The Field Theoretic Perspectives of the Wigner Function for the CME

Defu Hou
Central China Normal University, Wuhan

Wu, Hou, Ren, PRD 96 (2017)096015;
Feng, Hou, Ren, Liu, Wu PRD95 (2017)
Outline

I. Introduction

II. Subtlety of the Wigner function used for CME

III. CME on lattice

IV. Conclusion and outlook
1. Introduction: Anomalous Transports

(Fukushima-Kharzeev-Warringa, Son-Zhitnitsky, Vilenkin)

Micro-quantum anomaly + $B/\Omega \rightarrow$ macro-transport (CME/CVE)

Search in HIC

Nature Phys.12 (2016)
Science350 (2015) 413

• **Experimental situation in HIC**  
  (See H.Z.Huang, G.Wang, W.Li ‘s talks)

  1. **Off central collisions generate inhomogeneous & transient** \( B \)
  2. **axial charge produced via topological fluct. plus mass effect**
  3. **Beyond thermal equilibrium**

• **Theoretical investigations:**

  1. **Field theory**  
     Fukushima, Kharzeev, Landsteiner, Hou, Liu, Ren et al

  2. **Holography**  
     Yee, Rebhan, Landsteiner, Shu Lin et al.,  
     (see Landsteiner’s talk)

  3. **Hydrodynamics & Kinetic theory**

     Son, Stephanov, Yin, Gao, Pu, Wang, Shi, Liao, Zhuang

     (See talks by Zhuang, Wang, Liao, Pu)
II. Subtlety of the Wigner function used for CME

- Wigner function formulation
  Vasak, Gyulassy, Elze (1987), Heinz, Zhuang (96)

The Wigner function (WF) that links various hydrodynamic quantities of the system to the Green's function (see Zhuang's talk)

\textbf{CME} & \textbf{CVE} with a constant axial chemical potential was obtained

Gao, Liang, Pu, qWang, xnWang (2012)

inhomogeneous and transient axial chemical potential

Wu, Hou, Ren, 2017
The UV problem of the Wigner function
(without regularization)

○ Wigner function without regularization:

\[ W_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} < \overline{\psi}_\beta(x_+)U(x_+, x_-)\psi_\alpha(x_-) > \]

▲ gauge link: \[ U(x_+, x_-) = e^{i\epsilon \int_{x_-}^{x_+} d\xi A_\mu(\xi)} \]

▲ \[ x_\pm = x \pm \frac{y}{2} \]

▲ ensemble average: \[ < \Lambda > \]

○ The electric current:

\[ J_\mu(x) = ie \int \frac{d^4p}{(2\pi)^4} tr W(x, p) \gamma_\mu \]
○ The electric current:

\[
J_\mu (x) = ie \int d^4 y \delta^4(y) U(x_+, x_-) < \overline{\psi}(x_+) \gamma_\mu \psi(x_-) > \\
= \lim_{y \to 0} J_\mu (x, y)
\]

\[
J_\mu (x, y) = ie U(x_+, x_-) < \overline{\psi}(x_+) \gamma_\mu \psi(x_-) >
\]

○ The Lagrangian density:

\[
L = -\overline{\psi} \gamma_\mu (\partial_\mu - ieA_\mu - i\gamma_5A_{5\mu}) \psi
\]

▲ external electromagnetic field: \( A_\mu \)

▲ axial vector field: \( A_{5\mu} = (A_5, -i\mu_5) \)

▲ massless Dirac field

▲ closed time path Green function formation

\[
J_\mu (x) = e^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} e^{i(q_1+q_2) \cdot x} \Lambda_{\mu\rho\lambda}(q_1, q_2) A_\rho (q_1) A_{5\lambda} (q_2)
\]
* Expansion to the linear order in $A_{\mu}$ and $A_{5\mu}$

CTP full propagator:

$$S_{ab}(x_-, x_+) = S_{ab}(x_-, x_+) - \sum_c \int d^4 z S_{ac}(x_\mu - z) \gamma^c_{\rho 5} S_{cb}(z - x_+) A_{5\rho}(z)$$

$$- e \sum_c \int d^4 z S_{ac}(x_\mu - z) \gamma^c_{\rho} S_{cb}(z - x_+) A_{\rho}(z)$$

$$+ e \sum_{c,d} \int d^4 z_1 \int d^4 z_2 S_{ad}(x_\mu - z_2) \gamma^d_{\lambda 5} S_{dc}(z_2 - z_1) \gamma^c_{\rho} S_{ca}(z_1 - x_+) A_{\rho}(z_1) A_{5\lambda}(z_2)$$

$$+ e \sum_{c,d} \int d^4 z_1 \int d^4 z_2 S_{ac}(x_\mu - z_2) \gamma^c_{\rho} S_{cd}(z_2 - z_1) \gamma^d_{\lambda 5} S_{da}(z_1 - x_+) A_{\rho}(z_2) A_{5\lambda}(z_1)$$

with

$$\gamma^1_{\mu} = \gamma_{\mu}, \quad \gamma^2_{\mu} = -\gamma_{\mu}, \quad \gamma^1_{\mu 5} = \gamma_{\mu} \gamma_{5}, \quad \gamma^2_{\mu 5} = -\gamma_{\mu} \gamma_{5}$$

gauge link:

$$U(x_-, x_+) = 1 + ie \int_{x_-}^{x_+} d\xi \xi A_{\nu}(\xi) + O(A^2)$$
Two issues:

1. non-conserved electric current:

\[
\frac{\partial}{\partial x_\mu} J_\mu(x, y) = \frac{i}{8\pi^2} [\varepsilon_{\mu\rho\beta\lambda} F_{\mu\rho}(x) F_{5\beta\lambda}(x) \nonumber \\
+ 2\varepsilon_{\mu\rho\alpha\beta} \left( \frac{y_\alpha y_\beta}{y^2} \right) F_{\mu\rho}(x) \frac{\partial}{\partial x_\beta} A_{5\lambda}(x) ]
\]

\[
y \to 0
\]

\[\nabla \quad F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x_\mu} - \frac{\partial A_\mu(x)}{\partial x_\nu} \]

\[\nabla \quad F_{5\mu\nu}(x) = \frac{\partial A_{5\nu}(x)}{\partial x_\mu} - \frac{\partial A_{5\mu}(x)}{\partial x_\nu} \]

\[\nabla \quad \text{average the direction of } y
\]

\[
\frac{\partial}{\partial x_\mu} J_\mu(x) = \frac{\partial}{\partial x_\mu} J_\mu(x, 0) = \frac{3i}{32\pi^2} \varepsilon_{\mu\rho\beta\lambda} F_{\mu\rho}(x) F_{5\beta\lambda}(x)
\]
② inconsistent electric current:
(not being a functional derivative)

\[ \frac{\delta J_\mu(x, y)}{\delta A_\nu(x')} - \frac{\delta J_\nu(x, y)}{\delta A_\mu(x)} = \frac{i}{8\pi^2} \left( \varepsilon_{\mu\rho\beta\lambda} + \varepsilon_{\mu\rho\alpha\beta} \frac{y_\alpha y_\beta}{y^2} \right) \frac{\partial A_{5\lambda}(x)}{\partial x_\beta} \delta^4(x - x') \]

\[ y \to 0 \quad \text{\large ▲ average the direction of } y \]

\[ \frac{\delta J_\mu(x, y)}{\delta A_\nu(x')} - \frac{\delta J_\nu(x, y)}{\delta A_\mu(x)} = \frac{3i}{16\pi^2} \varepsilon_{\mu\rho\beta\lambda} F_{5\beta\lambda}(x) \delta^4(x - x') \]
Applying to a general chiral case, the present form of the Wigner function formulation needs to be revised!
The regularized Wigner function
(with Pauli-Villars regularization)

○ The electric current:

\[ J_\mu(x) = ie \int \frac{d^4 p}{(2\pi)^4} \text{tr}[W(x, p) + \sum_s C_s W(x, p | M_s)] \gamma_\mu \]

\[ = ie \lim_{y \to 0} U(x_+, x_-)[\langle \overline{\psi}(x_+) \gamma_\mu \psi(x_-) \rangle + \sum_s C_s \langle \overline{\psi}_s(x_+) \gamma_\mu \psi_s(x_-) \rangle] \]

▲ regulator mass \( M_s \to \infty, \quad \sum_s C_s = 1 \)
▲ regulator field \( \psi_s(x) \)
▲ regulator renders limit \( y \to 0 \) well defined

\[ J_\mu(x) = -ie \text{tr} \gamma_\mu \langle \overline{\psi}(x)\psi(x) \rangle - ie \sum_s C_s \text{tr} \gamma_\mu \langle \overline{\psi}_s(x)\psi_s(x) \rangle \]

The nonconservation and the inconsistency is solved
The electric current with PV regularized Wigner F.:

\[ J_{\mu}(x) = e^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} e^{i(q_1+q_2) \cdot x} \Lambda_{\mu \rho \lambda}^{\text{Reg}}(q_1, q_2) A_{\rho}(q_1) A_{5\lambda}(q_2) \]

\[ \Delta \text{ external electromagnetic field: } A_\rho(q_1) \]
\[ \Delta \text{ axial vector field: } A_{5\lambda}(q_2) = (A_5, -i\mu_5) \]

\[ \mu_5(q_2) \rightarrow \text{const.} \quad J_{\mu}(x) = ? \]

It depends on how the limit \( q_2 \rightarrow 0 \) is taken.
A homogeneous axial chemical potential:

\[
\lim_{q_{20} \to 0} \lim_{\vec{q}_2 \to 0} \Lambda_{ij4}(q_1, q_2) = -\frac{1}{2\pi^2} \epsilon_{ikj} q_{1k}
\]

\[
J = \frac{e^2}{2\pi^2} \mu_5 B
\]

A static axial chemical potential:

\[
\lim_{\vec{q}_2 \to 0} \lim_{q_{20} \to 0} \Lambda_{ij4}(q_1, q_2) = \frac{2f(0) - 1}{2\pi^2} \epsilon_{ikj} q_{1k} + O(q_1^2)
\]

The calculation is not relied on the explicit form of the distribution function.

CME current canceled at thermal equilibrium.
Phenom. implications of the limit subtleties

○ A stochastic equation:

\[
\left( \frac{\partial}{\partial t} - D \nabla^2 + \frac{1}{\tau} \right) n_5 = g(x)
\]

▲ \( < g(x) > = 0 \)

▲ \( < g(x)g(y) > = \kappa^2 \delta^4 (x - y) \)

◇ axial charge density: \( n_5 (k) = \frac{g(k)}{-ik_0 + Dk^2 + \frac{1}{\tau}} \)

◇ axial vector potential: \( A_{5\mu} (k) = -i \delta_{\mu 4} \mu_5 (k) = -i \delta_{\mu 4} \frac{n_5 (k)}{\chi (k)} \)

▲ axial susceptibility: \( \chi (k) \)
\[ A_{5\mu}(k) = -i\delta_{\mu\lambda}\mu_5(k) = -i\delta_{\mu\lambda}\frac{n_5(k)}{\chi(k)} \]

\[ J_\mu(x) = e^2\int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} e^{i(q_1+q_2)\cdot x} \Lambda^{\text{Reg}}_{\mu\rho\lambda}(q_1, q_2) A_{\rho}(q_1) A_{5\lambda}(q_2) \]

\[ \langle J(x) \rangle = 0 \]

\[ \langle J_i(x)J_j(y) \rangle \quad \text{dominated by diffusion pole} \]

\[ -iq_{20} + Dq_2^2 + \frac{1}{\tau} = 0 \]
Two limits

\[
\frac{|q_{20}|}{|\vec{q}_2|} = D|\vec{q}_2| + \frac{1}{|\vec{q}_2|\tau} \geq \sqrt{\frac{D}{\tau}}.
\]

▲ \(D \gg \tau\)

homog. \(\mu_5\) is a good approx. and classic form of CME current emerges ---Noneq. Phenom.

▲ \(D \ll \tau\) towards equilibrium, \(\tau \to \infty\),

\[
\frac{|q_{20}|}{|\vec{q}_2|} = D|\vec{q}_2| \sim \frac{D}{|\vec{x} - \vec{y}|} \to 0
\]

\[
\lim_{\vec{q}_2 \to 0} \lim_{q_{20} \to 0}
\]

and CME current disappears
III. CME on Lattice

using lattice QCD with Wilson term

\[ I = - \sum_x \sum_{\mu} \frac{1}{2a} \left[ \bar{\psi}(x) \left( \frac{1}{i} \gamma_{\mu} - r \right) U_{\mu}(x) \psi(x + a_{\mu}) \right. \]

\[ - \bar{\psi}(x + a_{\mu}) \left( \frac{1}{i} \gamma_{\mu} + r \right) U_{\mu}^{\dagger}(x) \psi(x) \]

\[ - \sum_x M \bar{\psi}(x) \psi(x) + \cdots \]

Yamamoto, PRL (2011)

Karsten and Smit (1981)
One-loop self-energy on lattice of size $N_s^3 \times N_t$

$J_i(p) = -\Pi_{ij}(p)A_j(p)$

$\Pi_{ij}^{(1)}(p) = \mathcal{I} \sum_k \epsilon_{ikj}p_k + O(a)$

CME vanishes at continu. limit.

At zero temperature

$\Pi_{ij}(q) = \Lambda_{ij4}(q)$

$= -\lim_{\rho \to 0} \frac{1}{q_4} \sum_{\rho} \frac{2}{a} \sin \frac{1}{2} a(Q_1 + Q_2)_{\rho} \Lambda_{ij\rho}(Q_1, Q_2)$

$\Pi_{ij}(q) = \frac{e^2}{2\pi^2} \sum_k \epsilon_{ijk} q_k$

Numerical calculations

<table>
<thead>
<tr>
<th>Lattice size</th>
<th>$\mathcal{I}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_s = 6, N_t = 4$</td>
<td>$1.347 \times 10^{-2}$</td>
</tr>
<tr>
<td>$N_s = 12, N_t = 4$</td>
<td>$2.439 \times 10^{-4}$</td>
</tr>
<tr>
<td>$N_s = 20, N_t = 4$</td>
<td>$8.886 \times 10^{-7}$</td>
</tr>
<tr>
<td>$N_s = 50, N_t = 8$</td>
<td>$4.512 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Analytical calculations (In the limit $N_s \rightarrow \infty$)

$$
\mathcal{I} = 12 \frac{1}{N_t} \sum_{l_4} \int \frac{d^3 l}{(2\pi)^3} \frac{\mathcal{N}(l)}{\left[\sin^2 l + \mathcal{M}^2(l)\right]^3} = 0
$$
IV. Concluding Remarks

▲ Naive Wigner function cannot be applicable to the case with non-constant $\mu_5$. The problem stems from axial anomaly. The PV regulated WF leads consistent results.

▲ the CME current depends on the limit order of momentum of $\mu_5$

\[
\lim_{q_{20} \to 0} \lim_{q_{2} \to 0} \frac{j}{q_{20}q_{2} \to 0} \quad J = \frac{e^2}{2\pi^2 \mu_5 B} \]

\[
\lim_{q_{2} \to 0} \lim_{q_{20} \to 0} \quad J = \frac{e^2}{2\pi^2 \mu_5 B} \]

Mutliplied by $2f(0) - 1$

vanishes in equilibrium

▲ We examine the issues raised here with lattice formulation; we obtained the same results as that in continuous case with QFT and Wigner function method.

▲ It is useful nonstatic response functions related to chiral anomaly (see M. Hovath’s poster)
Any higher order contribution to CME?

\[
\langle \partial_\mu J^{\mu 5} \rangle = \langle F_{\mu \nu} \tilde{F}^{\mu \nu} \rangle \frac{e_0^2}{8\pi^2} \left(1 - \frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2}\right)
\]

higher order contribution to CME?

Feng, Hou, Ren, in preparation
Thank you very much for your attention!