Chiral kinetic theory in noninertial frame

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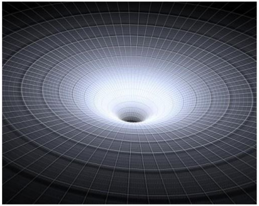
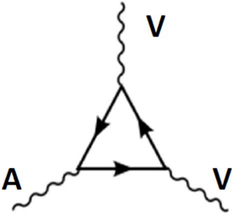
Y.C. Liu, L. Gao, K. Mameda, XGH, in preparation





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Chiral kinetic theory in curved spacetime

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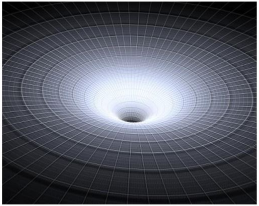

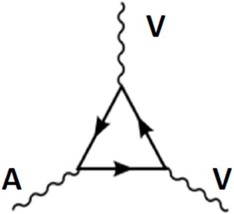

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Contents

Introduction and motivations

Chiral magnetic and vortical effects

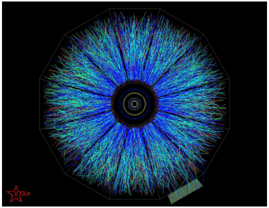
- Chiral fermions + magnetic field ⇒ chiral magnetic effect (CME) **(Kharzeev, Mclerran, Warringa, Fukushima 2008; Son, Zhitnitsky 2004; $\cdots$)**:

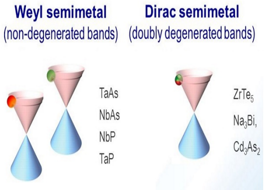
$$\vec{J}_R = \frac{1}{4\pi^2}\mu_R \vec{B}, \qquad \vec{J}_L = -\frac{1}{4\pi^2}\mu_L \vec{B}$$

- Chiral fermions + fluid vorticity ⇒ chiral vortical effect (CVE) **(Erdmenger etal 2008; Barnerjee etal 2008, Son, Surowka 2009; Landsteiner etal 2011)**:

$$\vec{J}_R = \frac{1}{4\pi^2}\mu_R^2 \vec{\omega} + \frac{T^2}{12}\vec{\omega}, \qquad \vec{J}_L = -\frac{1}{4\pi^2}\mu_L^2 \vec{\omega} - \frac{T^2}{12}\vec{\omega}$$

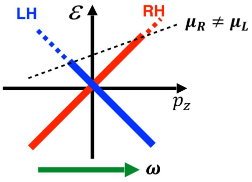
- Phenomenology: Heavy-ion collisions, Weyl/Dirac semi-metals, optically active materials, supernova, cold atoms, $\cdots$.





Coriolis force and CVE

- CVE is less understood than CME.
- CVE due to Coriolis force: The "lowest Landau level" problem



In magnetic field, Lorentz force:
$\boldsymbol{F} = e(\dot{\boldsymbol{x}} \times \boldsymbol{B})$

In rotating frame, Coriolis force:
$\boldsymbol{F} = 2\boldsymbol{\varepsilon}(\dot{\boldsymbol{x}} \times \boldsymbol{\omega}) + \boldsymbol{O}(\boldsymbol{\omega}^2)$

Larmor theorem: $e\boldsymbol{B} \sim 2\boldsymbol{\varepsilon}\boldsymbol{\omega}$

- This suggests the following replacement $\vec{B} \to 2\varepsilon\vec{\omega}$: **(Stephanov, Yin 2012)**

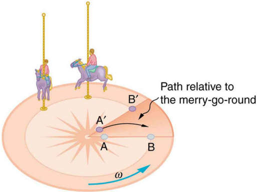
$$\vec{J}_{\rm CME} = \int \frac{d^3\vec{p}}{(2\pi)^3} e\vec{B}(\vec{\Omega}_B \cdot \hat{\vec{p}}) f \quad \Rightarrow \quad \vec{J}_{\rm CVE} = \int \frac{d^3\vec{p}}{(2\pi)^3} 2\varepsilon\vec{\omega}(\vec{\Omega}_B \cdot \hat{\vec{p}}) f$$

where Ω_B is the Berry curvature. This ($+$ antiparticle) indeed gives expected result.

- Can such a replacement be derived in chiral kinetic theory?

Rotating frame vs. rotating fluid

- CVE due to flow vorticity in inertial frame. **(Son, Surowka 2009; Gao, Pu, Wang, Wang 2012; Chen, Stephanov, Son, Yin 2014)**
- Is rotating frame equivalent to flow vorticity in generating CVE?



- Helpful if we formulate a chiral kinetic theory allowing both rotating-frame effect and flow vorticity
- Mysterious T^2 term in CVE. Is it due to gravitational anomaly?
- A kinetic theory in curved spacetime would be helpful.

Chiral kinetic theory in electromagnetic (EM) field

- The kinetic description of chiral fermions
  Son, Yamamoto 2012;
  Stephanov, Yin 2012;
  Gao, Liang, Pu, Wang, Wang 2012; Chen, Pu, Wang, Wang 2013
  Hidaka, Pu, Yang 2016;
  Muller, Venogopalan 2017;
  Huang, Shi, Jiang, Liao, Zhuang 2018;

  ......
- Based on adiabatic expansion or $\hbar$ expansion.
- Chiral anomaly is encoded.
- Valid for weak magnetic field.
- Friendly in application. Quark-gluon plasma, Weyl/Dirac semimetals, electroweak gases, $\cdots$

Semiclassical equations of motion

Heisenberg equation of motion

- Consider a Weyl fermion in a rotating frame:

$$H = \vec{p}\cdot\vec{\sigma} - \vec{\omega}\cdot\left(\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma}\right)$$

where $\vec{p} = -i\hbar\vec{\nabla}$.

- The Heisenberg equations:

$$\begin{aligned}
\dot{\vec{x}} &= \frac{1}{i\hbar}\left[\vec{x}, H\right] = \vec{\sigma} + \vec{x}\times\vec{\omega}, \\
\dot{\vec{p}} &= \frac{1}{i\hbar}\left[\vec{p}, H\right] = \vec{p}\times\vec{\omega}, \\
\dot{\vec{\sigma}} &= \frac{1}{i\hbar}\left[\vec{\sigma}, H\right] = \frac{2}{\hbar}\vec{p}\times\vec{\sigma} - \vec{\omega}\times\vec{\sigma}
\end{aligned}$$

- The EOM of for spin can be solved order by order in $\hbar$:

$$\vec{\sigma} = \hat{\vec{p}} + \hbar\frac{1}{2p}\left[\hat{\vec{p}}\times(\hat{\vec{p}}\times\vec{\omega}) - \hat{\vec{p}}\times\dot{\hat{\vec{p}}}\right] + O(\hbar^2)$$

with $\hat{\vec{p}} = \vec{p}/|\vec{p}|$. (We only show the particle branch.)

Semiclassical equation of motion

- Substitute this solution to the first two EOMs:

$$\dot{\vec{x}} = \hat{\vec{p}} + \vec{x}\times\vec{\omega} = \frac{\partial\varepsilon}{\partial\vec{p}} + \hbar\vec{p}\times\vec{\Omega}_B,$$
$$\dot{\vec{p}} = \vec{p}\times\vec{\omega} = -\frac{\partial\varepsilon}{\partial\vec{x}}$$

  where $\vec{\Omega}_B = \hat{\vec{p}}/(2p^2)$ is the Berry curvature and the single-particle energy $\varepsilon = p - \vec{\omega}\cdot(\vec{x}\times\vec{p}) - \frac{\hbar}{2}\vec{\omega}\cdot\hat{\vec{p}}$.
- The Coriolis force and centrifugal force:

$$\ddot{\vec{x}} = -2\vec{\omega}\times\dot{\vec{x}} - \vec{\omega}\times(\vec{\omega}\times\vec{x})$$

- Surprisingly the EOMs do not have $\hbar$ order correction. But the single-particle energy does.
- This suggests a kinetic equation up to $O(\hbar)$:

$$\partial_t f + \dot{\vec{x}}\cdot\vec{\nabla}_x f + \dot{\vec{p}}\cdot\vec{\nabla}_p f = C[f]$$

  with phase space measure 1.
- Similar procedure for EM field case gives the CKT in EM field as derived before. **(Son, Yamamoto 2012, Hidaka, Pu, Yang 2017, Huang, Shi, Jiang, Liao, Zhuang 2018)**

The dynamics of Wigner function

Wigner operator in curved spacetime

- Instead of Weyl fermions, we consider massless Dirac fermions so that the formalism can be potentially extended to massive case.
- Our Wigner operator is:

$$\hat{W}_{\alpha\beta}(x,p) = \int d^4y[-g(x)]^{1/2} e^{-ip\cdot y/\hbar} \left[\bar{\psi}(x) e^{y\cdot \overleftarrow{D}/2}\right]_\beta \left[e^{-y\cdot D/2}\psi(x)\right]_\alpha$$

  - The position of the spacetime manifold is x^μ, the position in the tangent space of x is y^μ, p_μ is in the cotangent space of x. The whole phase space is the cotangent bundle.
  - The derivative D_μ in the tangent bundle is lifted from the usual covariant derivative ∇_μ in the spacetime manifold:

$$D_\mu = \nabla_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda}$$

  - The derivative D_μ in the cotangent bundle is lifted from the usual covariant derivative ∇_μ in the spacetime manifold:

$$D_\mu = \nabla_\mu + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu}$$

  - The advantage of using D_μ, y^ν and p_ν are parallelly transported:

$$D_\mu y^\nu = 0 = D_\mu p_\nu$$

Wigner function and its dynamics

- Wigner function:

$$W(x,p) = \langle \hat{W}(x,p) \rangle$$

- Consider noninteracting fermions. Dirac equations:

$$i\hbar\gamma\cdot\nabla\psi(x) = i\hbar\bar{\psi}(x)\overleftarrow{\nabla}\cdot\gamma = 0$$

- Multiply D_μ to Wigner function and apply Dirac equations:

$$\begin{aligned}\gamma^\mu\left(\frac{i\hbar}{2}D_\mu + p_\mu\right)W(x,p) &= -i\gamma^\mu\hbar\hat{H}_\mu\left(x,-\frac{1}{2}i\hbar\partial_p\right)\otimes W(x,p) \\ &+ \frac{i\hbar}{2}\gamma^\mu\left[\hat{G}_\mu\left(x,-\frac{1}{2}i\hbar\partial_p\right),W(x,p)\right]_\otimes\end{aligned}$$

  - This equation is exact.
  - The tensor product is understood similarly as how we define $\hat{W}$.
  - The operators $\hat{G}$ and $\hat{H}$:

$$\begin{aligned}\hat{H}_\mu(x,y) &= \sum_{n=1}^{\infty}\frac{1}{n!}\overbrace{[y\cdot D,\cdots,[y\cdot D}^{n},D_\mu]\cdots], \\ \hat{G}_\mu(x,y) &= \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\underbrace{[y\cdot D,\cdots,[y\cdot D}_{n},D_\mu]\cdots]\end{aligned}$$

$\hbar$ expansion

- Up to $O(\hbar^2)$ order, here $\Delta_\mu = D_\mu - F_{\mu\lambda}\partial_p^\lambda$:

$$\gamma^\mu \left[p_\mu + \frac{i\hbar}{2}\Delta_\mu - \frac{\hbar^2}{8}R^\rho{}_{\sigma\lambda\mu}p_\rho\partial_p^\lambda\partial_p^\sigma + \frac{i\hbar^2}{8}R_{\lambda\mu ab}\sigma^{ab}\partial_p^\lambda + \frac{\hbar^2}{12}\left(\partial_\alpha F_{\beta\mu}\right)\partial_p^\alpha\partial_p^\beta\right] W(x,p)$$
$$= \frac{i\hbar^2}{32}\gamma^\mu R_{\lambda\mu ab}[\sigma^{ab}, \partial_p^\lambda W(x,p)]$$

- Dirac decomposition:

$$W(x,p) = \frac{1}{4}\left[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right]$$

- Focus on $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ or equivalently $\mathcal{R}_\mu/\mathcal{L}_\mu = (1/2)(\mathcal{V}_\mu \pm \mathcal{A}_\mu)$:

$$\begin{aligned} p_\mu\mathcal{R}^\mu &= O(\hbar^2), \\ p^{[\alpha}\mathcal{R}^{\beta]} + \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\Delta_\mu\mathcal{R}_\nu &= O(\hbar^2), \\ \Delta_\mu\mathcal{R}^\mu - \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}R_{\lambda\mu\alpha\beta}\partial_p^\lambda\mathcal{R}_\nu &= O(\hbar^2) \end{aligned}$$

  - $\mathcal{L}_\mu$ satisfies same equations but with opposite sign for $\hbar$ coefficients.
  - The first two equations fix $\mathcal{R}_\mu$ up to a scalar function (distribution function).
  - The third equation is kinetic equation for distribution function.

The kinetic equation

- The solutions to the first two equations:

$$\mathcal{R}_\mu = p_\mu f\delta(p^2) + \hbar\tilde{F}_{\mu\nu}p^\nu f\delta'(p^2) + \frac{\hbar}{2p\cdot n}\epsilon_{\mu\nu\rho\sigma}n^\nu p^\sigma \Delta^\rho f\delta(p^2) + O(\hbar^2)$$

  - n^μ is a frame choosing vector.
  - $f = f_0 + f_1 + O(\hbar^2)$ is the distribution function.

- The kinetic equation for f:

$$\delta\Big(p^2 - \frac{\hbar}{p\cdot n}\tilde{G}_{\alpha\beta}p^\alpha n^\beta\Big)\Big\{p\cdot\Delta + \frac{\hbar\epsilon_{\mu\rho\nu\sigma}n^\nu p^\sigma}{2(p\cdot n)^2}\left[D^\mu(p\cdot n) - G^{\mu\lambda}n_\lambda\right]\Delta^\rho$$
$$+\frac{\hbar\epsilon_{\mu\nu\rho\sigma}}{2p\cdot n}\left[p^\sigma(D^\mu n^\nu)\Delta^\rho - p^\sigma n^\nu(D^\mu G^{\rho\lambda})\partial^p_\lambda\right]\Big\}f = O(\hbar^2)$$

  where

$$\Delta_\mu = \partial_\mu + \Gamma^\lambda{}_{\mu\nu}p_\lambda\partial^\nu_p - G^{\rho\lambda}\partial^p_\lambda, \quad G_{\mu\nu} = F_{\mu\nu} - \frac{\hbar}{4}\epsilon_{\mu\lambda\alpha\beta}R_\nu{}^{\lambda\alpha\beta}$$

  - This equation is invariant under general coordinate transformation.
  - In Minkowski spacetime, $\Gamma^\lambda{}_{\mu\nu} = 0 = R_\nu{}^{\lambda\alpha\beta}$, it reduces to the kinetic equation in electromagnetic field. **(Huang, Shi, Jiang, Liao, Zhuang 2018)**
  - Let us focus on case of $F_{\mu\nu} = 0$. We consider the rotating frame.

Rotating frame

- An inertial frame with a Minkowski coordinate:

$$ds^2 = dt^2 - d\vec{x}^2$$

- A frame rotating with constant $\vec{\omega}$ w.r.t. the above inertial frame:

$$ds^2 = [1 - (\vec{\omega} \times \vec{x})^2]dt^2 - 2(\vec{\omega} \times \vec{x}) \cdot d\vec{x}dt - d\vec{x}^2$$

- Some key quantities for rotating frame:
  - Metric:

$$g_{00} = 1 - (\vec{\omega} \times \vec{x})^2, \quad g_{0i} = g_{i0} = -(\vec{\omega} \times \vec{x})^i, \quad g_{ij} = -\delta_{ij}$$

  - Inverse metric:

$$g^{00} = 1, \quad g^{0i} = g^{i0} = -(\vec{\omega} \times \vec{x})^i, \quad g^{ij} = -\delta^{ij} + (\vec{\omega} \times \vec{x})^i (\vec{\omega} \times \vec{x})^j$$

  - Nonzero components of Christoffel connection:

$$\Gamma^i{}_{00} = [\vec{\omega} \times (\vec{\omega} \times \vec{x})]^i, \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\epsilon^{ijk}\omega^k$$

  - Nonzero components of spin connection:

$$\Gamma_0 = -i\frac{\vec{\sigma}}{2} \cdot \vec{\omega}$$

  - Riemannian curvature:

$$R^\mu{}_{\nu\alpha\beta} = 0$$

$O(\hbar^0)$-order kinetic equation

- The classical kinetic equation:

$$\delta(p^2)p^\mu D_\mu f_0 = 0$$

- Integrating out "energy", we can write it in 3-dimensional form.
- Energy is associated with timelike Killing vectors:
  $\kappa_1 = \delta_0^\mu \partial_\mu = \partial_0$: time translation for a co-rotating observer
  $\kappa_2 = \partial_0 - (\vec{\omega} \times \vec{x})^i \partial_i$ for $|\vec{\omega} \times \vec{x}| < 1$:time translation for inertial observer
- Killing vector $\kappa_1 \Rightarrow \varepsilon_p = p_0 = |\vec{p}| - (\vec{\omega} \times \vec{x}) \cdot \vec{p}$ for particle, where $\vec{p} = -(p_1, p_2, p_3)$.
- Kinetic equation in 3-dimensional form:

$$\left[\partial_0 + \vec{v}_p \cdot \vec{\nabla}_x + \vec{p} \times \vec{\omega} \cdot \vec{\nabla}_p\right] f_0(t, \vec{x}, \vec{p}) = 0$$

  where $\vec{v}_p = \partial\varepsilon_p / \partial\vec{p} = \hat{\vec{p}} - \vec{\omega} \times \vec{x}$
- Single-particle equations of motion:

$$\dot{\vec{x}} = \hat{\vec{p}} - \vec{\omega} \times \vec{x} = \frac{\partial\varepsilon_p}{\partial\vec{p}},$$

$$\dot{\vec{p}} = \vec{p} \times \vec{\omega} = -\frac{\partial\varepsilon_p}{\partial\vec{x}}$$

  which is textbook result.

$O(\hbar)$-order kinetic equation

- The kinetic equation up to $O(\hbar)$:

$$\delta(p^2) D_\mu \left(p^\mu + \hbar s^{\mu\nu} D_\nu\right) f = 0$$

where $s_{\mu\nu} = -(1/2p\cdot n)\epsilon_{\mu\nu\rho\sigma} n^\rho p^\sigma$ is spin.
- The current:

$$J^\mu_R = \int \frac{d^4p}{(2\pi)^4} \delta(p^2) \left(p^\mu + \hbar s^{\mu\nu} D_\nu\right) f$$

where the second term represents the side-jump. **(Chen etal 2014)**
- Consider equilibrium state. $f = f(g)$ with g a linear combination of collisional conserved quantities.
- At $O(\hbar^0)$ order; classical:
  - Conserved quantities are particle number, momenta; angular momentum automatically conserved:

$$g = \alpha_0(x) + \beta_0^\mu(x) p_\mu$$

  - Substitute to kinetic equation:

$$\nabla_\mu \alpha_0 = 0, \quad \nabla_\mu \beta^0_\nu + \nabla_\nu \beta^0_\mu = \phi_0(x) g_{\mu\nu}$$

  - The functional form of $f(g)$ needs classical collision term to fix.

$O(\hbar)$-order kinetic equation

- Up to $O(\hbar)$ order:
  - Spin should be included in angular momentum:

$$g = \alpha(x) + \beta^\mu(x) p_\mu + \gamma^{\mu\nu}(x) s_{\mu\nu}$$
$$\alpha = \alpha_0 + \alpha_1 + O(\hbar^2), \quad \beta^\mu = \beta_0^\mu + \beta_1^\mu + O(\hbar^2), \quad \gamma^{\mu\nu} = O(\hbar)$$

  - Substitute to kinetic equation, the $\hbar$-order quantities:

$$\nabla_\mu \alpha_1 = 0, \quad \nabla_\mu \beta_{1\nu} + \nabla_\nu \beta_{1\mu} = \phi_1(x) g_{\mu\nu}$$
$$n^\mu = \beta_0^\mu, \quad \gamma_{\mu\nu} = \nabla_{[\mu}\beta_{0\nu]}, \quad \nabla_\mu \nabla_{[\nu}\beta_{0\lambda]} = 0$$

- Consider the co-rotating observer with a co-rotating fluid:
  - Killing vector and frame-choosing vector:

$$\kappa_1^\mu = \delta_0^\mu, n^\mu = \delta_0^\mu$$

  - Integrate out the energy $p \cdot \kappa_1$:

$$\partial_t f + \dot{\vec{x}} \cdot \vec{\nabla}_x f + \dot{\vec{p}} \cdot \vec{\nabla}_p f = 0$$
$$f = f(|\vec{p}| - (\vec{\omega} \times \vec{x}) \cdot \vec{p} - \hbar \vec{\omega} \cdot \hat{\vec{p}}/2)$$

    This coincides our warm-up calculation.
  - The CVE current:

$$\vec{J}_R = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\hat{\vec{p}} + \hbar |\vec{p}| \vec{\Omega}_B \times \vec{\nabla}_x \right) f$$

$O(\hbar)$-order kinetic equation

- Up to $O(\hbar)$ order:
  - Spin should be included in angular momentum:

$$g = \alpha(x) + \beta^{\mu}(x)p_{\mu} + \gamma^{\mu\nu}(x)s_{\mu\nu}$$
$$\alpha = \alpha_0 + \alpha_1 + O(\hbar^2), \quad \beta^{\mu} = \beta_0^{\mu} + \beta_1^{\mu} + O(\hbar^2), \quad \gamma^{\mu\nu} = O(\hbar)$$

  - Substitute to kinetic equation, the $\hbar$-order quantities:

$$\nabla_{\mu}\alpha_1 = 0, \quad \nabla_{\mu}\beta_{1\nu} + \nabla_{\nu}\beta_{1\mu} = \phi_1(x)g_{\mu\nu}$$
$$n^{\mu} = \beta_0^{\mu}, \quad \gamma_{\mu\nu} = \nabla_{[\mu}\beta_{0\nu]}, \quad \nabla_{\mu}\nabla_{[\nu}\beta_{0\lambda]} = 0$$

- Consider the co-rotating observer with an inertial fluid:
  - Killing vector and frame-choosing vector:

$$\kappa_1^{\mu} = \delta_0^{\mu}, n^{\mu} = g^{\mu 0}$$

  - Integrate out the energy $p \cdot \kappa_1$:

$$\left[\partial_0 + \vec{v}_p \cdot \vec{\nabla}_x + \vec{p} \times \vec{\omega} \cdot \vec{\nabla}_p\right] f = 0$$
$$f = f(|\vec{p}| - (\vec{\omega} \times \vec{x}) \cdot \vec{p})$$

  - There is no CVE.

Summary and outlook

- We derive generally covariant chiral kinetic theory up to $O(\hbar)$ order
- We discuss some aspects of CVE. The CVE appears when the fluid is rotating in inertial frame. A rotating frame itself does not generate CVE; CVE is due to rotating of the fluid.
- Derive gravitational anomaly relation. To reveal the relation between the T^2 term in CVE and gravitational anomaly.
- $\hbar$ correction to the collision term. Its influence to the CVE current.
- The interplay between electromagnetic field and inertial or gravitational effects
- Spin dynamics.

Thank you!