm \equiv \tfrac{1}{2}\left(\pm P_\mu\psi^\mu + \tfrac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right).$$

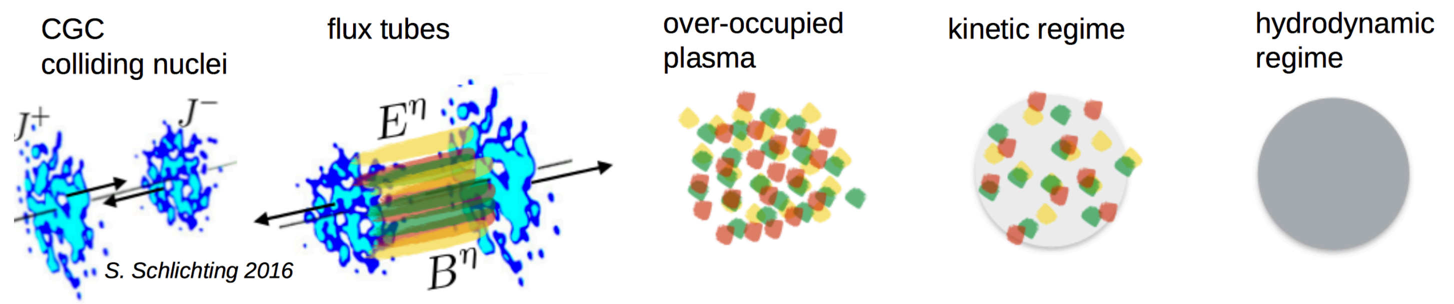
Phase space measure $$d^4\psi = (-i/(\sqrt{2})^4)d\psi^3 d\psi^2 d\psi^1 d\psi^0$$

$$\varepsilon\tilde{f}_\pm = 2i(\pm P_\mu\psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta)\epsilon^{ijk}\psi^i\psi^j\psi^k\,. \tag{22}$$

The above expression can be quantized by identifying $\psi^\mu \to \gamma^5\gamma^\mu/\sqrt{2}$. This gives

$$\varepsilon\tilde{f}_\pm \to \rho_\pm = \frac{1}{2}(\gamma \cdot P)(1 \pm \gamma^5)\gamma^0\,, \tag{23}$$

Back-up: Overview and Literature
Pre-equilibrium dynamics of the CME

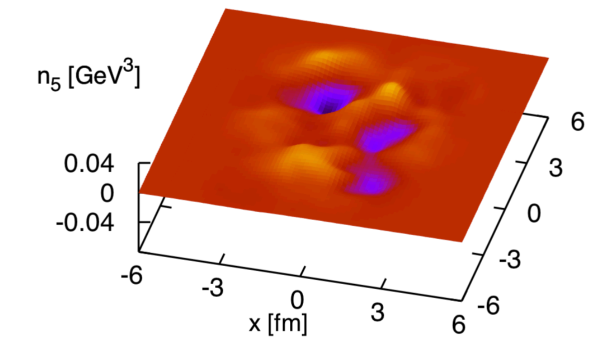


Large Magnetic Fields

Fermion production

'Anomalous transport'

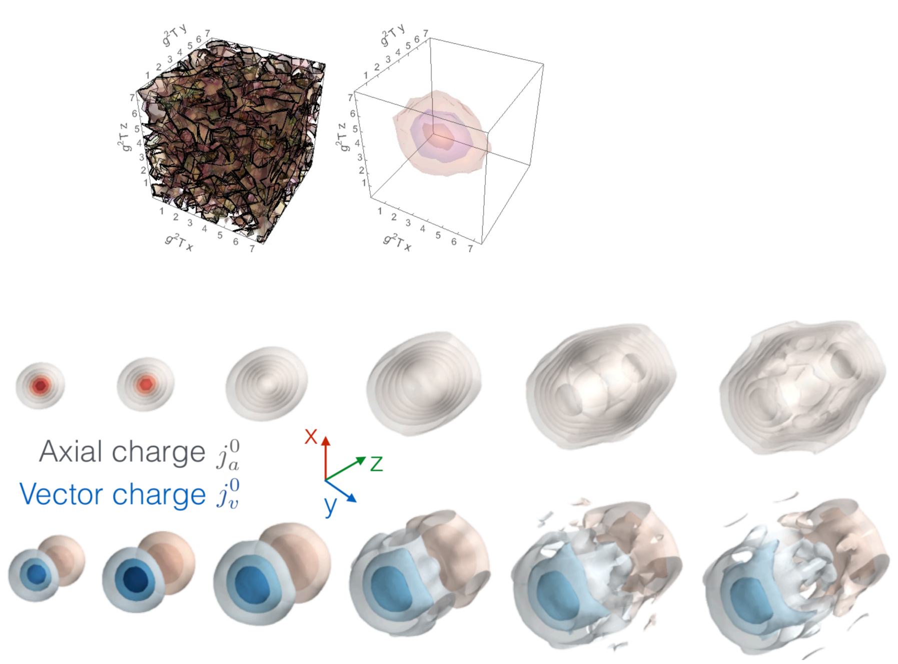
Anomalous Hydro



Son, Surowka 2009;
Nair, Ray, Roy 2012
Hongo, Hirnono, Hirano 2013,
Hirono, Hirano, Kharzeev 2014,
Florkowski, Friman, Jaiswal, Speranza 2017

See talk by W. Florkowski

Real-time lattice simulations



M. Mace, S. Schlichting, R. Venugopalan, PRD 93, 074036
M. Mace, NM, S. Schlichting, S. Sharma: PRL 117, 061601, PRD 95, 036023, NPA 967, 752
N. Tanji, NM, J. Berges: PRD 93, 074507
N. Tanji, J. Berges, PRD 97, 034013

See talk by N. Tanji

Chiral Kinetic Theory

?

Wigner functions, Gao, Pu, Wang, Huang, Shi, Jiang, Liao, Zhuang, Yang, …

Berry phases, Son & Yamamoto, Stephanov & Yi, Stone, Manuel, Torres-Rincon, Hidaka, Pu, Yang, Sun, Ko, Li, ….

See talks in this session