Relativistic hydrodynamics with spin

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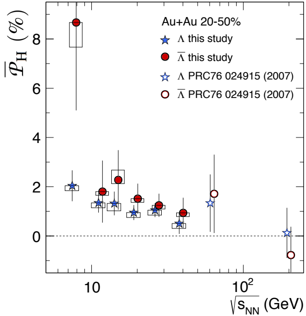
based on recent works with **B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza**
arXiv:1705.00587 , arXiv:1712.07676 (nucl-th)

Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions
Florence, March 19–22, 2018

- **Non-central heavy-ion collisions create fireballs with large global angular momenta** which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
- **Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions**, both from the experimental and theoretical point of view
  L. Adamczyk et al. (**STAR**), (2017), **Nature 548 (2017) 62-65**, arXiv:1701.06657 (nucl-ex)
  Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
  www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



Poincaré symmetry leads to the basic conservation laws:

Conservation of energy and momentum

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0, \qquad \text{(4 equations)}$$

Conservation of angular momentum

$$\partial_\mu \hat{J}^{\mu,\alpha\beta}(x) = 0, \qquad \hat{J}^{\mu,\alpha\beta}(x) = -\hat{J}^{\mu,\beta\alpha}(x) \quad \text{(6 equations)}$$

Angular momentum consists of orbital and spin parts:

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x),$$

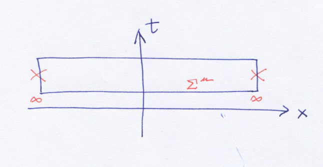
$$\hat{L}^{\mu,\alpha\beta}(x) = x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha},$$

$\partial_\mu \hat{S}^{\mu,\alpha\beta}(x) = \hat{T}^{\beta\alpha}(x) - \hat{T}^{\alpha\beta}(x) \neq 0,$ in most cases the spin tensor $\hat{S}^{\mu,\alpha\beta}(x)$ is not conserved

Density operator for any quantum mechanical system

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right)\right]$$

$d^3\Sigma_\mu$ is an element of a space-like, 3-dimensional hypersurface Σ_μ,
we can take, for example, $d^3\Sigma_\mu = (dV,0,0,0)$
in global equilibrium $\hat{\rho}(t)$ should be independent of time



$$\partial_\mu\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right) = \hat{T}^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\left(\partial_\mu\omega_{\alpha\beta}(x)\right) = 0$$

for **asymmetric** energy-momentum tensor:

$$b_\nu = \text{const.}\ , \quad \omega_{\alpha\beta} = \text{const.}$$

splitting angular momentum into its orbital and spin part

$$\begin{aligned}\hat{\rho}_{\rm EQ} &= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(\hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x)\right)\omega_{\alpha\beta}\right)\right] \\ &= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha} + \hat{S}^{\mu,\alpha\beta}(x)\right)\omega_{\alpha\beta}\right)\right] \\ &= \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\left(b_\nu + \omega_{\nu\alpha}x^\alpha\right) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}\right)\right]\end{aligned}$$

Introducing the notation

$$\beta_\nu = b_\nu + \varpi_{\nu\alpha}x^\alpha$$

we may write

$$\rho_{\rm EQ} = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\varpi_{\alpha\beta}\right)\right]$$

We note that β_ν is the Killing vector, satisfies the equation

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0, \quad \varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \text{const} \quad \text{(thermal vorticity)}$$

PRESENT PHENOMENOLOGY PRESCRIPTION USED TO DESCRIBE THE DATA:

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin
2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
(defined often by the condition T=const)
3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const
4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$
5) Make predictions about spin polarization

SUCH A METHOD WORKS WELL, DESCRIBES THE DATA, BUT...
M. LISA's talk: do not take that for granted!

THIS TALK:
1) in **(spin) local equilibrium** thermal vorticity and spin polarization tensor **are independent** — $\beta_\mu(x)$ and $\omega_{\mu\nu}(x)$ continue their independent lives
2) eventually, they may become related if the system reaches global equilibrium
3) **freedom in the initial conditions** should be realized by independent $\beta_\mu(t_0, \boldsymbol{x})$ and $\omega_{\mu\nu}(t_0, \boldsymbol{x})$, there must be a (relativistic) delay in the coupling between vorticity and polarization, **like shear stress/flow tensors**
4) spin polarization may be an early-stage effect that survives the whole evolution

F. BECATTINI: local equilibrium has $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$

G. TORRIERI: it is desirable to have space-time dependence of the polarization

Our starting point is the **phase-space distribution functions for spin-1/2 particles generalized from scalar functions to two by two spin density matrices** for each value of the space-time position x and momentum p, **F. Becattini et al., Annals Phys. 338 (2013) 32**

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p) X^{+} u_s(p), \quad f^{-}_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p) X^{-} v_r(p)$$

Following the notation used by F. Becattini et al., we introduce the matrices

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x) p^\mu\right] M^{\pm}$$

where

$$M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

Here we use the notation $\beta^\mu = u^\mu/T$ and $\xi = \mu/T$, with the temperature T, chemical potential μ and four velocity u^μ. The latter is normalized to $u^2 = 1$. Moreover, $\omega_{\mu\nu}$ is the spin polarization tensor, while $\hat{\Sigma}^{\mu\nu}$ is the spin operator expressed in terms of the Dirac gamma matrices, $\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$.

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma.$$

We can assume that both k_μ and ω_μ are orthogonal to u^μ, i.e., $k \cdot u = \omega \cdot u = 0$,

$$k_\mu = \omega_{\mu\nu} u^\nu, \quad \omega_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta}\, \omega^{\nu\alpha} u^\beta$$ electric- and magnetic-like components

It is convenient to introduce the dual spin tensor

$$\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta} = \omega_\mu u_\nu - \omega_\nu u_\mu + \epsilon^{\mu\nu\alpha\beta} k_\alpha u_\beta.$$

One finds $\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$ and $\frac{1}{2}\tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = 2k \cdot \omega$. Using the conditions $k \cdot \omega = 0$ and $k \cdot k - \omega \cdot \omega \geq 0$ we find the compact form with $\zeta = \frac{1}{2}\sqrt{k \cdot k - \omega \cdot \omega}$

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\, \omega_{\mu\nu} \hat{\Sigma}^{\mu\nu},$$

E. SPERANZA's talk: more about the properties of such distributions

Charge current

The **charge current** (S. de Groot, W. van Leeuwen, and C. van Weert)

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left[\mathrm{tr}_4(X^+) - \mathrm{tr}_4(X^-)\right] = n u^\mu$$

where 'tr_4' denotes the trace over spinor indices and n is the charge density

$$n = 4\cosh(\zeta)\sinh(\xi)\, n_{(0)}(T) = \left(e^\zeta + e^{-\zeta}\right)\left(e^\xi - e^{-\xi}\right) n_{(0)}(T)$$

Here $n_{(0)}(T) = \langle (u\cdot p)\rangle_0$ is the number density of spin 0, neutral Boltzmann particles, obtained using the thermal average

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\cdots)\, e^{-\beta\cdot p}, \qquad E_p = \sqrt{m^2 + \mathbf{p}^2}$$

simple thermodynamic interpretation
four species: particles and antiparticles with spin up and down

The **energy-momentum tensor** S. de Groot, W. van Leeuwen, and C. van Weert; F. Becattini et al., Annals Phys. 338 (2013) 32

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[\mathrm{tr}_4(X^+) + \mathrm{tr}_4(X^-)\right] = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu},$$

where the energy density and pressure are given by

$$\varepsilon = 4\cosh(\zeta)\cosh(\xi)\,\varepsilon_{(0)}(T)$$

and

$$P = 4\cosh(\zeta)\cosh(\xi)\,P_{(0)}(T),$$

respectively. In analogy to the density $n_{(0)}(T)$, we define the auxiliary quantities $\varepsilon_{(0)}(T) = \langle (u\cdot p)^2\rangle_0$ and $P_{(0)}(T) = -(1/3)\langle \left[p\cdot p - (u\cdot p)^2\right]\rangle_0$.

$T^{\mu\nu}$ is symmetric, expected for classical particles with $\boldsymbol{p} = \boldsymbol{v} E_p$, if $T^{\mu\nu} = \Delta p^\mu / \Delta\Sigma_\nu$
spin tensor is conserved in this case

The **entropy current** is given by an obvious generalization of the Boltzmann expression

$$S^\mu = -\int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\mu \left(\mathrm{tr}_4 \left[X^+(\ln X^+ - 1)\right] + \mathrm{tr}_4 \left[X^-(\ln X^- - 1)\right]\right)$$

This leads to the following entropy density

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu\, n - \Omega w}{T},$$

where Ω is defined through the relation $\zeta = \Omega/T$ and

$$w = 4 \sinh(\zeta) \cosh(\xi)\, n_{(0)}.$$

This suggests that Ω should be used as a thermodynamic variable of the grand canonical potential, in addition to T and μ. Taking the pressure P to be a function of T, μ and Ω, we find

$$s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\Omega}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\Omega}, \quad w = \left.\frac{\partial P}{\partial \Omega}\right|_{T,\mu}.$$

The conservation of energy and momentum requires that $\partial_\mu T^{\mu\nu} = 0$
This equation can be split into two parts, one longitudinal and the other transverse with respect to u^μ:

$$
\begin{aligned}
\partial_\mu[(\varepsilon + P)u^\mu] &= u^\mu \partial_\mu P \equiv \frac{dP}{d\tau}, \\
(\varepsilon + P)\frac{du^\mu}{d\tau} &= (g^{\mu\alpha} - u^\mu u^\alpha)\partial_\alpha P.
\end{aligned}
$$

Evaluating the derivative on the left-hand side of the first equation we find

$$T\,\partial_\mu(su^\mu) + \mu\,\partial_\mu(nu^\mu) + \Omega\,\partial_\mu(wu^\mu) = 0.$$

The middle term vanishes due to charge conservation,

$$\partial_\mu(nu^\mu) = 0.$$

Thus, in order to have entropy conserved in our system (for the perfect-fluid description we are aiming at), we demand that

$$\partial_\mu(wu^\mu) = 0.$$

Consequently, we self-consistently arrive at the conservation of entropy, $\partial_\mu(su^\mu) = 0$
Equations above form dynamic background for the spin dynamics.

Since we use a symmetric form of the energy-momentum tensor $T^{\mu\nu}$, the spin tensor $S^{\lambda,\mu\nu}$ satisfies the conservation law,

$$\partial_\lambda S^{\lambda,\mu\nu} = 0.$$

For $S^{\lambda,\mu\nu}$ we use the form (E. SPERANZA's talk):

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\lambda \, \mathrm{tr}_4 \left[(X^+ - X^-) \hat{\Sigma}^{\mu\nu} \right] = \frac{w u^\lambda}{4\zeta} \omega^{\mu\nu}$$

Using the conservation law for the spin density and introducing the rescaled spin tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$, we obtain

$$u^\lambda \partial_\lambda \, \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0,$$

with the normalization condition $\bar{\omega}_{\mu\nu}\, \bar{\omega}^{\mu\nu} = 2$.

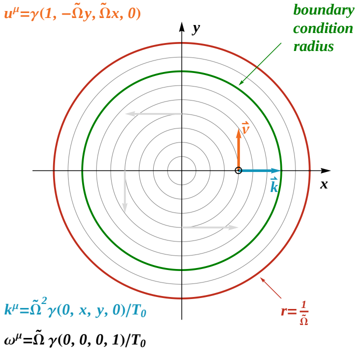
TRANSPORT OF THE SPIN POLARIZATION DIRECTION ALONG THE FLUID STREAM LINES

CHANGE OF THE POLARIZATION MAGNITUDE DESCRIBED BY THE BACKGROUND EQUATIONS

The hydrodynamic flow $u^\mu = \gamma(1, \boldsymbol{v})$ with the components

$$u^0 = \gamma, \quad u^1 = -\gamma\,\tilde\Omega\, y, \quad u^2 = \gamma\,\tilde\Omega\, x, \quad u^3 = 0,$$

$\tilde\Omega$ is a constant, $\gamma = 1/\sqrt{1-\tilde\Omega^2 r^2}$
r – distance from the vortex centre in the transverse plane, $r^2 = x^2 + y^2$
due to limiting light speed, $0 \le r \le R < 1/\tilde\Omega$.



Global equilibrium with rotation – stationary vortex 2

The total time (convective) derivative

$$\frac{d}{d\tau} = u^\mu \partial_\mu = -\gamma\tilde{\Omega}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right). \tag{1}$$

can be used to find the fluid (centripetal) acceleration

$$a^\mu = \frac{du^\mu}{d\tau} = -\gamma^2\tilde{\Omega}^2(0, x, y, 0).$$

It is easy to see that the equations of the hydrodynamic background are satisfied if T, μ and Ω are proportional to the Lorentz-γ factor

$$T = T_0\gamma, \quad \mu = \mu_0\gamma, \quad \Omega = \Omega_0\gamma,$$

with T_0, μ_0 and Ω_0 being constants. **One possibility is that the vortex represents an unpolarized fluid with $\omega_{\mu\nu} = 0$ and thus, with $\Omega_0 = 0$.**

Another possibility is that the particles in the fluid are polarized and $\Omega_0 \neq 0$. In the latter case we expect that the spin tensor has the structure

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{\Omega}/T_0 & 0 \\ 0 & \tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the parameter T_0 has been introduced to keep $\omega_{\mu\nu}$ dimensionless. This form yields $k^\mu = \tilde{\Omega}^2(\gamma/T_0)\,(0, x, y, 0)$ and $\omega^\mu = \tilde{\Omega}(\gamma/T_0)\,(0, 0, 0, 1)$. As a consequence, we find $\zeta = \tilde{\Omega}/(2T_0)$, which, for consistency with the hydrodynamic background equations, implies

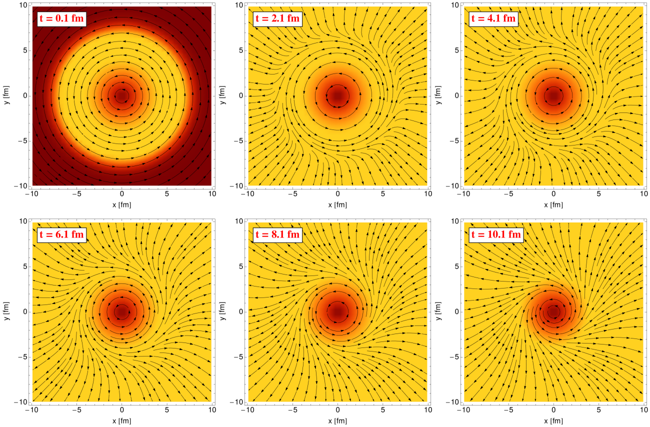
$$\tilde{\Omega} = 2\,\Omega_0.$$

In this case the spin polarization tensor agrees with the thermal vorticity, namely

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) = \omega_{\mu\nu}$$

as emphasised in the works by Becattini and collaborators.

What can happen if the external boundary is removed? Expansion into external vacuum.



Stream lines and temperature (color gradient), T_0 = 200 MeV, m = 1 GeV

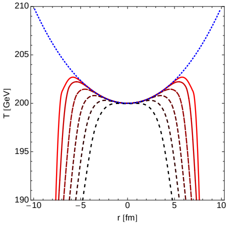


Figure: Temperature profiles

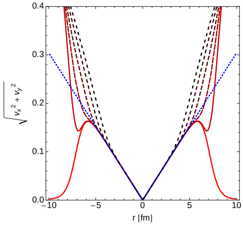


Figure: Velocity profiles

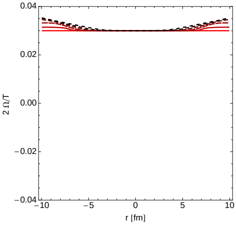


Figure: Spin chemical potential

time increases by 2 fm, red → black lines)

Initial gaussian temperature profile

$$T_{\rm i} = T_0 \exp\left(-\frac{x^2}{2x_0^2} - \frac{y^2}{2y_0^2} - \frac{z^2}{2z_0^2}\right)$$

$x_0 = 1$ (beam direction, one can possibly use the Landau model)
$y_0 = 2.6$ and $z_0 = 2$ (from GLISSANDO version of the Glauber Model, Au+Au, 20-30%)

Initial flow profile

$$\tilde{\Omega} \to \frac{1}{r} \tanh \frac{r}{r_0}, \quad v_x = -\frac{y}{r} \tanh \frac{r}{r_0}, \quad v_y = \frac{x}{r} \tanh \frac{r}{r_0}$$

the parameter r_0 controls the magnitude of the initial angular velocity, in this talk $r_0 = 1.0$

Quasi-realistic model for low-energy collisions 2

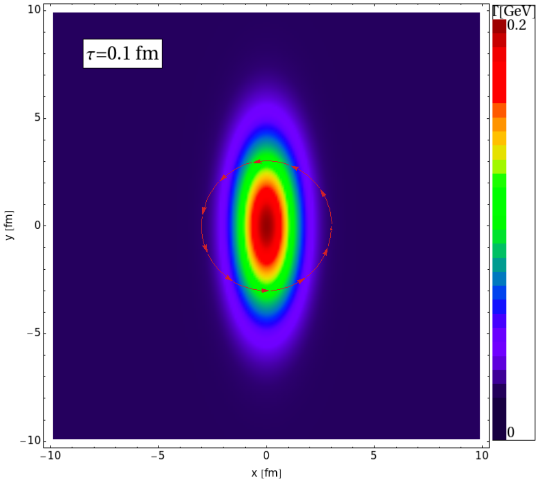
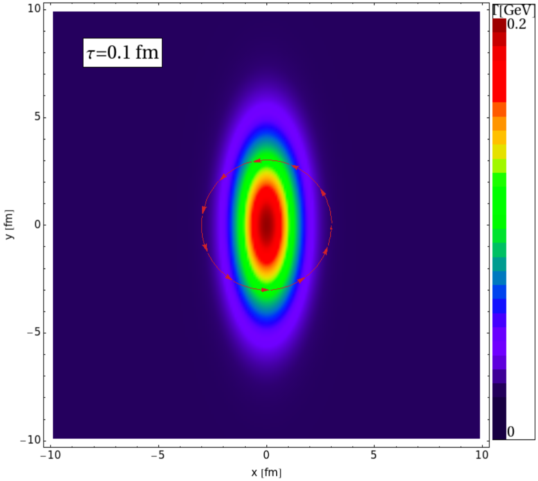


Figure: Initial conditions for the quasi-realistic model

Quasi-realistic model for low-energy collisions 3



We have introduced a hydrodynamic framework, which includes the evolution of the spin density in a consistent fashion. Equations that determine the dynamics of the system follow solely from conservation laws – minimal extension of the well established perfect-fluid picture.

Our framework can be used to determine the space-time dynamics of fluid variables, now including also the spin tensor, from initial conditions defined on an initial space-like hypersurface. This property makes them useful for practical applications in studies of polarization evolution in high-energy nuclear collisions and also in other physics systems exhibiting fluid-like, collective dynamics connected with non-trivial polarization phenomena.

The possibility to study the dynamics of systems in local thermodynamic equilibrium represents an important advance compared to studies, where global equilibrium was assumed.

Next steps: spin-orbit interactions, asymmetric $T_{\mu\nu}$, dissipation, ...

G. TORRIERI:

$$\zeta u^\lambda \partial_\lambda \bar{\omega}^{\mu\nu} = \zeta \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = -\frac{\omega_{\mu\nu} - \varpi_{\mu\nu}}{\tau_{\rm rel}}$$

consistent incorporation of the relaxation of the spin polarisation towards thermal vorticity defined with the help of arbitrary tensors $\Phi^{\lambda,\mu\nu}$ and $Z^{\alpha\lambda,\mu\nu}$

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}\right) \equiv T^{\mu\nu} + \frac{1}{2}\partial_\lambda G^{\lambda,\mu\nu}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu}$$

does not change global charges, new tensors, $T'^{\mu\nu}$ and $J'^{\lambda,\mu\nu}$, are also conserved

Bellinfante prescription: $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} \quad \rightarrow \quad S'^{\lambda,\mu\nu} = 0, \quad T'^{\mu\nu} = T'^{\nu\mu}$