Chiral kinetic theory in noninertial frame

Xu-Guang Huang
Fudan University
Y.C. Liu, L. Gao, K. Mameda, XGH, in preparation

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Chiral kinetic theory in curved spacetime

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Introduction and motivations
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- **Chiral fermions + magnetic field ⇒ chiral magnetic effect (CME)**
  
  $(\text{Kharzeev, McLerran, Warringa, Fukushima 2008; Son, Zhitnitsky 2004; \cdots})$:
  
  $\vec{J}_R = \frac{1}{4\pi^2} \mu_R \vec{B}$, \quad $\vec{J}_L = -\frac{1}{4\pi^2} \mu_L \vec{B}$

- **Chiral fermions + fluid vorticity ⇒ chiral vortical effect (CVE)**
  
  $(\text{Erdmenger et al 2008; Barnerjee et al 2008, Son, Surowka 2009; Landsteiner et al 2011})$:
  
  $\vec{J}_R = \frac{1}{4\pi^2} \mu_R^2 \vec{\omega} + \frac{T^2}{12} \vec{\omega}$, \quad $\vec{J}_L = -\frac{1}{4\pi^2} \mu_L^2 \vec{\omega} - \frac{T^2}{12} \vec{\omega}$

- **Phenomenology:** Heavy-ion collisions, Weyl/Dirac semi-metals, optically active materials, supernova, cold atoms, \cdots.
Coriolis force and CVE

- CVE is less understood than CME.
- CVE due to Coriolis force: The “lowest Landau level” problem

\[ \vec{J}_{\text{CME}} = \int \frac{d^3\vec{p}}{(2\pi)^3} e\vec{B}(\vec{\Omega}_B \cdot \hat{\vec{p}})f \quad \Rightarrow \quad \vec{J}_{\text{CVE}} = \int \frac{d^3\vec{p}}{(2\pi)^3} 2\varepsilon\vec{\omega}(\vec{\Omega}_B \cdot \hat{\vec{p}})f \]

where $\Omega_B$ is the Berry curvature. This (+ antiparticle) indeed gives expected result.

- Can such a replacement be derived in chiral kinetic theory?
Rotating frame vs. rotating fluid

- CVE due to flow vorticity in inertial frame. (Son, Surowka 2009; Gao, Pu, Wang, Wang 2012; Chen, Stephanov, Son, Yin 2014)

- Is rotating frame equivalent to flow vorticity in generating CVE?

- Helpful if we formulate a chiral kinetic theory allowing both rotating-frame effect and flow vorticity

- Mysterious $T^2$ term in CVE. Is it due to gravitational anomaly?

- A kinetic theory in curved spacetime would be helpful.
Chiral kinetic theory in electromagnetic (EM) field

- The kinetic description of chiral fermions
  - Son, Yamamoto 2012;
  - Stephanov, Yin 2012;
  - Hidaka, Pu, Yang 2016;
  - Muller, Venogopalan 2017;
  - Huang, Shi, Jiang, Liao, Zhuang 2018;
  .......

- Based on adiabatic expansion or $\hbar$ expansion.

- Chiral anomaly is encoded.

- Valid for weak magnetic field.

- Friendly in application. Quark-gluon plasma, Weyl/Dirac semimetals, electroweak gases, ....
Semiclassical equations of motion
Heisenberg equation of motion

Consider a Weyl fermion in a rotating frame:

\[ H = \vec{p} \cdot \vec{\sigma} - \vec{\omega} \cdot \left( \vec{x} \times \vec{p} + \frac{\hbar}{2} \vec{\sigma} \right) \]

where \( \vec{p} = -i\hbar \nabla \).

The Heisenberg equations:

\[ \dot{\vec{x}} = \frac{1}{i\hbar} [\vec{x}, H] = \vec{\sigma} + \vec{x} \times \vec{\omega}, \]

\[ \dot{\vec{p}} = \frac{1}{i\hbar} [\vec{p}, H] = \vec{p} \times \vec{\omega}, \]

\[ \dot{\vec{\sigma}} = \frac{1}{i\hbar} [\vec{\sigma}, H] = \frac{2}{\hbar} \vec{p} \times \vec{\sigma} - \vec{\omega} \times \vec{\sigma} \]

The EOM of for spin can be solved order by order in \( \hbar \):

\[ \vec{\sigma} = \hat{\vec{p}} + \hbar \frac{1}{2p} \left[ \hat{\vec{p}} \times (\hat{\vec{p}} \times \vec{\omega}) - \hat{\vec{p}} \times \hat{\vec{p}} \right] + O(\hbar^2) \]

with \( \hat{\vec{p}} = \frac{\vec{p}}{|\vec{p}|} \). (We only show the particle branch.)
Semiclassical equation of motion

- Substitute this solution to the first two EOMs:

\[
\dot{\vec{x}} = \hat{\vec{p}} + \vec{x} \times \vec{\omega} = \frac{\partial \varepsilon}{\partial \vec{p}} + \hbar \vec{p} \times \vec{\Omega}_B,
\]

\[
\dot{\vec{p}} = \vec{p} \times \vec{\omega} = -\frac{\partial \varepsilon}{\partial \vec{x}}
\]

where \(\vec{\Omega}_B = \hat{\vec{p}}/(2p^2)\) is the Berry curvature and the single-particle energy \(\varepsilon = p - \vec{\omega} \cdot (\vec{x} \times \vec{p}) - \frac{\hbar}{2} \vec{\omega} \cdot \hat{\vec{p}}\).

- The Coriolis force and centrifugal force:

\[
\ddot{\vec{x}} = -2\vec{\omega} \times \dot{\vec{x}} - \vec{\omega} \times (\vec{\omega} \times \vec{x})
\]

- Surprisingly the EOMs do not have \(\hbar\) order correction. But the single-particle energy does.

- This suggests a kinetic equation up to \(O(\hbar)\):

\[
\partial_t f + \dot{\vec{x}} \cdot \nabla_x f + \dot{\vec{p}} \cdot \nabla_p f = C[f]
\]

with phase space measure 1.

- Similar procedure for EM field case gives the CKT in EM field as derived before. (Son, Yamamoto 2012, Hidaka, Pu, Yang 2017, Huang, Shi, Jiang, Liao, Zhuang 2018)
The dynamics of Wigner function
Wigner operator in curved spacetime

- Instead of Weyl fermions, we consider massless Dirac fermions so that the formalism can be potentially extended to massive case.
- Our Wigner operator is:

\[
\hat{W}_{\alpha\beta}(x, p) = \int d^4y \left[-g(x)\right]^{1/2} e^{-ip \cdot y / \hbar} \left[\bar{\psi}(x) e^{y \cdot \hat{D}/2} \right]_\beta \left[e^{-y \cdot \hat{D}/2} \psi(x)\right]_\alpha
\]

- The position of the spacetime manifold is \(x^\mu\), the position in the tangent space of \(x\) is \(y^\mu\), \(p_\mu\) is in the cotangent space of \(x\). The whole phase space is the cotangent bundle.
- The derivative \(D_\mu\) in the tangent bundle is lifted from the usual covariant derivative \(\nabla_\mu\) in the spacetime manifold:

\[
D_\mu = \nabla_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda}
\]

- The derivative \(D_\mu\) in the cotangent bundle is lifted from the usual covariant derivative \(\nabla_\mu\) in the spacetime manifold:

\[
D_\mu = \nabla_\mu + \Gamma^\lambda_{\mu\nu} p^\lambda \frac{\partial}{\partial p^\nu}
\]

- The advantage of using \(D_\mu\), \(y^\nu\) and \(p_\nu\) are parallely transported:

\[
D_\mu y^\nu = 0 = D_\mu p_\nu
\]
Wigner function and its dynamics

- **Wigner function:**

\[
W(x, p) = \langle \hat{W}(x, p) \rangle
\]

- Consider noninteracting fermions. Dirac equations:

\[
i\hbar \gamma \cdot \nabla \psi(x) = i\hbar \bar{\psi}(x) \nabla \cdot \gamma = 0
\]

- Multiply \( D_{\mu} \) to Wigner function and apply Dirac equations:

\[
\gamma^{\mu} \left( \frac{i\hbar}{2} D_{\mu} + p_{\mu} \right) W(x, p) = -i\gamma^{\mu} \hbar \hat{H}_{\mu} \left( x, -\frac{1}{2} i\hbar \partial_{p} \right) \otimes W(x, p)
\]

\[
+ \frac{i\hbar}{2} \gamma^{\mu} \left[ \hat{G}_{\mu} \left( x, -\frac{1}{2} i\hbar \partial_{p} \right), W(x, p) \right] \otimes \]

- This equation is exact.
- The tensor product is understood similarly as how we define \( \hat{W} \).
- The operators \( \hat{G} \) and \( \hat{H} \):

\[
\hat{H}_{\mu}(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[ y \cdot D, \cdots, [y \cdot D, D_{\mu}] \cdots \right],
\]

\[
\hat{G}_{\mu}(x, y) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left[ y \cdot D, \cdots, [y \cdot D, D_{\mu}] \cdots \right]
\]
**$\hbar$ expansion**

- Up to $O(\hbar^2)$ order, here $\Delta_\mu = D_\mu - F_{\mu\lambda} \partial_\lambda^p$:

$$
\gamma^\mu \left[ p_\mu + \frac{i\hbar}{2} \Delta_\mu - \frac{\hbar^2}{8} R^\rho \sigma_{\lambda\mu} p_\rho \partial_\lambda^p \partial_\rho^p + \frac{i\hbar^2}{8} R_{\lambda\mu ab} \sigma^{ab} \partial_\rho^\lambda + \frac{\hbar^2}{12} (\partial_\alpha F_{\beta\mu}) \partial_\rho^\alpha \partial_\rho^\beta \right] W(x, p) 
$$

$$
= \frac{i\hbar^2}{32} \gamma^\mu R_{\lambda\mu ab} [\sigma^{ab}, \partial_\rho^\lambda] W(x, p)
$$

- Dirac decomposition:

$$
W(x, p) = \frac{1}{4} \left[ \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} S_{\mu\nu} \right]
$$

- Focus on $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ or equivalently $\mathcal{R}_\mu / \mathcal{L}_\mu = (1/2)(\mathcal{V}_\mu \pm \mathcal{A}_\mu)$:

$$
p_\mu \mathcal{R}^\mu = O(\hbar^2),
$$

$$
p^\alpha [\mathcal{R}_{\beta}] + \frac{\hbar}{4} \epsilon^{\mu\nu\alpha\beta} \Delta_\mu \mathcal{R}_{\nu} = O(\hbar^2),
$$

$$
\Delta_\mu \mathcal{R}^\mu - \frac{\hbar}{4} \epsilon^{\mu\nu\alpha\beta} R_{\lambda\mu\alpha\beta} \partial_\rho^\lambda \mathcal{R}_{\nu} = O(\hbar^2)
$$

- $\mathcal{L}_\mu$ satisfies same equations but with opposite sign for $\hbar$ coefficients.
- The first two equations fix $\mathcal{R}_\mu$ up to a scalar function (distribution function).
- The third equation is kinetic equation for distribution function.
The kinetic equation

- The solutions to the first two equations:

\[ R_\mu = p_\mu f \delta(p^2) + \hbar \tilde{F}_{\mu\nu} p^\nu f \delta'(p^2) + \frac{\hbar}{2p \cdot n} \epsilon_{\mu\nu\rho\sigma} n^\nu p^\sigma \Delta^\rho f \delta(p^2) + O(\hbar^2) \]

  - \( n^\mu \) is a frame choosing vector.
  - \( f = f_0 + f_1 + O(\hbar^2) \) is the distribution function.

- The kinetic equation for \( f \):

\[
\delta \left( p^2 - \frac{\hbar}{p \cdot n} \tilde{G}_{\alpha\beta} p^\alpha n^\beta \right) \left\{ p \cdot \Delta + \frac{\hbar \epsilon_{\mu\rho\nu\sigma} n^\nu p^\sigma}{2(p \cdot n)^2} \left[ D^\mu (p \cdot n) - G^{\mu\lambda} n_\lambda \right] \Delta^\rho 
+ \frac{\hbar \epsilon_{\mu\nu\rho\sigma}}{2p \cdot n} \left[ p^\sigma (D^\mu n^\nu) \Delta^\rho - p^\sigma n^\nu (D^\mu G^{\rho\lambda}) \partial^\rho_\lambda \right] \right\} f = O(\hbar^2)
\]

where

\[ \Delta_\mu = \partial_\mu + \Gamma^\lambda_{\mu\nu} p_\lambda \partial^\nu - G^{\rho\lambda} \partial^p_\lambda, \quad G_{\mu\nu} = F_{\mu\nu} - \frac{\hbar}{4} \epsilon_{\mu\lambda\alpha\beta} R^\nu_\lambda \alpha\beta \]

- This equation is invariant under general coordinate transformation.
- In Minkowski spacetime, \( \Gamma^\lambda_{\mu\nu} = 0 = R^\nu_\lambda \alpha\beta \), it reduces to the kinetic equation in electromagnetic field. (Huang, Shi, Jiang, Liao, Zhuang 2018)
- Let us focus on case of \( F_{\mu\nu} = 0 \). We consider the rotating frame.
Rotating frame

- An inertial frame with a Minkowski coordinate:
  \[ ds^2 = dt^2 - d\vec{x}^2 \]

- A frame rotating with constant \( \vec{\omega} \) w.r.t. the above inertial frame:
  \[ ds^2 = [1 - (\vec{\omega} \times \vec{x})^2] dt^2 - 2(\vec{\omega} \times \vec{x}) \cdot d\vec{x} dt - d\vec{x}^2 \]

- Some key quantities for rotating frame:
  - Metric:
    \[ g^{00} = 1 - (\vec{\omega} \times \vec{x})^2, \quad g^{0i} = g_{i0} = -(\vec{\omega} \times \vec{x})^i, \quad g_{ij} = -\delta_{ij} \]
  - Inverse metric:
    \[ g^{00} = 1, \quad g^{0i} = g^{i0} = -(\vec{\omega} \times \vec{x})^i, \quad g^{ij} = -\delta^{ij} + (\vec{\omega} \times \vec{x})^i (\vec{\omega} \times \vec{x})^j \]
  - Nonzero components of Christoffel connection:
    \[ \Gamma^i_{00} = [\vec{\omega} \times (\vec{\omega} \times \vec{x})]^i, \quad \Gamma^i_{0j} = \Gamma^i_{j0} = -\epsilon^{ijk} \omega^k \]
  - Nonzero components of spin connection:
    \[ \Gamma_0 = -i \frac{\vec{\sigma}}{2} \cdot \vec{\omega} \]
  - Riemannian curvature:
    \[ R^{\mu}_{\nu\alpha\beta} = 0 \]
$O(\hbar^0)$-order kinetic equation

- The classical kinetic equation:

$$\delta(p^2)p^\mu D_\mu f_0 = 0$$

- Integrating out “energy”, we can write it in 3-dimensional form.

- Energy is associated with timelike Killing vectors:
  \begin{align*}
  \kappa_1 &= \delta^\mu_0 \partial_\mu = \partial_0: \text{time translation for a co-rotating observer} \\
  \kappa_2 &= \partial_0 - (\vec{\omega} \times \vec{x})^i \partial_i \quad \text{for } |\vec{\omega} \times \vec{x}| < 1: \text{time translation for inertial observer}
  \end{align*}

- Killing vector $\kappa_1 \Rightarrow \varepsilon_p = p_0 = |\vec{p}| - (\vec{\omega} \times \vec{x}) \cdot \vec{p}$ for particle, where $\vec{p} = -(p_1, p_2, p_3)$.

- Kinetic equation in 3-dimensional form:

$$\left[ \partial_0 + \vec{v}_p \cdot \vec{\nabla}_x + \vec{p} \times \vec{\omega} \cdot \vec{\nabla}_p \right] f_0(t, \vec{x}, \vec{p}) = 0$$

where $\vec{v}_p = \partial \varepsilon_p / \partial \vec{p} = \hat{\vec{p}} - \vec{\omega} \times \vec{x}$

- Single-particle equations of motion:

$$\dot{\vec{x}} = \hat{\vec{p}} - \vec{\omega} \times \vec{x} = \frac{\partial \varepsilon_p}{\partial \vec{p}},$$

$$\dot{\vec{p}} = \vec{p} \times \vec{\omega} = -\frac{\partial \varepsilon_p}{\partial \vec{x}}$$

which is textbook result.
$O(\hbar)$-order kinetic equation

- The kinetic equation up to $O(\hbar)$:

$$\delta(p^2) D_\mu (p^\mu + \hbar s^{\mu\nu} D_\nu) f = 0$$

where $s_{\mu\nu} = -(1/2p \cdot n) \epsilon_{\mu\nu\rho\sigma} n^\rho p^\sigma$ is spin.

- The current:

$$J_R^\mu = \int \frac{d^4p}{(2\pi)^4} \delta(p^2) (p^\mu + \hbar s^{\mu\nu} D_\nu) f$$

where the second term represents the side-jump. (Chen et al 2014)

- Consider equilibrium state. $f = f(g)$ with $g$ a linear combination of collisional conserved quantities.

- At $O(\hbar^0)$ order; classical:
  - Conserved quantities are particle number, momenta; angular momentum automatically conserved:

$$g = \alpha_0(x) + \beta_0(x)p_\mu$$

  - Substitute to kinetic equation:

$$\nabla_\mu \alpha_0 = 0, \quad \nabla_\mu \beta_0^0 + \nabla_\nu \beta_0^\nu = \phi_0(x) g_{\mu\nu}$$

  - The functional form of $f(g)$ needs classical collision term to fix.
$O(\hbar)$-order kinetic equation

- Up to $O(\hbar)$ order:
  - Spin should be included in angular momentum:
    \[
    g = \alpha(x) + \beta^\mu(x)p_\mu + \gamma^{\mu\nu}(x)s_{\mu\nu},
    \]
    \[
    \alpha = \alpha_0 + \alpha_1 + O(\hbar^2), \quad \beta^\mu = \beta^\mu_0 + \beta^\mu_1 + O(\hbar^2), \quad \gamma^{\mu\nu} = O(\hbar)
    \]
  - Substitute to kinetic equation, the $\hbar$-order quantities:
    \[
    \nabla_\mu \alpha_1 = 0, \quad \nabla_\mu \beta_{1\nu} + \nabla_\nu \beta_{1\mu} = \phi_1(x)g_{\mu\nu}
    \]
    \[
    n^\mu = \beta^\mu_0, \quad \gamma_{\mu\nu} = \nabla_{[\mu} \beta_{0\nu]}, \quad \nabla_\mu \nabla_{[\nu} \beta_{0\lambda]} = 0
    \]
- Consider the co-rotating observer with a co-rotating fluid:
  - Killing vector and frame-choosing vector:
    \[
    \kappa^\mu_1 = \delta^\mu_0, \quad n^\mu = \delta^\mu_0
    \]
- Integrate out the energy $p \cdot \kappa_1$:
  \[
  \partial_t f + \hat{x} \cdot \vec{\nabla}_x f + \hat{p} \cdot \vec{\nabla}_p f = 0
  \]
  \[
  f = f(|\hat{p}| - (\vec{\omega} \times \vec{x}) \cdot \hat{p} - \hbar \vec{\omega} \cdot \hat{p}/2)
  \]
  This coincides our warm-up calculation.
- The CVE current:
  \[
  \vec{J}_R = \int \frac{d^3 \hat{p}}{(2\pi)^3} \left( \hat{p} \left( \hat{p} + \hbar |\hat{p}| \vec{\Omega}_B \times \vec{\nabla}_x \right) \right) f
  \]
**$O(\hbar)$-order kinetic equation**

- **Up to $O(\hbar)$ order:**
  - Spin should be included in angular momentum:
    \[
    g = \alpha(x) + \beta^\mu(x)p_\mu + \gamma^{\mu\nu}(x)s_{\mu\nu}
    \]
    \[
    \alpha = \alpha_0 + \alpha_1 + O(\hbar^2), \quad \beta^\mu = \beta_0^\mu + \beta_1^\mu + O(\hbar^2), \quad \gamma^{\mu\nu} = O(\hbar)
    \]
  - Substitute to kinetic equation, the $\hbar$-order quantities:
    \[
    \nabla_\mu \alpha_1 = 0, \quad \nabla_\mu \beta_{1\nu} + \nabla_\nu \beta_{1\mu} = \phi_1(x)g_{\mu\nu}
    \]
    \[
    n^\mu = \beta_0^\mu, \quad \gamma_{\mu\nu} = \nabla_{[\mu} \beta_{0\nu]}, \quad \nabla_\mu \nabla_{[\nu} \beta_{0\lambda]} = 0
    \]
- **Consider the co-rotating observer with an inertial fluid:**
  - Killing vector and frame-choosing vector:
    \[
    \kappa_1^\mu = \delta_0^\mu, \quad n^\mu = g^{\mu0}
    \]
  - Integrate out the energy $p \cdot \kappa_1$:
    \[
    \left[ \partial_0 + \vec{v}_p \cdot \nabla_x + \vec{p} \times \vec{\omega} \cdot \nabla_p \right] f = 0
    \]
    \[
    f = f(|\vec{p}| - (\vec{\omega} \times \vec{x}) \cdot \vec{p})
    \]
- There is no CVE.
Summary and outlook

- We derive generally covariant chiral kinetic theory up to $O(\hbar)$ order.
- We discuss some aspects of CVE. The CVE appears when the fluid is rotating in inertial frame. A rotating frame itself does not generate CVE; CVE is due to rotating of the fluid.
- Derive gravitational anomaly relation. To reveal the relation between the $T^2$ term in CVE and gravitational anomaly.
- $\hbar$ correction to the collision term. Its influence to the CVE current.
- The interplay between electromagnetic field and inertial or gravitational effects.
- Spin dynamics.

Thank you!