The Field Theoretic Perspectives of the Wigner Function for the CME

Defu Hou

Central China Normal University, Wuhan

Wu, Hou, Ren, PRD 96 (2017)096015;

Feng, Hou, Ren, Liu, Wu PRD95 (2017)

4th Workshop on Chirality, Vorticity and Magnetic Field in HIC, GGI, Florence, 2018

Outline

I. Introduction

II. Subtlety of the Wigner function used for CME

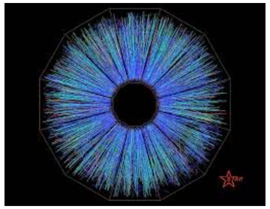
III CME on lattice

IV. Conclusion and outlook

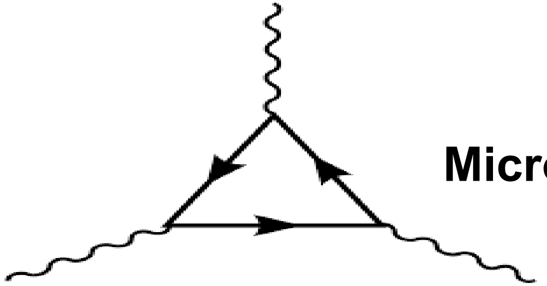
I. Introduction : Anomalous Transports

(Fukushima-Kharzeev-Warringa, Son-Zhitnitsky, Vilenkin)

Micro-quantum anomaly + B/Ω → macro-transport (CME/CVE)



Search in HIC



Micro

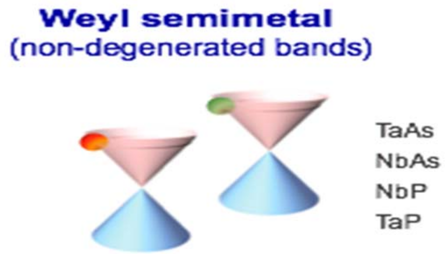
B ↓ Ω

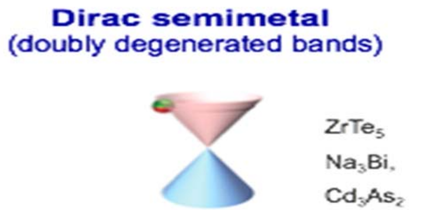
$$\vec{J} = \sigma_5 \mu_5 \vec{B}$$

Macro

↓

Astrophysics, cosmology





Nature Phys.12（2016）

Phys. Rev. X.5（2015）

Science350（2015）413

X.Huang， Rep.prog. Phys. (2016)
Kharzeev,Liao,Voloshin,Wang: Prog.Part.Nucl.Phys.2016

- **Experimental situation in HIC** (See H.Z.Huang, G.Wang, W.Li 's talks)

  ① **Off central collisions generate inhomogeneous & transient** $\vec{B}$

  ② **axial charge produced via topological fluct. plus mass effect**

  ③ **Beyond thermal equilibrium**

- **Theoretical investigations:**

  ① **Field theory** Fukushima , Kharzeev, Landsteiner , Hou , Liu , Ren et al

  ② **Holography** Yee. Rebhan , Landsteiner, Shu Lin et. al., (see Landsteiner's talk)

  ③ **Hydrodynamics & Kinetic theory**

  Son , Stephanov, Yin, Gao, Pu, Wang , Shi, Liao , Zhuang

  (See talks by Zhuang, Wang, Liao , Pu)

II. Subtlety of the Wigner function used for CME

- **Wigner function formulation**

Vasak, Gyulassy , Elze (1987), Heinz , Zhuang (96)

The WF that links various hydrodynamic quantities of the system to the Green′s function **(see Zhuang's talk)**

CME& CVE with a constant axial chemical potential was obtained

Gao, Liang, Pu, qWang, xnWang (2012)

inhomogeneous and transient axial chemical potential ?

Wu,Hou, Ren, 2017

The UV problem of the Wigner function

(without regularization)

○ Wigner function without regularization:

$$W_{\alpha\beta} = \int \frac{d^4 y}{(2\pi)^4} e^{-ip\cdot y} < \overline{\psi}_\beta(x_+) U(x_+, x_-) \psi_\alpha(x_-) >$$

▲ gauge link: $U(x_+, x_-) = e^{ie\int_{x_-}^{x_+} d\xi_\mu A_\mu(\xi)}$

▲ $x_\pm = x \pm \frac{y}{2}$

▲ ensemble average: $<\Lambda>$

○ The electric current:

$$J_\mu(x) = ie\int \frac{d^4 p}{(2\pi)^4} \mathbf{tr} W(x,p)\gamma_\mu$$

○ The electric current:

$$J_\mu(x) = ie\int d^4y\,\delta^4(y)U(x_+,x_-) < \overline{\psi}(x_+)\gamma_\mu\psi(x_-) >$$

$$= \lim_{y\to 0} J_\mu(x,y)$$

$$J_\mu(x,y) = ieU(x_+,x_-) < \overline{\psi}(x_+)\gamma_\mu\psi(x_-) >$$

○ The Lagrangian density:

$$L = -\overline{\psi}\gamma_\mu(\partial_\mu - ieA_\mu - i\gamma_5 A_{5\mu})\psi$$

- ▲ external electromagnetic field: A_μ
- ▲ axial vector field: $A_{5\mu} = (\mathbf{A}_5, -i\mu_5)$
- ▲ massless Dirac field
- ▲ closed time path Green function formation

$$J_\mu(x) = e^2\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\, e^{i(q_1+q_2)\cdot x}\Lambda_{\mu\rho\lambda}(q_1,q_2)A_\rho(q_1)A_{5\lambda}(q_2)$$

* Expansion to the linear order in A_μ and $A_{5\mu}$

CTP full propagator:

free propagator

$$S_{ab}(x_-,x_+)$$

$$= S_{ab}(x_-,x_+) - \sum_c \int d^4z S_{ac}(x_- - z)\gamma^c_{\rho 5} S_{cb}(z - x_+) A_{5\rho}(z)$$

$$- e\sum_c \int d^4z S_{ac}(x_- - z)\gamma^c_\rho S_{cb}(z - x_+) A_\rho(z)$$

$$+ e\sum_{cd} \int d^4z_1 \int d^4z_2 S_{ad}(x_- - z_2)\gamma^d_{\lambda 5} S_{dc}(z_2 - z_1)\gamma^c_\rho S_{ca}(z_1 - x_+) A_\rho(z_1) A_{5\lambda}(z_2)$$

$$+ e\sum_{cd} \int d^4z_1 \int d^4z_2 S_{ac}(x_- - z_2)\gamma^c_\rho S_{cd}(z_2 - z_1)\gamma^d_{\lambda 5} S_{da}(z_1 - x_+) A_\rho(z_2) A_{5\lambda}(z_1)$$

with $\gamma^1_\mu = \gamma_\mu, \quad \gamma^2_\mu = -\gamma_\mu, \quad \gamma^1_{\mu 5} = \gamma_\mu\gamma_5, \quad \gamma^2_{\mu 5} = -\gamma_\mu\gamma_5$

gauge link:

$$U(x_-,x_+) = 1 + ie\int_{x_-}^{x_+} d\xi_\upsilon A_\upsilon(\xi) + O(A^2)$$

Two issues:

① non-conserved electric current:

$$\frac{\partial}{\partial x_\mu} J_\mu(x,y) = \frac{i}{8\pi^2}[\varepsilon_{\mu\rho\beta\lambda} F_{\mu\rho}(x) F_{5\beta\lambda}(x)$$

$$+2\varepsilon_{\mu\rho\alpha\beta} \boxed{\frac{y_\alpha y_\beta}{y^2}} F_{\mu\rho}(x) \frac{\partial}{\partial x_\beta} A_{5\lambda}(x)]$$

$y \to 0$

▲ $F_{\mu\upsilon}(x) = \frac{\partial A_\upsilon(x)}{\partial x_\mu} - \frac{\partial A_\mu(x)}{\partial x_\upsilon}$

▲ $F_{5\mu\upsilon}(x) = \frac{\partial A_{5\upsilon}(x)}{\partial x_\mu} - \frac{\partial A_{5\mu}(x)}{\partial x_\upsilon}$

▲ average the direction of y

$$\frac{\partial}{\partial x_\mu} J_\mu(x) = \frac{\partial}{\partial x_\mu} J_\mu(x,0) = \frac{3i}{32\pi^2} \varepsilon_{\mu\rho\beta\lambda} F_{\mu\rho}(x) F_{5\beta\lambda}(x)$$

② inconsistent electric current:

(not being a functional derivative)

$$\frac{\delta J_\mu(x,y)}{\delta A_\upsilon(x')} - \frac{\delta J_\upsilon(x,y)}{\delta A_\mu(x)}$$

$$= \frac{i}{8\pi^2}\left(\varepsilon_{\mu\rho\beta\lambda} + \varepsilon_{\mu\rho\alpha\beta}\boxed{\frac{y_\alpha y_\beta}{y^2}}\right)\frac{\partial A_{5\lambda}(x)}{\partial x_\beta}\delta^4(x-x')$$

$y \to 0$ ⬇ ▲ average the direction of y

$$\frac{\delta J_\mu(x,y)}{\delta A_\upsilon(x')} - \frac{\delta J_\upsilon(x,y)}{\delta A_\mu(x)} = \frac{3i}{16\pi^2}\varepsilon_{\mu\rho\beta\lambda}F_{5\beta\lambda}(x)\delta^4(x-x')$$

Applying to a general chiral case, the present form of the Wigner function formulation needs to be revised!

The regularized Wigner function

(with Pauli-Villars regularization)

○ The electric current:

$$J_\mu(x) = ie\int \frac{d^4p}{(2\pi)^4}\,\mathbf{tr}[W(x,p) + \sum_s C_s W(x,p\,|\,M_s)]\,\gamma_\mu$$

$$= ie\,\lim_{y\to 0} U(x_+,x_-)[<\overline{\psi}(x_+)\gamma_\mu\psi(x_-)> + \sum_s C_s <\overline{\psi}_s(x_+)\gamma_\mu\psi_s(x_-)>]$$

- ▲ regulator mass $M_s \to \infty,\quad \sum_s C_S = 1$
- ▲ regulator field $\psi_s(x)$
- ▲ regulator renders limit $y \to 0$ well defined

$$J_\mu(x) = -ie\,\mathbf{tr}\gamma_\mu <\overline{\psi}(x)\psi(x)> - ie\sum_s C_s \mathbf{tr}\gamma_\mu <\overline{\psi}_s(x)\psi_s(x)>$$

The nonconservation and the inconsistency is solved

○ The electric current with PV regularized Wigner F. :

$$J_\mu(x) = e^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} e^{i(q_1+q_2)\cdot x} \Lambda^{\mathbf{Reg}}_{\mu\rho\lambda}(q_1, q_2) A_\rho(q_1) A_{5\lambda}(q_2)$$

▲ external electromagnetic field: $A_\rho(q_1)$

▲ axial vector field: $A_{5\lambda}(q_2) = (\mathbf{A}_5, -i\mu_5)$

$$\mu_5(q_2) \to const. \qquad J_\mu(x) = ?$$

It depends on how the limit $q_2 \to 0$ is taken.

A homogeneous axial chemical potential :

$$\lim_{q_{20}\to 0}\lim_{\vec{q}_2\to 0}\Lambda_{ij4}(q_1,q_2) = -\frac{1}{2\pi^2}\epsilon_{ikj}q_{1k}$$

$$\mathbf{J} = \frac{e^2}{2\pi^2}\mu_5\mathbf{B}$$

Wu,Hou, Ren, 2017

A static axial chemical potential :

$$\lim_{\vec{q}_2\to 0}\lim_{q_{20}\to 0}\Lambda_{ij4}(q_1,q_2) = \frac{2f(0)-1}{2\pi^2}\epsilon_{ikj}q_{1k}+O(q_1^2)$$

The calculation is not relied on the explicit form of the distribution function

CME current canceled at thermal equilibrium.

Phenom. implications of the limit subtleties

○ A stochastic equation:

$$\left(\frac{\partial}{\partial t} - D\nabla^2 + \frac{1}{\tau}\right) n_5 = g(x)$$

▲ $< g(x) >= 0$

▲ $< g(x) g(y) >= \kappa^2 \delta^4 (x-y)$

◇ axial charge density: $n_5(k) = \dfrac{g(k)}{-ik_0 + Dk^2 + \frac{1}{\tau}}$

◇ axial vector potential: $A_{5\mu}(k) = -i\delta_{\mu 4}\mu_5(k) = -i\delta_{\mu 4}\dfrac{n_5(k)}{\chi(k)}$

▲ axial susceptibility: $\chi(k)$

◇ axial vector potential: $A_{5\mu}(k) = -i\delta_{\mu 4}\mu_5(k) = -i\delta_{\mu 4}\dfrac{n_5(k)}{\chi(k)}$

$k \rightarrow q_2$

◇ electric current:

$$J_\mu(x) = e^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \; e^{i(q_1+q_2)\cdot x} \Lambda^{\mathbf{Reg}}_{\mu\rho\lambda}(q_1, q_2) A_\rho(q_1) A_{5\lambda}(q_2)$$

◇ average current: $<\vec{J}(x)> = 0$

◇ observation: $<J_i(x) J_j(y)>$

dominatied by diffusion pole →

$$-iq_{20} + D\vec{q}_2^{\,2} + \frac{1}{\tau} = 0$$

Two limits

$$\frac{|q_{20}|}{|\vec{q}_2|} = D|\vec{q}_2| + \frac{1}{|\vec{q}_2|\tau} \geq \sqrt{\frac{D}{\tau}}.$$

▲ $D >> \tau$ ⟶ $\lim_{q_{20}\to 0} \lim_{\vec{q}_2 \to 0}$

homog. \mu_5 is a good approx. and classic form of CME current emerges ---Noneq. Phenom.

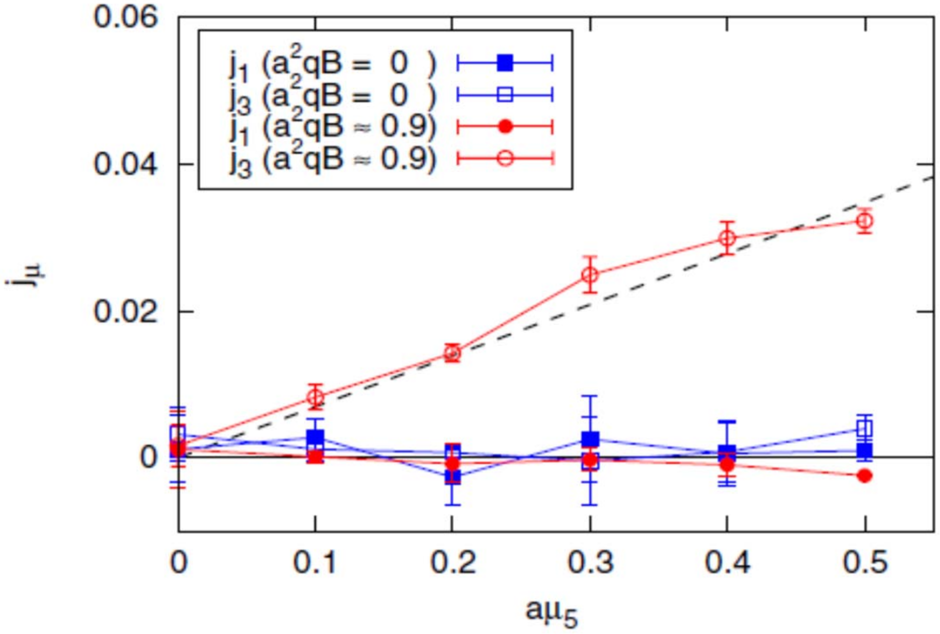
▲ $D << \tau$ towards equilibrium, $\tau \to \infty$,

$$\frac{|q_{20}|}{|\vec{q}_2|} = D|\vec{q}_2| \sim \frac{D}{|\vec{x}-\vec{y}|} \to 0$$

⟶ $\lim_{\vec{q}_2\to 0} \lim_{q_{20} \to 0}$

and CME current disappears

III. CME on Lattice



Yamamoto,PRL(2011)

using lattice QCD with Wilson term

$$I = -\sum_x \sum_\mu \frac{1}{2a} \left[\bar{\psi}(x) \left(\frac{1}{i}\gamma_\mu - r \right) U_\mu(x)\psi(x + a_\mu) \right. \\ \left. - \bar{\psi}(x + a_\mu) \left(\frac{1}{i}\gamma_\mu + r \right) U_\mu^\dagger(x)\psi(x) \right] \\ - \sum_x M\bar{\psi}(x)\psi(x) + \cdots$$

Karsten and Smit (1981)

$$J_i(p) = -\Pi_{ij}(p)A_j(p)$$

One-loop self-energy on lattice of size $N_s^3 \times N_t$

$$\Pi_{ij}^{(1)}(p) = \mathcal{I}\sum_k \epsilon_{ikj} p_k + \mathcal{O}(a)$$

CME vanishes at continu. limit .

At zero temperature

$$\Pi_{ij}(q) \equiv \Lambda_{ij4}(q)$$
$$= -\lim_{q_4 \to 0} \frac{1}{q_4} \sum_\rho \frac{2}{a} \sin \frac{1}{2} a(Q_1 + Q_2)_\rho \Lambda_{ij\rho}(Q_1, Q_2)$$

$$\Pi_{ij}(q) = \frac{e^2}{2\pi^2} \sum_k \epsilon_{ijk} q_k$$

Feng,Hou, Liu, Ren, Wu, PRD95,(2017)

umerical calculations

Lattice size	$\mathcal{I}$
$N_s = 6, N_t = 4$	1.347×10^{-2}
$N_s = 12, N_t = 4$	2.439×10^{-4}
$N_s = 20, N_t = 4$	8.886×10^{-7}
$N_s = 50, N_t = 8$	4.512×10^{-9}

nalytical calculations(In the limit $N_s \to \infty$)

$$\mathcal{I} = 12 \frac{1}{N_t} \sum_{l_4} \int \frac{d^3\mathbf{l}}{(2\pi)^3} \frac{\mathcal{N}(l)}{\left[\sin^2 l + \mathcal{M}^2(l)\right]^3} = 0$$

IV. Concluding Remarks

▲ Naive Wigner function can not be applicable to the case with non-constant μ_5 The problem stems from axial anomaly .
The PV regulated WF leads consistent results

▲ the CME current depends on the limit order of momentum of μ_5

$$\lim_{q_{20}\to 0}\lim_{\vec{q}_2\to 0} \qquad \mathbf{J} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B}$$

$$\lim_{\vec{q}_2\to 0}\lim_{q_{20}\to 0} \qquad \mathbf{J} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B}$$

Mutiplied by $2f(0)-1$

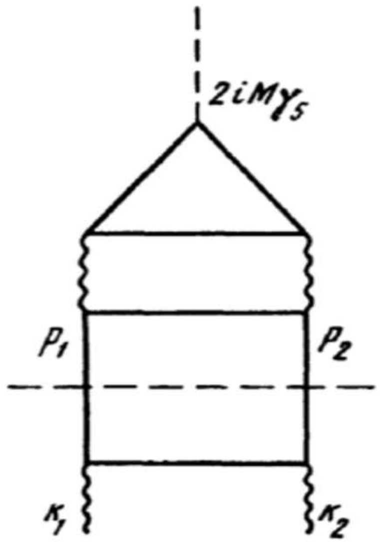
vanishes in equilibrium

▲ We examine the issues raised here with lattice formulation we obtained the same results as that in continuous case with QFT and Wigner function method

▲ It is useful nonstatic response functions related to chiral anomaly (see M. Hovath's poster)

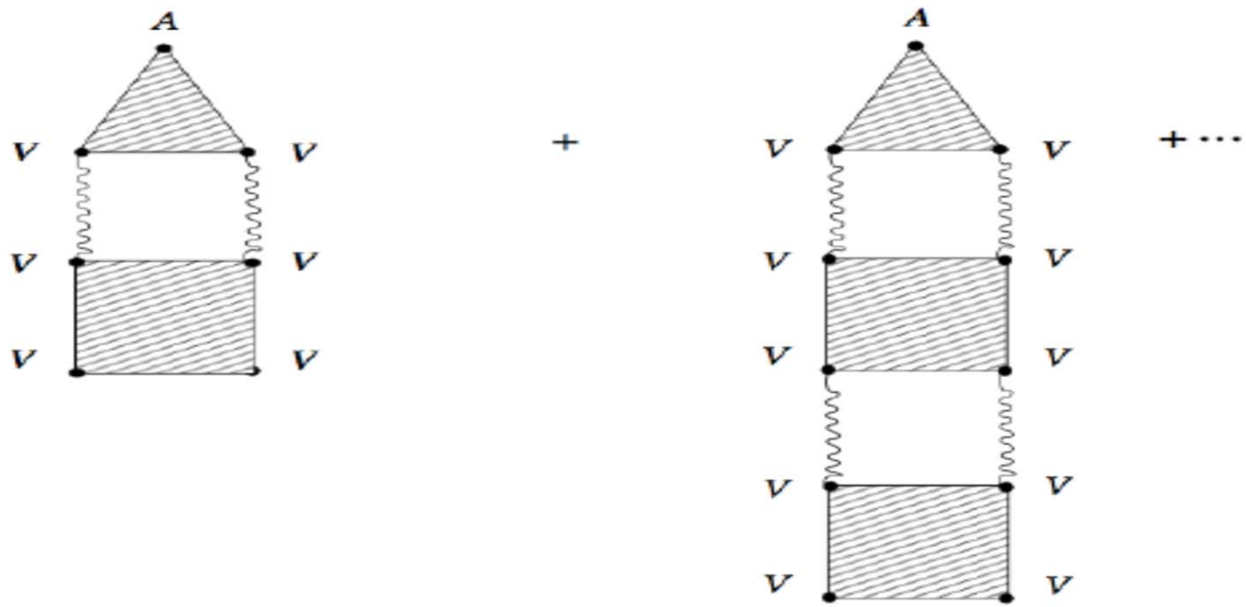
Any higher order contribution to CME?

photon scattering diagram(three-loop)



$$\langle \partial_\mu J^{\mu 5} \rangle = \langle F_{\mu\nu} \tilde{F}^{\mu\nu} \rangle \frac{e_0^2}{8\pi^2} \left(1 - \frac{3 e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} \right)$$

A.Ansel'm and A.Iogensen Sov. Phys. JETP ('1989); S.Adler arXiv:0405040 ('2004) higher order contribution to CME ?



Feng, Hou , Ren , in preparation

Thank you very much for your attention!