Non-Equilibrium Anomalous Transport of Chiral Fluids from Kinetic Theory

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Yoshimasa Hidaka, Shi Pu, DY, arXiv:1612.04630, PRD 95 (2017) no.9, 091901
Yoshimasa Hidaka, Shi Pu, DY, arXiv:1710.00278, PRD 97 (2018) no.1, 016004
Yoshimasa Hidaka, DY, arXiv:1801.08253

Non-equilibrium anomalous transport

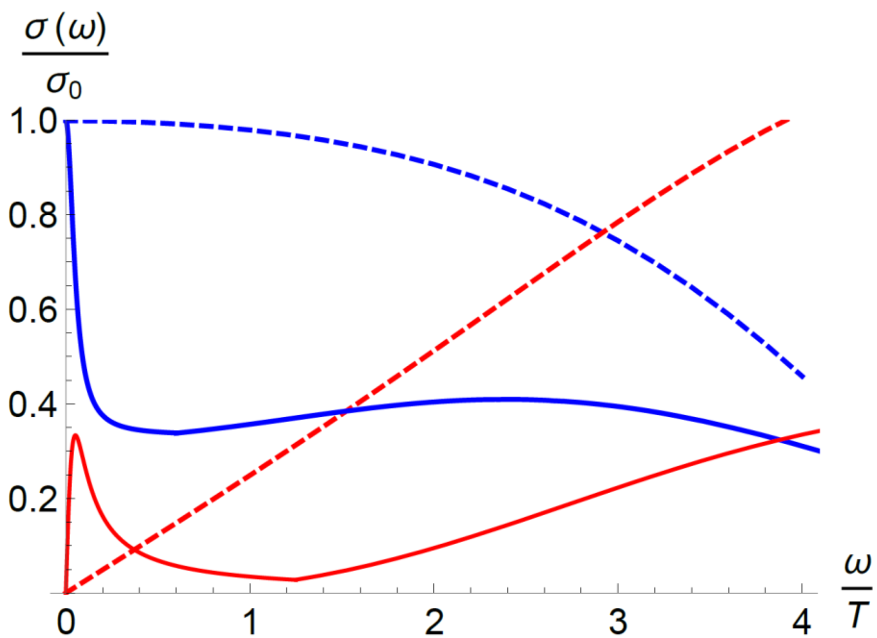
- CME in thermal equilibrium (BF) : independent of interactions
- Non-equilibrium cases : interactions could be involved
- e.g. AC conductivity of CME :

D. Kharzeev & H. Warringa, 09

D. Satow & H. U. Yee, 14.

D. Kharzeev, et.al., 17,

(time-dep. B field)



blue : real part

red : imaginary part

solid : kinetic theory

dashed : AdS/CFT

$$\sigma(\omega) = \sigma_0 \left(1 - \frac{2}{3}\frac{\omega}{\omega + i\tau_R^{-1}}\right)$$

$$\tau_R^{-1} \approx 1.3\,\alpha_s^2 \log(1/\alpha_s) T$$

$$(2-\text{flavor QCD})$$

Chiral kinetic theory

- Chiral kinetic theory (CKT): kinetic theory + chiral anomaly
- Limitation : weakly coupled systems (mean free path≫typical length scale)
- Strongly coupled systems :AdS/CFT (see M. Kamisnski & K. Lansteiner's talks)
- Different derivations :

❖ The semi-classical (non-QFT) approach : classical action + Berry phase $\mathcal{O}(\hbar)$ (quantum correction)

D. T. Son and N. Yamamoto, 12

M. Stephanov and Y. Yin, 12

Berry curvature : $\mathbf{\Omega_p} = \frac{\mathbf{p}}{2|\mathbf{p}|^3}$

➢ EOM :

$$\begin{aligned} \dot{\mathbf{x}} &= \hat{\mathbf{p}} + \hbar\dot{\mathbf{p}} \times \mathbf{\Omega_p}, \\ \dot{\mathbf{p}} &= \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}. \end{aligned} \Longrightarrow \frac{d}{dt} f(x,\mathbf{p}) = \left(\partial_t + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}}\right) f = 0$$

(on-shell) (collisionless)

❖ Not manifestly Lorentz covariance :

➢ f is not a Lorentz scalar (frame dependent)

➢ Modified Lorentz transformation for f : side jumps

J.-Y. Chen, et.al. 14

J.-Y. Chen, D. T. Son, and M. Stephanov, 15

❖ QFT derivation : Wigner-function approach

steady state & large μ :
J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, 13
D. T. Son and N. Yamamoto, 13

❖ World-line formalism (see N. Mueller's talk)

Outline

- CKT from QFT (Wigner-function approach, semi-classical) :
  - Side-jump terms : modified Lorentz (frame) transformation
  - General CKT with collisions Hidaka, Pu, DY, 16

- Applications : non-equilibrium transport for chiral fluids
  - Wigner functions & anomalous transport in global/local equilibrium
  - Non-linear (2^{nd}-order) responses for anomalous transport

  Hidaka, Pu, DY, 17
  Hidaka, DY, 18

(I will focus on R-handed fermions.)

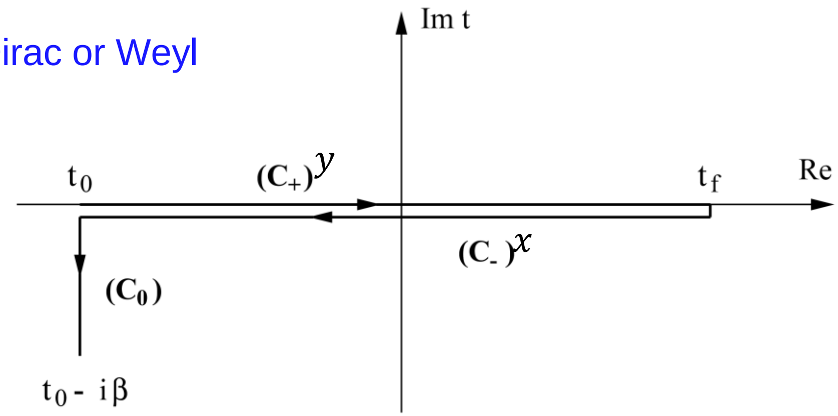
Wigner functions (WF)

- less (greater) propagators :

Dirac or Weyl

$$S^{>}(x,y) = \langle \psi(x) \mathcal{P}\mathcal{U}^{\dagger}(A_\mu, x, y) \psi^{\dagger}(y) \rangle$$
$$S^{<}(x,y) = \langle \psi^{\dagger}(y) \mathcal{P}\mathcal{U}(A_\mu, x, y) \psi(x) \rangle$$

gauge link



review : J. Blaizot, E. Iancu, Phys.Rept. 359 (2002) 355-528

(See also
D. Hou' talk)

$$X = \frac{x+y}{2},\ Y = x - y$$

Wigner functions : $\grave{S}^{<(>)}(q, X) = \int d^4Y e^{\frac{iq\cdot Y}{\hbar}} S^{<(>)}\left(X + \frac{Y}{2}, X - \frac{Y}{2}\right)$

- Wigner functions are always covariant : $J^\mu = \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left(\sigma^\mu \grave{S}^{<}\right)$

- Kadanoff-Baym-like equations up to $\mathcal{O}(\hbar)$: ($q \gg \partial$: weak fields)

$$\sigma^\mu \left(q_\mu + \frac{i\hbar}{2}\Delta_\mu\right) \grave{S}^{<} = \frac{i\hbar}{2}\left(\Sigma^{<}\grave{S}^{>} - \Sigma^{>}\grave{S}^{<}\right), \qquad \Delta_\mu = \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu$$

systematically include collisions

$$\left(q_\mu - \frac{i\hbar}{2}\Delta_\mu\right) \grave{S}^{<}\sigma^\mu = -\frac{i\hbar}{2}\left(\grave{S}^{>}\Sigma^{<} - \grave{S}^{<}\Sigma^{>}\right).$$

(I will focus on R-handed fermions.)

Quantum corrections for WF

- WF up to $\mathcal{O}(\hbar)$: Hidaka, Pu, DY, 16

CME in equilibrium

key eq.

$$\grave{S}^{<\mu}(q,X)=2\pi\bar{\epsilon}(q\cdot n)\left(q^{\mu}\delta(q^2)f_q^{(n)}+\hbar\delta(q^2)S^{\mu\nu}_{(n)}\mathcal{D}_{\nu}f_q^{(n)}+\hbar\epsilon^{\mu\nu\alpha\beta}q_{\nu}F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}f_q^{(n)}\right),$$

$$\mathcal{D}_{\beta}f_q^{(n)}=\Delta_{\beta}f_q^{(n)}-\mathcal{C}_{\beta},\quad C_{\beta}[f]=\Sigma^{<}_{\beta}\bar{f}-\Sigma^{>}_{\beta}f$$

spin tensor : $S^{\mu\nu}_{(n)}=\frac{\epsilon^{\mu\nu\alpha\beta}}{2(q\cdot n)}q_{\alpha}n_{\beta}$

side-jump term :
magnetization current
CVE or non-equilibrium effects
$S^{\mu\nu}_{(n)}\partial_{\nu}f_q^{(n)}$: break translational inv.

- Frame vector n^{μ} : choice of the spin basis $\sigma^0=I$ ($n^{\mu}=(1,\mathbf{0})$) ⟹ $n_{\mu}\sigma^{\mu}=I$
- L.T. of phase-space coordinates is equivalent to inverse L.T. of frames.

$$q'^{\mu}=\Lambda^{\mu}{}_{\nu}q^{\nu},\ X'^{\mu}=\Lambda^{\mu}{}_{\nu}X^{\nu}. \leftrightarrow\ n'^{\mu}=(\Lambda^{-1})^{\mu}{}_{\nu}n^{\nu}$$

- The full WF has to be frame independent ⟹ $f_q^{(n)}$ is frame dependent

$$(\Lambda^{-1})_{\mu}{}^{\nu}\grave{S}'^{<}_{\nu}(q',X')-\grave{S}^{<}_{\mu}(q,X)=0$$

- The modified frame transformation :

$$f_q^{(n')}=f_q^{(n)}+\frac{\hbar\epsilon^{\nu\mu\alpha\beta}q_{\alpha}n'_{\beta}n_{\mu}}{2(q\cdot n)(q\cdot n')}\mathcal{D}_{\nu}f_q^{(n)}$$

Covariant CKT with collisions

- A general form of CKT (for $n^\mu = n^\mu(X)$) :

$$\Delta_\mu \grave{S}^{<\mu} = \Sigma^<_\mu \grave{S}^{>\mu} - \Sigma^>_\mu \grave{S}^{<\mu}$$

Hidaka, Pu, DY, 17

key eq.

$$\delta\left(q^2 - \hbar\frac{B\cdot q}{q\cdot n}\right)\left[q\cdot\mathcal{D} + \frac{\hbar S^{\mu\nu}_{(n)}E_\mu}{q\cdot n}\mathcal{D}_\nu + \hbar S^{\mu\nu}_{(n)}(\partial_\mu F_{\rho\nu})\partial^\rho_q + \hbar(\partial_\mu S^{\mu\nu}_{(n)})\mathcal{D}_\nu\right] f^{(n)}_q = 0$$

energy shift

from X-dep. of the frame

(previous expression of CKT in e.g. Son & Yamamoto, 12 : $n^\mu = (1, \mathbf{0})$ & onshell)

Hidaka, Pu, DY, 16

- Energy-momentum tensor and current :

$$T^{\mu\nu} = \int \frac{d^4q}{(2\pi)^4}\left[q^\mu \grave{S}^{<\nu} + q^\nu \grave{S}^{<\mu}\right], \quad J^\mu = 2\int \frac{d^4q}{(2\pi)^4}\grave{S}^{<\mu}.$$

Global/local equilibrium

- Global equilibrium (T, μ =const. $E = 0$) such that $C[f] = 0$:

equilibrium f for arbitrary n^{μ} :

J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

$$f_q^{\mathrm{eq}(n)} = (e^g + 1)^{-1}, \quad g = \left(\beta q \cdot u - \bar{\mu} + \frac{\hbar S^{\mu\nu}_{(n)}}{2} \partial_\mu (\beta u_\nu)\right)$$

u^{μ}: fluid velocity

- Local equilibrium ($T, \mu \neq$const.) : f can be nontrivially defined in $n^{\mu} = u^{\mu}$.

⇨ $$f_q^{\mathrm{eq}(u)} = \left(\exp\left[\beta(q \cdot u - \mu) + \frac{\hbar q \cdot \omega}{2q \cdot u}\right] + 1\right)^{-1}, \quad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu (\partial_\alpha u_\beta).$$

Hidaka, Pu, DY, 17

- Near local equilibrium :

$$T^{\mu\nu} = u^\mu u^\nu \epsilon - p\Theta^{\mu\nu} + \Pi^{\mu\nu}_{\mathrm{non}} + \Pi^{\mu\nu}_{\mathrm{dis}}, \quad J^\mu = N_0 u^\mu + v^\mu_{\mathrm{non}} + v^\mu_{\mathrm{dis}}, \quad \Theta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$$

- Equilibrium anomalous transport : $F_{\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} B^\mu u^\nu + u_\beta E_\alpha - u_\alpha E_\beta$

$$v^\mu_{\mathrm{non}} = \hbar \sigma_B B^\mu + \hbar \sigma_\omega \omega^\mu \qquad \Pi^{\mu\nu}_{\mathrm{non}} = \hbar \xi_\omega \left(\omega^\mu u^\nu + \omega^\nu u^\mu\right) + \hbar \xi_B \left(B^\mu u^\nu + B^\nu u^\mu\right)$$

$$\sigma_\omega = \frac{T^2}{12}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right), \quad \sigma_B = \frac{\mu}{4\pi^2}, \quad \xi_\omega = \frac{T^3}{6}\left(\bar{\mu} + \frac{\bar{\mu}^3}{\pi^2}\right) = N_0, \quad \xi_B = \frac{T^2}{24}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right) = \frac{\sigma_\omega}{2}.$$

CVE CME Hidaka, Pu, DY, 17

(agree with different approaches, e.g. Son & Surowka, 09.
K. Landsteiner, et.al. Lect. Notes, 13.)

RT approximation & matching conditions

- 2nd-order anomalous responses :
  - Solve CKT for $f_q^{(u)} - f_q^{\rm eq} = \delta f_q = \delta f_q^{(c)} + \hbar \delta f_q^{(Q)}$
  - Constrains from hydrodynamic EOM : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$

$\Rightarrow$ $DT, D\mu, Du^\mu. \qquad D = u\cdot\partial$

- RT approximation for CKT :

$$\Box(q,X) f_q^{(u)} = \mathcal{C}_{\rm full}, \quad \mathcal{C}_{\rm full} = -\frac{1}{\tau_R}\left(q\cdot u + \boxed{\hbar \frac{q^\mu \mathcal{A}_\mu}{(q\cdot u)^2}}\right)\delta f_q$$

For classical RT, we set it to zero.

- Matching conditions :

Hidaka, Pu, DY, 17
Hidaka, DY, 18

From CKT :

$$\partial_\mu J^\mu = -\frac{\hbar}{4\pi^2} E_\mu B^\mu + 2\int_q \left[\delta(q^2) q^\mu + \hbar \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \frac{\partial_{q\nu}\delta(q^2)}{4}\right]\mathcal{C}_\mu$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu + 2\int_q \delta(q^2)\left[q^\nu q^\mu + \frac{\hbar \epsilon^{\sigma\mu\alpha\beta}}{4}\left(\delta^\nu_\sigma\left(q_\beta \partial_\alpha + F_{\alpha\beta}\right) + q^\nu F_{\alpha\beta}\partial_{q\sigma}\right)\right]\mathcal{C}_\mu$$

Energy-momentum cons. $\Rightarrow$ $\boxed{u_\mu \delta J^\mu = u_\mu \delta T^{\mu\nu} = 0}$

classical RT

No non-equilibrium corrections on charge density & energy-density current (same as standard KT)

Non-equilibrium currents

- 2^{nd}-order corrections on currents : $V_\perp^\mu \equiv P^\mu_\nu V^\nu \quad P^{\mu\nu} \equiv \eta^{\mu\nu} - u^\mu u^\nu$

$J^\mu_{Q\perp}$	$\mathcal{O}(\partial)$	$\mathcal{O}(\partial^2)$
$\mathcal{O}(\hbar)$	$\sigma_B B^\mu + \sigma_\omega \omega^\mu$	$\tau_R\Big[\epsilon^{\mu\nu\alpha\beta}u_\nu\big(\hat{\gamma}_E\partial_\alpha E_\beta + \hat{\gamma}_\mu E_\alpha \partial_\beta \mu + \hat{\gamma}_T E_\alpha \partial_\beta T + \hat{\gamma}_{T\mu}(\partial_\alpha T)(\partial_\beta \mu)\big) + \delta\hat{\sigma}_{BL}\theta B^\mu + \delta\hat{\sigma}_{BH}\pi^{\mu\nu}B_\nu + \delta\hat{\sigma}_{\omega L}\theta\omega^\mu + \delta\hat{\sigma}_{\omega H}\pi^{\mu\nu}\omega_\nu\Big]$

$\theta \equiv \partial\cdot u$

$\pi^{\mu\nu} \equiv P^\mu_\rho P^\nu_\sigma(\partial^\rho u^\sigma + \partial^\sigma u^\rho - 2\eta^{\rho\sigma}\theta/3)/2$

key findings: Hidaka, Pu, DY, 17
Hidaka, DY, 18

Viscous corrections for CME/CVE

- 2^{nd}-order quantum transport coefficients : $\mathcal{P}, \mathcal{T} - odd$
- The origin of viscous corrections : time-dep. B

$$\partial_\nu \tilde{F}^{\mu\nu} = 0$$

$$\Rightarrow \boxed{u\cdot\partial B^\rho + B^\rho \partial\cdot u - B\cdot\partial u^\rho + u^\rho B^\mu u\cdot\partial u_\mu} + \epsilon^{\rho\mu\alpha\beta}\left(u_\beta \partial_\mu E_\alpha + u_\mu E_\alpha u\cdot\partial u_\beta\right) = 0$$

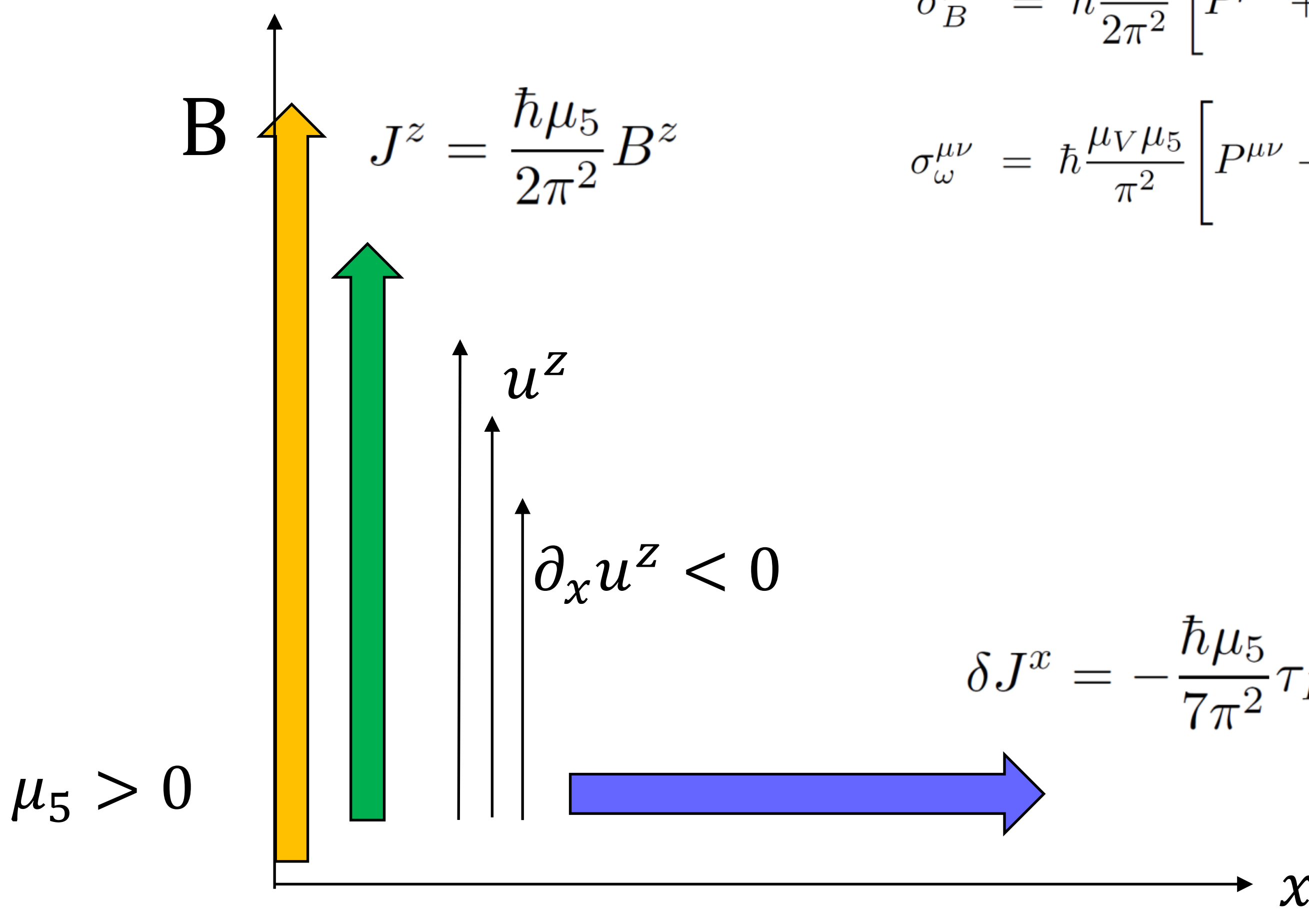
High-temperature limit

- Viscous corrections on CME/CVE ($\bar{\mu} \ll 1$) : $J^{\mu}_{Q\perp} = \sigma^{\mu\nu}_{B} B_{\nu} + \sigma^{\mu\nu}_{\omega}\omega_{\nu},$

Hidaka, DY, 18

$$\sigma^{\mu\nu}_{B} = \hbar\frac{\mu_5}{2\pi^2}\left[P^{\mu\nu} + \tau_R\left(\frac{10P^{\mu\nu}}{9}\theta - \frac{2\pi^{\mu\nu}}{7}\right)\right],$$

$$\sigma^{\mu\nu}_{\omega} = \hbar\frac{\mu_V\mu_5}{\pi^2}\left[P^{\mu\nu} - \tau_R\left(\frac{37P^{\mu\nu}}{264}\theta + \frac{10\pi^{\mu\nu}}{63}\right)\right].$$



$$\delta J^x = -\frac{\hbar\mu_5}{7\pi^2}\tau_R B^z \partial_x u^z$$

Conclusions & outlook

- We have presented a self-consistent formalism for CKT as a useful tool to study non-equilibrium anomalous transport for Weyl fermions.
- In such a formalism, Lorentz covariance is manifested and both background fields and collisions are involved.
- Novel 2nd –order non-equilibrium anomalous transport including the viscous corrections on CME/CVE is found.

- Phenomenological applications : HIC, Weyl semimetals, etc.
  - More realistic collisions?
- Theoretical issues : how to go beyond the weak-field limit?
  - Landau-level WF : subjected to constant B (no Lorentz sym.).
  - Perturbative sol. up to $\mathcal{O}(\hbar^2)$: new phenomena expected, interpolation with L.L. (IR singularity could be more severe).

relevant work :
J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, 18

e.g. $\Delta j_V^0 = q_f \frac{C_A}{2} (\boldsymbol{B} \cdot \boldsymbol{\omega})$

K. Hattori, Y. Yin, 16

Thank you!

Solving WF perturbatively

- To derive anomalous(quantum) corrections : Hidaka, Pu, DY, 16
  - work in Weyl bases ($\grave{S}^{<}=\bar{\sigma}^{\mu}\grave{S}^{<}_{\mu}$, for R-handed fermions).
  - solve for the perturbative solution : $\grave{S}^{<}_{\mu}=2\pi q_{\mu}\delta(q^2)f+\hbar\delta S^{<}_{\mu}$
  - usually choosing a basis for $\sigma^0=I$.
  - a more general choice : $n_{\mu}\sigma^{\mu}=I$ with n^{μ} being a frame vector.
- The frame vector can be regarded as the zeroth component of a vierbein.

Local coordinates : $\sigma^a=(I,\sigma^i)$, $\bar{\sigma}^a=(I,-\sigma^i)$ (local spin direction) ⟶ Global spacetime : $\sigma^{\mu}=e^{\mu}_a\sigma^a$

$$n^{\mu}\equiv e^{\mu}_0$$

$n^2=1$

flat : $e^{\mu}_a(X)=\delta^{\mu}_a$, $n^{\mu}=(1,\mathbf{0})$

- Global spacetime coordinate transf. ⟷ Frame transf. $n^{\mu}\rightarrow n'^{\mu}$
- KB eq. :

$$\mathcal{D}\cdot\grave{S}^{<}=0,\quad q\cdot\grave{S}^{<}=0,$$

$$\mathcal{D}_{\beta}f^{(n)}_q=\Delta_{\beta}f^{(n)}_q-\mathcal{C}_{\beta},$$

$$\Delta_{\mu}=\partial_{\mu}+F_{\nu\mu}\partial/\partial q_{\nu}$$

$$C_{\beta}[f]=\Sigma^{<}_{\beta}\bar{f}-\Sigma^{>}_{\beta}f$$

give WF

$$\begin{cases} 2\pi\delta(q^2)\left(q\cdot n\mathcal{D}_{\mu}-q_{\mu}n\cdot\mathcal{D}\right)f=-2\epsilon_{\alpha\mu\nu\beta}n^{\alpha}q^{\nu}\delta\grave{S}^{<\beta}, \\ 2\pi\epsilon_{\alpha\mu\nu\beta}\delta(q^2)n^{\alpha}q^{\nu}\mathcal{D}^{\beta}f=2\left(q\cdot n\delta\grave{S}^{<}_{\mu}-q_{\mu}n\cdot\delta\grave{S}^{<}\right). \end{cases}$$

Origin of side-jumps

- An alternative way to derive WF (no background fields & no collisions) with a "scalar f". Hidaka, Pu, DY, 16
- Nontrivial phase for massless particles with helicity: Helicity dep. phase

S. Weinberg, QFT, Vol. I

$$\text{L. T. :}\ v_+(\Lambda p) = e^{i\Phi(p,\Lambda)}U(\Lambda)v_+(p)$$

- Second quantization : $\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2|\mathbf{p}|}} e^{-ip\cdot x} v_+(p) a_{\mathbf{p}}$ (neglect anti-fermions)

$$\grave{S}^<(x,y) = \langle \psi^\dagger(y)\psi(x)\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2|\mathbf{p}|}} \int \frac{d^3p'}{(2\pi)^3\sqrt{2|\mathbf{p}'|}} v_+(p)v_+^\dagger(p')\langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle e^{i(p'-p)\cdot X - \frac{i}{2}(p'+p)\cdot Y}$$

Helicity dep. phase

- Under the L.T. : $N(p',p) \equiv \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle \to e^{-i\left(\Phi(\Lambda,p)-\Phi(\Lambda,p')\right)}\langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle$
- Could we define a scalar distribution function?
- Introduce a phase field : $\phi(p) \to \phi'(\Lambda p) = \phi(p) - \Phi(p,\Lambda)$
- Reparametrize the wave function and annihilation operator :

$$v_+(p) \to e^{i\phi(p)}v_+(p)$$
$$a_{\mathbf{p}} \to e^{-i\phi(p)}a_{\mathbf{p}}$$

Manifestation of Lorentz symmetry

- From $a_{\mathbf{p}} \to e^{-i\phi(p)} a_{\mathbf{p}}$, we may define a scalar distribution function :

scalar — non-scalar

$$\check{N}(p',p) \equiv e^{-i(\phi(p)-\phi(p'))} N(p',p) \quad \Rightarrow \quad \check{f}(q,X) \equiv \int \frac{d^3\bar{p}}{(2\pi)^3} \check{N}\left(q-\frac{\bar{p}}{2}, q+\frac{\bar{p}}{2}\right) e^{-i\bar{p}\cdot X}$$

$$N(p',p) \equiv \langle a^\dagger_{\mathbf{p}'} a_{\mathbf{p}} \rangle$$

- The derivation of WF implicitly involves the contribution from anti-fermions.

Key eq. : $\mathrm{Im}\left[c^\dagger_\pm(q)\sigma^k \frac{\partial}{\partial q_\beta} c_\pm(q)\right] = \mp a^\beta_\pm v^k - \frac{1}{2|\mathbf{q}|}\epsilon^{kj\beta} v_j$ from $c_+(p)c^\dagger_+(p) + c_-(p)c^\dagger_-(p) = I$

$$\Rightarrow \grave{S}^<_\mu(q,X) = (2\pi)\theta(q^0)\delta(q^2)\left(q_\mu\left(1-\hbar(\partial^\nu_q\phi - a^\nu)\partial_\nu\right) + \hbar\delta_{\mu i}\epsilon_{ijk}\frac{q_j}{2|\mathbf{q}|}\partial_k\right)\check{f}(q,X),$$

covariant — scalar

- Compare to the previous expression :

$$\grave{S}^{<\mu} = 2\pi\theta(q^0)\delta(q^2)\left(q^\mu + \hbar\delta^{\mu i}\epsilon^{ijk}\frac{q_j}{2|\mathbf{q}|}\partial_k\right) f(q,X)$$

"the origin of side-jumps"

$$\Rightarrow f(q,X) = \check{f}\left(q_\mu, X^\mu - \hbar\partial^\mu_q\phi(q) + \hbar a^\mu\right)$$

non-scalar

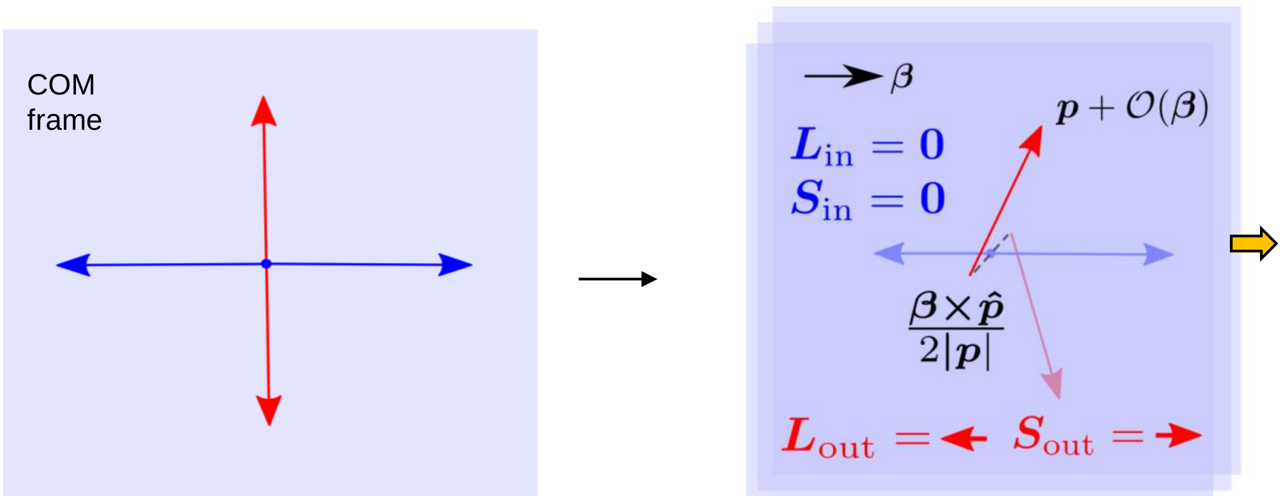
- Choices of phase field corresponds to the gauge degrees of freedom for the Berry connection.
- The perturbative solution could be uniquely determined by Lorentz symmetry.

A no-jump frame

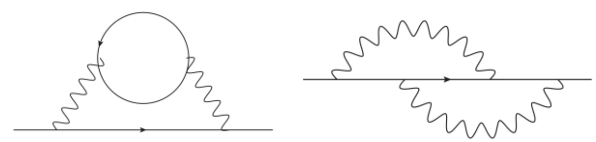
- Conservation of the angular momentum : COM frame = no-jump frame

  J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

- A phenomenological argument for side jumps in collisions :



⇨ side-jump corrections for C

E.g. 2-2 Coulomb scattering ($n_c^\mu = (q^\mu + q'^\mu)/\sqrt{s}$) : $\Sigma =$ 

$$q^\mu C_\mu\left[f^{(n_c)}\right] = \frac{1}{4}\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} |\mathcal{M}|^2\left[\bar{f}^{(n_c)}(q)\bar{f}^{(n_c)}(q')f^{(n_c)}(k)f^{(n_c)}(k') - f^{(n_c)}(q)f^{(n_c)}(q')\bar{f}^{(n_c)}(k)\bar{f}^{(n_c)}(k')\right]$$

Hidaka, Pu, DY, 16

"no side-jump corrections"

The no-jump frame in 2-2 scattering

- Introducing a frame : $\tilde{S}^{<}_{\mu} = 2\pi\delta(q^2)\left[q_\mu f - \frac{\hbar}{2q\cdot u}\epsilon_{\mu\nu\alpha\beta}u^\nu q^\alpha\left(\partial^\beta f + \Sigma^{>\beta} f - \Sigma^{<\beta}\bar{f}\right)\right]$
- Conservation of the angular momentum : COM frame = no-jump frame

  J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)
- Choosing the COM frame : $u_c^\mu = (q+q')^\mu/\sqrt{s}$

$$\mathcal{P}(q',k,k') = 4e^4\left(\frac{1}{(q-k)^2} + \frac{1}{(q-k')^2}\right)^2,$$

$$\Sigma^{<}_{\mu} = \int_{q',k,k'} \mathcal{P}(q',k,k')\tilde{S}^{>}_{\mu}(q')\left(\tilde{S}^{<}(k)\cdot\tilde{S}^{<}(k')\right),$$

$$\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} = \int \frac{d^3\mathbf{q}' d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^5}\frac{\delta^{(4)}(q+q'-k-k')}{8E_{q'}E_kE_{k'}}.$$

$$\Rightarrow \tilde{S}^{>\mu}\Sigma^{<}_{\mu} = 2\pi\delta(q^2)\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'}\mathcal{P}(q',k,k')(k\cdot k')(q\cdot q') \times \bar{f}^{(u_c)}(q)\bar{f}^{(u_c)}(q')f^{(u_c)}(k)f^{(u_c)}(k'),$$

- No "explicit" $\mathcal{O}(\hbar)$ corrections in C_μ : $\partial_\mu \tilde{S}^{\mu<} = 2\pi\delta(q^2)q^\mu C_\mu\left[f^{(u_c)}\right],$

$$q^\mu C_\mu\left[f^{(u_c)}\right] = \frac{1}{4}\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} |\mathcal{M}|^2 \times \left[\bar{f}^{(u_c)}(q)\bar{f}^{(u_c)}(q')f^{(u_c)}(k)f^{(u_c)}(k') - f^{(u_c)}(q)f^{(u_c)}(q')\bar{f}^{(u_c)}(k)\bar{f}^{(u_c)}(k')\right]$$

no side-jumps

- The final expression is concise but not pragmatic.
- u_c is momentum-dependent : hard to write down $f^{(u_c)}$ with different q'

Anomalous Hydrodynamics

- Anomalous hydro : ($T, \bar{\mu} = \frac{\mu}{T}, u^\mu$: free parameters)

E & B : $n^\nu F_{\mu\nu} = E_\mu,$

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} n_\nu F_{\alpha\beta} = B^\mu$$

constrains : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$

in the local rest frame $u^\mu \approx (1, \mathbf{0})$

$$\frac{\partial_0 T}{T} = \hbar \boldsymbol{\mathcal{E}} \cdot \left(\tilde{T}_B \mathbf{B} + \tilde{T}_\omega \boldsymbol{\omega} T\right), \quad \partial_0 \bar{\mu} = \hbar \boldsymbol{\mathcal{E}} \cdot \left(\tilde{\mu}_B \mathbf{B} + \tilde{\mu}_\omega \boldsymbol{\omega} T\right), \quad \mathcal{E}_\nu = E_\nu + T\partial_\nu \bar{\mu}$$

$$\partial_0 \mathbf{u} = \partial_0 \mathbf{u}^{(0)} + \hbar \partial_0 \delta \mathbf{u},$$

Hidaka, Pu, DY, 17

$$\partial_0 \mathbf{u}^{(0)} = -\frac{\nabla T}{T} + \frac{N_0 \boldsymbol{\mathcal{E}}}{4p}, \quad \hbar \partial_0 \delta \mathbf{u} = \hbar \left(\tilde{U}_E \nabla \times \mathbf{E} + \tilde{U}_T \frac{\mathbf{E} \times \nabla T}{T} + \tilde{U}_{\bar{\mu}} \mathbf{E} \times \nabla \bar{\mu}\right)$$

- Free to choose a different frame in hydrodynamics :

e.g. Landau frame :

$$\tilde{u}^\mu = u^\mu + \hbar \frac{\xi_\omega \omega^\mu + \xi_B B^\mu}{\epsilon + p},$$

$$T^{\mu\nu}_{\rm leq} = \tilde{u}^\mu \tilde{u}^\nu \epsilon - p\tilde{\Theta}^{\mu\nu},$$

$$J^\mu_{\rm leq} = N_0 \tilde{u}^\mu + \hbar \tilde{\sigma}_\omega \omega^\mu + \hbar \tilde{\sigma}_B B^\mu,$$

$$\partial_0 \tilde{\mathbf{u}} = -\frac{\nabla T}{T} + \frac{N_0 \boldsymbol{\mathcal{E}}}{4p} - \frac{\hbar \sigma_\omega}{8p} \boldsymbol{\omega} \times \mathbf{B}$$

(yields Chiral Alfven waves)

N. Yamamoto, Phys. Rev. Lett. 115, 141601 (2015)

Non-equilibrium distribution functions

- Solving CKT for non-equilibrium fluctuations of f : $f^{(u)}_q - f^{\rm eq}_q = \delta f_q = \delta f^{(c)}_q + \hbar \delta f^{(Q)}_q$

classical quantum

$$\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial^\rho_q + \hbar\big(\hat{\Pi}^\mu_1 + \hat{\Pi}^\mu_2\big)\Delta_\mu\right] f^{(u)}_q = \mathcal{C}_{\rm full},\ \ \partial_\mu S^{\mu\nu}_{(u)} = \hat{\Pi}^\nu_1 + \hat{\Pi}^\nu_2.$$

- $\mathcal{C}_{full}$: $\mathcal{C}_{\rm full} = q^\mu \mathcal{C}_\mu + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\mathcal{C}_\nu + \hbar(\partial_\rho S^{\rho\mu}_{(u)})\mathcal{C}_\mu$ $\xrightarrow{\text{RTA}}$ $\mathcal{C}_{\rm full} = -\frac{1}{\tau_R}\left(q\cdot u + \hbar\frac{q^\mu \mathcal{A}_\mu}{(q\cdot u)^2}\right)\delta f_q$

$$\partial_\rho S^{\rho\mu}_{(u)} = \frac{1}{2}\left[\omega^\mu - \frac{(q\cdot\omega)u^\mu}{q\cdot u} - \frac{(q\cdot\omega)q^\mu}{(q\cdot u)^2}\right] + \frac{\epsilon^{\mu\nu\alpha\beta}}{2q\cdot u}\left(q_\alpha\kappa_{\beta\nu} - \frac{u_\nu q_\alpha q^\rho}{q\cdot u}(\sigma_{\beta\rho}+\kappa_{\beta\rho})\right)$$

- The perturbative solution : Hidaka, Pu, DY, 17

$$\delta f_q = -\frac{\tau_R}{q\cdot u}\left(1 - \frac{\hbar q\cdot\mathcal{A}}{q\cdot u}\right)\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial^\rho_q + \hbar\big(\hat{\Pi}^\mu_1 + \hat{\Pi}^\mu_2\big)\Delta_\mu\right] f^{\rm eq}_q$$

- Different contributions for $\hbar\delta f^{(Q)}_q = \delta f^{\mathcal{K}}_q + \delta f^{\mathcal{H}}_q + \delta f^{\mathcal{C}}_q$

from CKT, from hydro EOM, from collisions

(neglect nonlinear classical responses)

- The non-equilibrium four current :

$$\delta J^\mu_Q = 2\hbar\int\frac{d^4q}{(2\pi)^3}\bar{\epsilon}(q\cdot u)\delta(q^2)\left[q^\mu\delta f^{(Q)}_q - \frac{1}{2}\left(\epsilon^{\mu\nu\alpha\beta}\frac{u_\nu q_\alpha}{q\cdot u}\Delta_\beta + \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}}{2}\right)\delta f^{(c)}_q\right]$$

Anomalous Hall currents

- Non-equilibrium (quantum) charge currents : Hidaka, Pu, DY, 17

$$\delta \boldsymbol{J}_{QR/L} = \mp \frac{\hbar \tau_R}{84\pi^2}\left[\mu_{R/L}\nabla\times\boldsymbol{E} + \frac{7\mu_{R/L}}{2T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{1}{2}\boldsymbol{E}\times\nabla\mu_{R/L} + \frac{12\mu_{R/L}}{T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \ll 1$$

emerge from hydro.

$$\delta \boldsymbol{J}_{QR/L} = \pm \frac{\hbar \tau_R}{4\pi^2}\left[\frac{\pi^2 T^2}{3\mu_{R/L}}\nabla\times\boldsymbol{E} + \frac{2\mu_{R/L}}{3T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{2}{3}\boldsymbol{E}\times\nabla\mu_{R/L} - \frac{2\mu_{R/L}}{3T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \gg 1$$

- Some observations :
  - B and ω do not contribute to nonlinear corrections in the inviscid case.
  - The transport coefficients are parity-odd (charged currents from chiral imbalance).
- A caveat : part of anomalous Hall currents vanishes without hydro in the "naïve" RT ($\tau_R =$ const.) approximation.

$$\tau_R(T,\mu):\ \delta \boldsymbol{J}_{\tau_R} = \frac{\hbar}{12\pi^2}\Big((\mu\partial_T\tau_R - I_1\partial_{\bar{\mu}}\tau_R)\boxed{(\nabla\mu)\times(\nabla T)} + \bar{\mu}(\bar{\mu}\partial_{\bar{\mu}}\tau_R - T\partial_T\tau_R)\boxed{\mathbf{E}\times\nabla T} - \bar{\mu}(\partial_{\bar{\mu}}\tau_R)\mathbf{E}\times\nabla\mu\Big)$$

exist without hydro.
(could exist in Weyl semimetals?)

Matching Conditions & Entropy Production

- From CKT :

$$\partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B}) + 2\int_q \Big[\delta(q^2)q^\mu + \hbar\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}\delta(q^2)}{4}\Big]\mathcal{C}_\mu$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu}J_\mu + 2\int_q \delta(q^2)\Big[q^\nu q^\mu + \frac{\hbar\epsilon^{\sigma\mu\alpha\beta}}{4}\Big(\eta^\nu_\sigma\left(q_\beta\partial_\alpha + F_{\alpha\beta}\right) + q^\nu F_{\alpha\beta}\partial_{q\sigma}\Big)\Big]\mathcal{C}_\mu$$

- Matching conditions for the classical RT approximation : $u_\mu \delta J^\mu = 0,$ $u_\mu \delta T^{\mu\nu} = 0$

- Entropy currents : $s^\mu = \frac{1}{T}\Big(pu^\mu + T^{\mu\nu}u_\nu - \mu J^\mu\Big) + \hbar(D_B B^\mu + D_\omega \omega^\mu)$

$$T^{\mu\nu} = u^\mu u^\nu \epsilon - pP^{\mu\nu} + \Pi^{\mu\nu}_{\text{dis}} + \Pi^{\mu\nu}_{\text{non}}$$

- 2nd law is protected by classical parts :

$$\partial\cdot s = \frac{1}{T}\Big[\Pi^{\mu\nu}_{\text{dis}}\partial_\mu u_\nu - (E_\mu + T\partial_\mu\bar{\mu})\delta J^\mu\Big]$$

Hidaka, DY, 18