Kirill Tuchin

based on work done with Yang Li arXiv:1708.08536

QCD Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions
Florence, March 21, 2018
**MOTIVATION & OUTLINE**

**Motivation:**
Magnetic monopoles at $T=0$: dual superconductor, color confinement.

The condensate may not melt away at $T_c$  
→ Important part of QGP dynamics

**Outline**
Chiral evolution without magnetic monopoles  
Chiral evolution with magnetic monopoles (an Abelian model)
**QED WITH CHIRAL ANOMALY WITHOUT MONOPOLES**

Maxwell-Chern-Simons theory:

\[
\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \\
\nabla \cdot \mathbf{E} = \rho - c_A \nabla \theta \cdot \mathbf{B}, \quad \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \mathbf{j} + c_A (\partial_t \theta \mathbf{B} + \nabla \theta \times \mathbf{E})
\]

\[c_A = N_c \sum_f q_f^2 e^2 / 2\pi^2\]

Pseudo-scalar field \(\theta\) describes coupling of QED to the topological vacuum fluctuations of the nuclear matter (e.g. QGP).

Assumption: \(\theta\) is spatially homogeneous (single topological sector)

Anomalous current \(\mathbf{j} = \sigma_\chi \mathbf{B}\) \(\mathbf{j}, \mathbf{B}\) are T-odd \(\Rightarrow \sigma_\chi\) is T-even \(\Rightarrow\) non-dissipative

\[\sigma_\chi = c_A \dot{\theta} = c_A \mu_5 \quad \text{chiral conductivity}\] induced by the chiral anomaly of QED
HELIACY CONSERVATION WITHOUT MONOPOLES

Chiral anomaly \[ \partial_\mu j^\mu_A = c_A E \cdot B \]

Magnetic helicity: \[ \mathcal{H}_{em} = \int A \cdot B \, d^3x \]

Integrate over time ➩ helicity conservation

\[ \frac{2V}{c_A} \langle n_A \rangle + \mathcal{H}_{em} = \mathcal{H}_{tot} \]

Equation of state (hot medium) \[ \langle n_A \rangle = \chi \mu_5 \]

\[ \frac{\sigma_\chi(t)}{\alpha} = 1 - \frac{\mathcal{H}_{em}(t)}{\mathcal{H}_{tot}} \]

Helicity flows between the magnetic field and medium. Total helicity is conserved.

Fraction of the total helicity in the plasma

Fraction of the total helicity in the field
ADIABATIC SOLUTION WITHOUT MONPOLES

Radiation gauge: \[ \nabla \cdot A = 0, \ A^0 = 0 \quad -\nabla^2 A = -\partial_t^2 A + j + \sigma_\chi(t) \nabla \times A \]

Expand \[ A = \sum_{k,\lambda} [a_{k\lambda}(t) W_{k'\lambda'}(x) + a^*_{k'\lambda'}(t) W^*_{k'\lambda'}(x)] \]

where \[ W_{k\lambda}(x) = \frac{\epsilon^\lambda}{\sqrt{2kV}} e^{ik\cdot x} \quad \nabla \times W_{k\lambda}(x) = \lambda k W_{k\lambda}(x) \]

Adiabatic approximation \[ a_{k\lambda} = a_{k\lambda}(0) e^{-i \int_0^t \omega_{k\lambda}(t') dt'} \]

\[ \omega_{k\lambda}(t) = \left[ -\frac{i\sigma}{2} + \lambda_1 \sqrt{k^2 - \sigma_\chi \lambda k - \sigma^2/4} \right] \quad \lambda_1 = \pm 1 \]

When \( k < \sigma_\chi \quad Im \ \omega > 0 \quad \Rightarrow \) instability \[ \text{inverse cascade} \]
THE FASTEST GROWING STATE (FGS) MODEL

The fastest growing state has the largest imaginary $\omega_{k,\lambda}(t)$ which happens when

$$k_0 = \frac{\sigma \chi \lambda}{2}$$

for which

$$\omega_0(t) = -\frac{i\sigma}{2} + \frac{i}{2} \sqrt{\sigma^2 + \sigma^2_\chi(t)}$$

The fastest growing state is then given by

$$a_0(t) = e^{\gamma(t)/2} \quad \gamma(t) = \int_0^t \left[ \sqrt{\sigma^2 + \sigma^2_\chi(t')} - \sigma \right] dt'$$

Within this model the helicity conservation equation:

$$\dot{\sigma}_\chi = -\left( \sqrt{\sigma^2 + \sigma^2_\chi} - \sigma \right) (\alpha - \sigma_\chi)$$

can be solved analytically ⇒ chiral evolution
Since $0^\ddagger(t)$ is monotonically decreasing from its initial value $0^\ddagger(0)=0$, the magnetic field has maximum only if $\sigma_{\lambda}(0)/\alpha > 1/2$, i.e. if most of the initial helicity is in the medium. Otherwise, $B(t)$ is a monotonically decreasing function of time (despite the fact that $A$ always grows). This is shown in Fig. 2.

Magnetic field has a peak (instability) only if $\sigma_{\lambda}(0)/\alpha > 1/2$
VALIDITY OF ADIABATIC APPROXIMATION

Works very well!
\[ \nabla \cdot B = 0, \quad -\nabla \times E = \partial_t B + j_m \\
\nabla \cdot E = 0, \quad \nabla \times B = \partial_t E + j_e + \sigma_\chi B, \]

Assume linear response: \[ j_e = \sigma_e E, \quad j_m = \sigma_m B, \]

\[ -\nabla^2 B + \partial^2_t B = -(\sigma_e + \sigma_m)\partial_t B - \sigma_e\sigma_m B + \sigma_\chi(t)\nabla \times B \]

Define “vector potential” \[ E = -\partial_t A - \sigma_m A \quad B = \nabla \times A. \quad \nabla \cdot A = 0 \]

Chiral anomaly equation \[ \partial_t \sigma_\chi = c^2_A/(\chi V) \int E \cdot B \, d^3x, \]

\[ \beta^{-1} \partial_t \sigma_\chi = -\partial_t \mathcal{H}_{em} - 2\sigma_m \mathcal{H}_{em} \]

⇒ Magnetic current dissipates magnetic helicity, but conserves energy because is \( \sigma_m \) is T-even
SUPERCONDUCTING PHASE

\[ E = -\partial_t A - \sigma_m A \] reminiscent of the London theory of superconductivity

Note, however, is that \( A \) does not transform under the gauge transform as a vector potential has to.

Define the “normal” and “super” fields

\[ E_n = -\partial_t A, \quad E_s = -\sigma_m A. \]

and currents

\[ j_n = \sigma_e E_n, \quad j_s = \sigma_e E_s = -\sigma_e \sigma_m A. \]

charge conservation

\[ \nabla \cdot j_n = \nabla \cdot j_s = 0. \]

London equations:

\[ \nabla \times j_s = -\sigma_e \sigma_m B, \quad \partial_t j_s = +\sigma_e \sigma_m E_n, \]

\[ \Rightarrow \text{when } E_n=0, j_s \text{ is finite} \]
MEISSNER EFFECT

Stationary limit: \[ \nabla^2 B = \sigma_e \sigma_m B - \sigma_\chi \nabla \times B, \]
\[ \nabla^2 E_s = \sigma_e \sigma_m E_s - \sigma_\chi \nabla \times E_s \]

Spectrum:
\[ k = \frac{\lambda \sigma_\chi}{2} \pm \sqrt{\frac{1}{4} \sigma_\chi^2 - \sigma_e \sigma_m}. \]

Meissner effect appears if \[ \sigma_\chi^2 < 4 \sigma_e \sigma_m \]

London penetration depth \[ L = (\sigma_e \sigma_m - \sigma_\chi^2/4)^{-1/2} \]

Same is true of electric field. Moreover, one can introduce \( A \) such that \( E = \text{curl} \ A \). Everything is dual!

FIG. 1. Meissner effect in a chiral medium.
ADIABATIC SOLUTION OF MCS WITH MONOPOLES

\[-\nabla^2 B + \partial_t \partial_t B = -(\sigma_e + \sigma_m) \partial_t B - \sigma_e \sigma_m B + \sigma_\chi(t) \nabla \times B\]

Dispersion: \[\omega_{k\lambda}(t) = -\frac{i(\sigma_e + \sigma_m)}{2} + \lambda_1 \frac{i}{2} \sqrt{(\sigma_e + \sigma_m)^2 + 4(\sigma_\chi \lambda k - \sigma_e \sigma_m - k^2)} \]

\[\lambda_1 = \pm 1\]

Instability (exponential increase of \(H\)) occurs for \(\text{Im} \ \omega > 0\):

\[\frac{\sigma_\chi}{2} - \frac{1}{2} \sqrt{\sigma_\chi^2 - 4\sigma_e \sigma_m} \leq k \leq \frac{\sigma_\chi}{2} + \frac{1}{2} \sqrt{\sigma_\chi^2 - 4\sigma_e \sigma_m}.\]

\(\Rightarrow\) only possible if \(\sigma_\chi^2 > 4\sigma_e \sigma_m.\)

\(\Rightarrow\) chiral instability is incompatible with the Meissner effect

BTW, the Fastest Growing State model captures the essential qualitative features.
FASTEST GROWING STATE WITH MONOPOLES

Use FGS model to illustrate the general features the chiral evolution.

\[ \omega_{k\lambda}(t) = -\frac{i(\sigma_e + \sigma_m)}{2} + \lambda \frac{i}{2} \sqrt{(\sigma_e + \sigma_m)^2 + 4(\sigma_{\chi} \lambda k - \sigma_e \sigma_m - k^2)} \]

The growth rate is largest for \( k_* = \frac{\sigma_{\chi} \lambda}{2} \).

Amplitude \( a_*(t) = a_*(0) e^{\frac{1}{2} \gamma(t)} \)

\[ \gamma(t) = \int_0^t \left[ \sqrt{(\sigma_e - \sigma_m)^2 + \sigma_{\chi}^2(t') - (\sigma_e + \sigma_m)} \right] dt' \]

Magnetic helicity \( \mathcal{H}_{em}(t) = f \mathcal{H}_{tot}(0) e^{\gamma(t)} \) \( f \) is the initial fraction of the helicity stored in \( B \)

One can derive a closed equation governing time-evolution of the chiral conductivity

\[ \dot{\sigma}_{\chi} = - [f + F(1 - f) - F(\sigma_{\chi})] \left( \sqrt{(\sigma_e - \sigma_m)^2 + \sigma_{\chi}^2} - \sigma_e + \sigma_m \right) \]

\[ F(\sigma_{\chi}) = \frac{1}{\sigma_{\chi}} \left\{ \sigma_{\chi}^2 + 2\sigma_m \left[ \sqrt{\sigma_{\chi}^2 + (\sigma_e - \sigma_m)^2} + \sigma_e - \sigma_m \right] - 2\sigma_{\chi} \sigma_m \ln \left[ \sigma_{\chi} + \sqrt{\sigma_{\chi}^2 + (\sigma_e - \sigma_m)^2} \right] \right\} , \]
EXAMPLES OF EVOLUTION WITH MONOPOLES

(a) \( f = 0.5, \sigma_e = 10\sigma_m \)

(c) \( \sigma_e = 1\alpha, \sigma_m = 0.1\alpha \)

Notice that \( \sigma_{\chi} \) does not vanish at \( t \to \infty \) unlike no-monopole case. This is because part of magnetic helicity leaks out and evolution stops when the system runs out of the magnetic helicity, but the medium helicity is finite.
**FINAL VALUES OF CHIRAL CONDUCTIVITY**

At \( t \to \infty \) \( \sigma_\chi \) approaches a constant that satisfies

\[
\sigma^2_\chi(t \to \infty) < 4\sigma_e \sigma_m
\]

🤔 Final state is a stable and exhibits the Meisner effect!

Stability = exponential decay of small magnetic helicity fluctuations
When Magnetic Helicity Is Stable

Using FGS one can derive a more stringent stability condition. Instability of magnetic helicity = period of growth → stability condition is \( \dot{\mathcal{H}}_{em} < 0 \).

\[
\dot{\mathcal{H}}_{em} = \mathcal{H}_{tot}(0) \left[ f + F(1 - f) - F(\sigma_\chi) \right].
\]

\[
\rightarrow F'(\sigma_\chi) \dot{\sigma}_\chi > 0.
\]

\[
\dot{\sigma}_\chi = - \left[ f + F(1 - f) - F(\sigma_\chi) \right] \left( \sqrt{(\sigma_e - \sigma_m)^2 + \sigma_\chi^2} - \sigma_e + \sigma_m \right) \rightarrow \dot{\sigma}_\chi < 0
\]

\[
\rightarrow F'(\sigma_\chi) < 0
\]

\[
F(\sigma_\chi) = \frac{1}{\sigma_\chi} \left\{ \sigma_\chi^2 + 2\sigma_m \left[ \sqrt{\sigma_\chi^2 + (\sigma_e - \sigma_m)^2} + \sigma_e - \sigma_m \right] - 2\sigma_\chi \sigma_m \ln \left[ \sigma_\chi + \sqrt{\sigma_\chi^2 + (\sigma_e - \sigma_m)^2} \right] \right\},
\]

\[
\rightarrow \sigma_\chi^2 \leq 4\sigma_e \sigma_m
\]

In particular,

\[
\sigma_\chi^2(0) \leq 4\sigma_e \sigma_m.
\]
SUMMARY

1. Focus of this talk: dual QED with linear response.

2. There is a superconducting phase; Meissner effect for both $E$ and $B$.

3. Magnetic current dissipates helicity but conserves energy.

4. Chiral evolution ends up always in a superconducting state satisfying $\sigma^2_\chi < 4\sigma_e\sigma_m$.

5. A superconducting state is always stable: a small fluctuation of magnetic helicity decays exponentially with time.

6. Superfluid component may explain very low viscosity of QGP.

Close CMP analogue: spin ice, but has quadratic response.

🤔 Interesting: what is the role of the Dirac quantization $eg=N/2$?