Disentangle covariant Wigner functions for chiral fermions

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Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions
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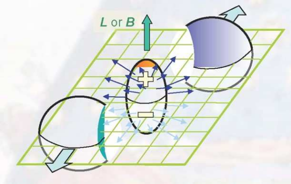
Outline

- Introduction
- Wigner functions of chiral fermions in external electromagnetic field
  [J. Gao, et al., 1203.0725; J. Gao, QW, 1504.07334]
- Disentangle covariant Wigner functions for chiral fermions
  [J. Gao, Z.T. Liang, QW, X.N. Wang, 1802.06216]

Introduction

- High energy HIC:

$$v \sim 1 - \frac{m_p^2}{2s}, \quad \gamma \sim \frac{\sqrt{s}}{m_p}$$



- Electric field in the rest frame of one nucleus

$$\mathbf{E} = \frac{Ze}{R^2}\hat{\mathbf{r}}$$

Boost to Lab frame ($v_z \approx 0.99995c$ for 200 GeV)

$$\begin{aligned} \mathbf{B} &= -\gamma \mathbf{v}_z \times \mathbf{E} \\ eB &\sim 2\gamma v_z \frac{Ze^2}{R^2} \sim m_\pi^2 \sim 10^{18}\mathrm{Gs} \end{aligned}$$

Fukushima, Kharzeev, Warringa (2008)

Skokov (2009), Deng, Huang (2012)

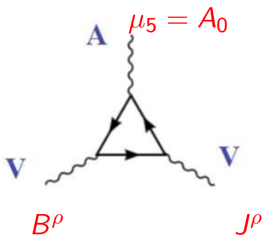
many others ...

Introduction

- Anomalies: the classical symmetry of the Lagrangian broken by quantum effects:
  (a) chiral symmetry by axial anomaly
  (b) scale symmetry by scale anomaly
- Anomalies imply correlations between currents. VAA coupling: if A^μ ($\mu_5 = A_0$) and V^μ (Magnetic field B^μ) are background field, then V^μ (vector current J^μ) is induced.
- Chiral Magnetic effect:

$$J^\rho = \frac{Q^2\hbar}{2\pi^2}\mu_5 B^\rho$$

It's a quantum and topological effect (no higher order corrections).



Kharzeev, McLerran, Warringa (2008)

Fukushima, Kharzeev, Warringa (2008)

Recent review, e.g.:

Kharzeev, Liao, Voloshin, Wang (2015)

Talks: H.Z.Huang, G.Wang, W.Li, D.F.Hou, K.Landsteiner, R.Lacey, etc.

Introduction

- The Wigner function

$$W_{\alpha\beta}(X,p)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot y/\hbar}\left\langle\bar{\psi}_\beta\left(x_+\right)U\left(x_+,x_-\right)\psi_\alpha\left(x_-\right)\right\rangle$$

  where $x_\pm=X\pm y/2$.
- The EOM for Wigner function

$$\left(\gamma\cdot p+i\frac{1}{2}\gamma\cdot\nabla_X\right)W(X,p)=0$$

- $W(X,p)$ (4×4 complex matrix) can be expanded in 16 Clifford algebra (Scalar, Pseudoscalar, Vector, Axial vector, Tensor) $\Rightarrow$ 16 variables $\Rightarrow$ 8 variables for chiral fermions (Vector, Axial vector) $\Rightarrow$ 4 for RH and 4 for LH.

Talks in this conference:
P.F. Zhuang, D.L. Yang, X.G. Huang

Introduction

- The question has long been asked: to what extent that a chiral fermion quantum system in electromagnetic fields can be described by the quasi-classical distribution function.
- This puzzle has been solved by a rigorous proof of a theorem based on the semi-classical expansion in $\hbar$ for the covariant Wigner functions.
- This remarkable property of chiral fermions will significantly simplify the kinetic simulation of chiral effects in heavy ion collisions and Dirac/Weyl semimetals.

Chiral fermions in Pauli spinor

- Dirac equation for right-handed (left-handed) fermions $[D_\rho = \hbar\partial_\rho + iQA_\rho]$

$\gamma^\rho = \begin{pmatrix} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{pmatrix}$, $\sigma^\rho = (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\rho = (1, -\boldsymbol{\sigma})$

γ^ρ in Weyl basis

$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$

$\bar{\psi} = (\bar{\chi}_R, \bar{\chi}_L)$

$$\begin{aligned} \mathscr{L} &= \underline{\bar{\psi} i\gamma^\rho D_\rho \psi} = \chi_R^\dagger i\sigma\cdot D\chi_R + \chi_L^\dagger i\bar{\sigma}\cdot D\chi_L \\ \sigma\cdot D\chi_R &= 0, \quad \chi_R^\dagger \sigma\cdot \overleftarrow{D}^\dagger = 0 \\ \bar{\sigma}\cdot D\chi_L &= 0, \quad \chi_L^\dagger \bar{\sigma}\cdot \overleftarrow{D}^\dagger = 0 \end{aligned} \tag{1}$$

- Two-point Green function for right-handed fermions

$$S_{ab}(x_1, x_2) = \left\langle \chi_b^\dagger(x_2)\chi_a(x_1) \right\rangle \tag{2}$$

$$\sigma\cdot D_{x_1} S(x_1, x_2) = 0 \tag{3}$$

Blaizot, Iancu (2002); Hidaka, Pu, Yang (2017, 2018);

Equation for 2-point function: gradient expansion

- Change of variables

$$X = \frac{1}{2}(x_1 + x_2),\ y = x_2 - x_1 \tag{4}$$

$$\partial_{x_1} = \frac{1}{2}\partial_X - \partial_y,\ \partial_{x_2} = \frac{1}{2}\partial_X + \partial_y \tag{5}$$

- Equation for 2-point function

$$\begin{aligned} 0 &= \sigma \cdot D_{x_1} S(x_1, x_2) = \sigma \cdot [\hbar\partial_{x_1} + \underbrace{iQA(x_1)}]S(x_1, x_2) \\ &\approx \sigma \cdot \Bigg[\frac{1}{2}\hbar\partial_X - \hbar\partial_y \\ &\quad \overbrace{+ iQA(X) - iQ\frac{1}{2}y \cdot \partial_X A(X)}\Bigg] S(X, y) \end{aligned} \tag{6}$$

Gauge invariant 2-point function

- Gauge invariant 2-point Green function with gauge link

$$W(X,y) = U(X, X-\frac{y}{2})S(X-\frac{y}{2}, X+\frac{y}{2})U(X+\frac{y}{2}, X) \quad (7)$$

$$U(x_1, x_2) = \exp\left[-i\frac{Q}{\hbar}\int_{x_2}^{x_1} dz \cdot A(z)\right] \quad (8)$$

- Weak background field $Qy \cdot A \ll 1$

$$\begin{aligned} W(X,y) &\simeq e^{-iQyA(X)/\hbar}S(X,y) \\ S(X,y) &\simeq e^{iQyA(X)/\hbar}W(X,y) \end{aligned} \quad (9)$$

Gauge invariant 2-point function and Wigner function

- Fourier transform

$$\begin{aligned} W(X,p) &= \int d^4y e^{-ip\cdot y/\hbar} W(X,y) \\ &= \int d^4y e^{-ip_{\mathrm{c}}\cdot y/\hbar} S(X,y) \\ &= S(X,p_{\mathrm{c}}) \\ S(X,y) &= \frac{1}{(2\pi)^4}\int d^4p_{\mathrm{c}} e^{ip_{\mathrm{c}}\cdot y/\hbar} S(X,p_{\mathrm{c}}) \end{aligned} \tag{10}$$

where $p_{\mathrm{c}} = p + QA$ is the canonical momentum.

Some early works on Wigner functions of QED/QCD:

Heinz (1983); Elze, Gyulassy, Vasak (1986); Vasak, Gyulassy, Elze (1987);

Zhuang, Heinz (1996); Blaizot, Iancu (2002); QW, Redlich, Stoecker, Greiner (2002)

Equation for Wigner function

- Equation for 2-point function

$$\sigma\cdot\left[\frac{1}{2}\hbar\partial_X-\hbar\partial_y+iQA-iQ\frac{1}{2}y\cdot\partial_X A\right]S(X,y)=0 \quad (11)$$

- Momentum space, $\hbar\partial_y^\mu\rightarrow ip_c^\mu$ and $y^\mu\rightarrow i\hbar\frac{\partial}{\partial p_\mu^c}$

$$\sigma\cdot\left[\frac{1}{2}\hbar\partial_X+\underbrace{\frac{1}{2}\hbar Q(\partial_\nu^X A)\frac{\partial}{\partial p_\nu^c}}_{=-\hbar\frac{1}{2}QF_{\mu\nu}\partial_{p_c}^\nu+\hbar\frac{1}{2}Q(\partial_\mu^X A_\nu)\frac{\partial}{\partial p_\nu^c}}\underbrace{-ip_c+iQA}_{-ip}\right]S(X,p_c) = 0 \quad (12)$$

Equation for Wigner function

- Using

$$\frac{1}{2}\sigma\cdot\left(\partial_X + Q\partial_X A_\nu \frac{\partial}{\partial p_\nu^{\mathrm{c}}}\right) S(X,p_c)$$
$$\rightarrow\quad \frac{1}{2}\sigma\cdot\partial_X S(X,p+QA(X)) = \frac{1}{2}\sigma\cdot\partial_X W(X,p) \qquad (13)$$

- we obtain

$$\left[\sigma\cdot p + \frac{1}{2}i\hbar\sigma\cdot\nabla\right] W_R(X,p) \quad = \quad 0 \qquad (14)$$

$\nabla_\mu = \partial_\mu^X - QF_{\mu\nu}\partial_p^\nu$

- In the same way, we have

$$W_R(X,p)\left[-\sigma\cdot p + \frac{1}{2}i\hbar\sigma\cdot\overleftarrow{\nabla}\right] = 0 \qquad (15)$$

- For left-handed, replace $\sigma^\mu \rightarrow \bar{\sigma}^\mu$.

Equation for Wigner function

- The Wigner functions for right- and left-handed fermions

$$\begin{aligned} W_R(x,p) &= \bar{\sigma}^\mu \mathscr{J}_\mu^+, \\ W_L(x,p) &= \sigma^\mu \mathscr{J}_\mu^-, \end{aligned} \tag{16}$$

where $\mathscr{J}_\mu^s$ can be extracted by taking traces

$$\mathscr{J}_\mu^+ = \frac{1}{2}\mathrm{Tr}\,(\sigma_\mu W_R),\ \mathscr{J}_\mu^- = \frac{1}{2}\mathrm{Tr}\,(\bar{\sigma}_\mu W_L). \tag{17}$$

- Take the sum and difference of Eqs. (14,15) and take trace of resulting equations we have

$$\begin{aligned} \mathrm{Tr}\,(\sigma^\mu\bar{\sigma}^\rho)\,\nabla_\mu \mathscr{J}_\rho^+ &= \nabla^\rho \mathscr{J}_\rho^+ = 0 \\ \mathrm{Tr}\,(\sigma^\mu\bar{\sigma}^\rho)\,p_\mu \mathscr{J}_\rho^+ &= p^\rho \mathscr{J}_\rho^+ = 0 \end{aligned} \tag{18}$$

Equation for Wigner function

- Eq. (14) can be rewritten as

$$\sigma^{\mu}\bar{\sigma}^{\nu}\left[p_{\mu}\mathscr{J}_{\nu}^{+}+\frac{1}{2}i\hbar\nabla_{\mu}\mathscr{J}_{\nu}^{+}\right] = 0$$

$\rightarrow$ Multiply $\sigma^{\lambda}\bar{\sigma}^{\rho}$ and take trace

$$\mathrm{Tr}\left(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\lambda}\bar{\sigma}^{\rho}\right)\left[p_{\mu}\mathscr{J}_{\nu}^{+}+\frac{1}{2}i\hbar\nabla_{\mu}\mathscr{J}_{\nu}^{+}\right] = 0$$

$\rightarrow$ Take real part

$$2(p^{\lambda}\mathscr{J}_{+}^{\rho}-p^{\rho}\mathscr{J}_{+}^{\lambda})+\hbar\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}_{\nu}^{+} = 0 \quad (19)$$

where we have used

$$\mathrm{Tr}\left(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\lambda}\bar{\sigma}^{\rho}\right) = 2(g^{\mu\nu}g^{\lambda\rho}+g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho}-i\epsilon^{\mu\nu\lambda\rho})$$

Equation for Wigner function at $O(\hbar)$

- The set of equations for the vector component $\mathscr{J}_s^{\mu}(x,p)$ for chiral fermions (with chirality $s=\pm$)

$\nabla_{\mu}=\partial_{\mu}^{X}-QF_{\mu\nu}\partial_{p}^{\nu}$

$$
\begin{aligned}
p^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0\\
\nabla^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0\\
2s(p^{\lambda}\mathscr{J}_{s}^{\rho}-p^{\rho}\mathscr{J}_{s}^{\lambda}) &= -\hbar\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}_{\nu}^{s} \qquad (20)
\end{aligned}
$$

- where $\mathscr{J}_{\mu}^{s}(x,p)$ are defined as

$$
\mathscr{J}_{\mu}^{s}(x,p) = \frac{1}{2}[\mathscr{V}_{\mu}(x,p)+s\mathscr{A}_{\mu}(x,p)] \qquad (21)
$$

Vasak, Gyulassy, Elze (1987); Gao, Liang, Pu, QW, Wang (2012); Chen, Pu, QW, Wang (2013); Gao, QW (2015); Gao, Pu, QW (2017)

Equation for Wigner function at any order of $\hbar$

- The set of equations for the vector component $\mathscr{J}_s^{\mu}(x,p)=(\mathscr{J}_0,\mathscr{J}^i)$ for chiral fermions (with chirality $s=\pm$)

Equation 3-8:
3 evolution eqs for $\mathscr{J}^i$
3 constraint eqs for $\mathscr{J}^i$ and $\mathscr{J}_0$

$$
\begin{aligned}
\Pi^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \\
G^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \\
2s(\Pi^{\lambda}\mathscr{J}_s^{\rho}-\Pi^{\rho}\mathscr{J}_s^{\lambda}) &= -\hbar\epsilon^{\mu\nu\lambda\rho}G_{\mu}\mathscr{J}_{\nu}^{s}
\end{aligned} \tag{22}
$$

$\longrightarrow$ Equation 1: mass-shell condition

$\longrightarrow$ Equation 2: evolution equation for $\mathscr{J}_0$

- where

$j_1(z)=(\sin z - z\cos z)/z^2$

$$
\begin{aligned}
\Pi^{\mu} &= p^{\mu}-\hbar\frac{1}{2}j_1\left(\frac{\hbar}{2}\partial_x\cdot\partial_p\right)QF^{\mu\nu}\partial_{\nu}^{p} \\
G^{\mu} &= \partial_x^{\mu}-j_0\left(\frac{\hbar}{2}\partial_x\cdot\partial_p\right)QF^{\mu\nu}\partial_{\nu}^{p}
\end{aligned} \tag{23}
$$

$j_0(z)=\sin z/z$

Vasak, Gyulassy, Elze (1987)

Semiclassical expansion in $\hbar$

- Semiclassical expansion in powers of $\hbar$

$$
\begin{aligned}
\mathscr{J}_\mu &= \sum_{n=0}^{\infty} \hbar^n \mathscr{J}_\mu^{(n)} \\
\Pi^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} \Pi^\mu_{(2n)} \\
G^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} G^\mu_{(2n)}
\end{aligned} \tag{24}
$$

- where

∂_x acts only on $F^{\mu\nu}$

$$
\begin{aligned}
G^\mu_{(2n)} &= \frac{(-1)^{n+1}}{2^{2n}(2n+1)!} (\partial_x \cdot \partial_p)^{2n} F^{\mu\nu} \partial^p_\nu \\
\Pi^\mu_{(2n)} &= \frac{(-1)^n n}{2^{2n-1}(2n+1)!} (\partial_x \cdot \partial_p)^{2n-1} F^{\mu\nu} \partial^p_\nu
\end{aligned} \tag{25}
$$

∂_x acts only on $F^{\mu\nu}$

The 0-th and 2nd order operators

- 0-th order operators (in comoving frame)

$$\begin{aligned} G^{\mu}_{(0)} &= \partial^{\mu}_{x} - QF^{\mu\nu}\partial^{p}_{\nu} = (G^{(0)}_{0}, -\mathbf{G}^{(0)}) \\ G^{(0)}_{0} &= \partial_t + Q\mathbf{E}\cdot\nabla_p \\ \mathbf{G}^{(0)} &= \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_p \\ \Pi^{\mu}_{(0)} &= (p_0, \mathbf{p}) \end{aligned} \tag{26}$$

Only $G^{(0)}_{0}$ contains time derivative ∂_t on the Wigner function

- 2nd order operators

$$\begin{aligned} G^{\mu}_{(2)} &= \frac{Q^2}{24}(\partial_x\cdot\partial_p)^2 F^{\mu\nu}\partial^{p}_{\nu} \\ \Pi^{\mu}_{(2)} &= -\frac{Q^2}{12}(\partial_x\cdot\partial_p)F^{\mu\nu}\partial^{p}_{\nu} \end{aligned} \tag{27}$$

Convention

- From now on we work in the comoving frame where a four-vector can be decomposed into time- and space-component, one can work in a general frame by inroducing fluid velocity u^μ to rewrite the time- and space-component.

$$
\begin{aligned}
\mathscr{J}_s^\mu(x,p) &= (\mathscr{J}_0, \boldsymbol{\mathscr{J}}) \\
&\to (u\cdot\mathscr{J}, \Delta^\mu_\nu \mathscr{J}^\nu)
\end{aligned}
$$

The n-th order evolution equations

- The evolution equations at $O(\hbar^n)$ read

Contains $\partial_t \mathscr{J}_0^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \mathscr{J}_0^{(n-2i)} + \mathbf{G}^{(2i)} \cdot \boldsymbol{\mathscr{J}}^{(n-2i)} \right] = 0 \tag{28}$$

Contains $\partial_t \boldsymbol{\mathscr{J}}^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \boldsymbol{\mathscr{J}}^{(n-2i)} + \mathbf{G}^{(2i)} \mathscr{J}_0^{(n-2i)} \right] = 2s \sum_{i=0}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \boldsymbol{\mathscr{J}}^{(n-2i+1)} \tag{29}$$

The n-th order constraint equations

- The constraint equations at $O(\hbar^n)$ read

Mass-shell condition

$$\sum_{i=0}^{[n/2]}\left[\Pi_0^{(2i)}\mathscr{J}_0^{(n-2i)}-\mathbf{\Pi}^{(2i)}\cdot\boldsymbol{\mathscr{J}}^{(n-2i)}\right]=0 \quad (30)$$

$$2s\sum_{i=0}^{[(n+1)/2]}\left[\mathbf{\Pi}^{(2i)}\mathscr{J}_0^{(n-2i+1)}-\Pi_0^{(2i)}\boldsymbol{\mathscr{J}}^{(n-2i+1)}\right]$$

Can solve $\boldsymbol{\mathscr{J}}^{(n+1)}$

Constraint equation to relate $\boldsymbol{\mathscr{J}}$ in terms of $\mathscr{J}_0$

$$=-\sum_{i=0}^{[n/2]}\mathbf{G}^{(2i)}\times\boldsymbol{\mathscr{J}}^{(n-2i)} \quad (31)$$

Solve $\mathscr{J}^{(n+1)}$ as function of $\mathscr{I}_0^{(0)}, \cdots \mathscr{I}_0^{(n+1)}$

- From Eq. (31) we can solve

for $n=0$: side-jump $\mathbf{G}^{(0)} \times \mathscr{J}^{(0)}$

$$\begin{aligned}\mathscr{J}^{(n+1)} &= \frac{\mathbf{p}}{p_0}\mathscr{I}_0^{(n+1)} + \frac{s}{2p_0}\sum_{i=0}^{[n/2]}\mathbf{G}^{(2i)} \times \underbrace{\mathscr{J}^{(n-2i)}}_{\mathscr{J}^{(n)},\, \mathscr{J}^{(n-2)} \ldots} \\ &+ \frac{1}{p_0}\sum_{i=1}^{[(n+1)/2]}\left[\mathbf{\Pi}^{(2i)}\underbrace{\mathscr{I}_0^{(n-2i+1)}}_{\mathscr{I}_0^{(n-1)},\, \mathscr{I}_0^{(n-3)} \ldots} - \Pi_0^{(2i)}\underbrace{\mathscr{J}^{(n-2i+1)}}_{\mathscr{J}^{(n-1)},\, \mathscr{J}^{(n-3)} \ldots}\right] \qquad (32)\end{aligned}$$

- By recursively using the above for $\mathbf{G}^{(2i)} \times \mathscr{J}^{(n-2i)}$ and $\Pi_0^{(2i)}\mathscr{J}^{(n-2i+1)}$ one can finally express $\mathscr{J}^{(n+1)}\left[\mathscr{I}_0^{(0)}, \cdots, \mathscr{I}_0^{(n)}, \mathscr{I}_0^{(n+1)}\right]$.

Evolution equation for $\mathscr{J}^{(n)}$

- Evolution equation for $\partial_t \mathscr{J}^{(n)}$ in (29) becomes

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots] = 2s\,(\mathbf{p} \times \mathscr{J}^{(n+1)})$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \underbrace{\mathscr{J}^{(n-2i)}}_{\mathscr{J}^{(n)},\, \mathscr{J}^{(n-2)} \cdots} + \mathbf{G}^{(2i)} \underbrace{\mathscr{J}_0^{(n-2i)}}_{\mathscr{J}_0^{(n)},\, \mathscr{J}_0^{(n-2)} \cdots} \right]$$

$$= 2s\,\mathbf{p} \times \mathscr{J}^{(n+1)} + 2s \sum_{i=1}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \underbrace{\mathscr{J}^{(n-2i+1)}}_{\mathscr{J}^{(n-1)},\, \mathscr{J}^{(n-3)} \cdots} \qquad (33)$$

- We use Eq. (32)

$$\mathscr{J}^{(n+1)} = \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(n+1)} + \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots , \right] \qquad (34)$$

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots]$

$= 2s\,\mathbf{p} \times \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \cdots \right]$

Evolution equation for $\boldsymbol{\mathscr{J}}^{(n)}$

- The evolution equation for $\partial_t \boldsymbol{\mathscr{J}}^{(n)}$ in (29) is now converted to another evolution for $\partial_t \mathscr{J}_0^{(n)}$

$$\mathbf{F}\left[\frac{\mathbf{p}}{p_0}\partial_t \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \cdots\right] = 2s\,\mathbf{p}\times\mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots\right] \tag{35}$$

- The original evolution equation (28)

$$\left(\partial_t + Q\mathbf{E}\cdot\nabla_p\right)\mathscr{J}_0^{(n)} + \mathbf{G}^{(0)}\cdot\boldsymbol{\mathscr{J}}^{(n)} \sum_{i=1}^{[n/2]}\left[G_0^{(2i)}\mathscr{J}_0^{(n-2i)} + \mathbf{G}^{(2i)}\cdot\boldsymbol{\mathscr{J}}^{(n-2i)}\right] = 0 \tag{36}$$

Two evolution equations for $\mathscr{J}_0^{(n)}$

- There are two evolution equations for $\mathscr{J}_0^{(n)}$: the original one (28) and the one converted from that of $\boldsymbol{\mathscr{J}}^{(n)}$ in (29).
- The question naturally arises: are these evolution equations consistent to each other?
- We can prove that to any order of $\hbar$ the evolution equation for $\boldsymbol{\mathscr{J}}^{(n)}$ is automatically satisfied once the original evolution equation for $\mathscr{J}_0^{(n)}$ and the mass-shell equation are satisfied.
- This means: the time component $\mathscr{J}_0^{(n)}$ is sufficient to describe the kinetics of chiral fermions in EM fields.

The proof of the statement: Gao, Liang, QW, Wang, 1802.06216

Second order results

- Solving mass shell equations we obtain $\mathscr{J}_0^{(0,1,2)}$ up to $O(\hbar^2)$

$\delta'(y) \equiv \frac{d\delta(y)}{dy} = -\frac{1}{y}\delta(y)$

$\delta''(y) = \frac{2}{y^2}\delta(y)$

$$
\begin{aligned}
\mathscr{J}_0^{(0)} &= p_0 f^{(0)}\delta(p^2), \\
\mathscr{J}_0^{(1)} &= p_0 f^{(1)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(0)}\delta'(p^2), \\
\mathscr{J}_0^{(2)} &= p_0 f^{(2)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(1)}\delta'(p^2) + Q^2\frac{(\mathbf{p}\cdot\mathbf{B})^2}{2p_0}f^{(0)}\delta''(p^2) \\
&\quad + \frac{1}{4p^2}\mathbf{p}\cdot\left\{\mathbf{G}^{(0)}\times\left[\frac{1}{p_0}\mathbf{G}^{(0)}\times\left(\mathbf{p}f^{(0)}\delta(p^2)\right)\right]\right\} \\
&\quad - \frac{p_0}{p^2}\Pi_\mu^{(2)}p^\mu f^{(0)}\delta(p^2) \\
&\quad + \frac{1}{p^2}\mathbf{p}\cdot\left(\mathbf{\Pi}^{(2)}p_0 - \Pi_0^{(2)}\mathbf{p}\right)f^{(0)}\delta(p^2) \qquad (37)
\end{aligned}
$$

Mass-shell condition up to $O(\hbar)$

- We collect first three lines of Eq. (37), $\mathscr{J}_0^{(0)} + \hbar \mathscr{J}_0^{(1)} + \hbar^2 \mathscr{J}_0^{(2)}$, to obtain

$$\mathscr{J}_0 \approx p_0 f(x,p)\delta(\tilde{p}^2) \tag{38}$$

where

Quantum effect

$$\begin{aligned} \tilde{p}^2 &\equiv p^2 + \hbar s Q \frac{\mathbf{p}\cdot\mathbf{B}}{p_0} \\ f(x,p) &\equiv f^{(0)} + \hbar f^{(1)} + \hbar^2 f^{(2)} \end{aligned} \tag{39}$$

- The mass-shell condition $\delta(\tilde{p}^2)$ gives

$$E_p^{(\pm)} = \pm E_p(1 \mp \hbar s Q \mathbf{B}\cdot\mathbf{\Omega}_p) \tag{40}$$

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Gao, QW (2015); Hidaka, Yang, Pu (2017);

Huang, Shi, Jiang, Liao, Zhuang (2018); Gao, Liang, QW, Wang (2018)

CKE in three-momentum

- For non-zero energy, $|\mathbf{p}| \neq 0$, we obtain the previous result

$$
\begin{aligned}
& \left(1+\hbar s Q \boldsymbol{\Omega}_p \cdot \mathbf{B}\right) \partial_t f(x, E_p, \mathbf{p}) \\
& + \Big[\overset{\color{red}{\mathbf{v} \equiv \nabla_p E_p^{(+)}}}{\mathbf{v}} + \hbar s Q(\mathbf{E} \times \boldsymbol{\Omega}_p) + \hbar s Q \frac{1}{2|\mathbf{p}|^2} \mathbf{B}\Big] \cdot \nabla_x f(x, E_p, \mathbf{p}) \qquad \color{red}{\boldsymbol{\Omega}_p \equiv \frac{\mathbf{p}}{2|\mathbf{p}|^3}} \\
& + \Big[Q \underset{\color{red}{\tilde{\mathbf{E}} \equiv \mathbf{E} - Q^{-1}\nabla_x E_p^{(+)}}}{\tilde{\mathbf{E}}} + Q \mathbf{v} \times \mathbf{B} + \hbar s Q^2 (\mathbf{E} \cdot \mathbf{B}) \boldsymbol{\Omega}_p\Big] \cdot \nabla_p f(x, E_p, \mathbf{p}) = 0 \quad (41)
\end{aligned}
$$

- At $|\mathbf{p}| = 0$, there are two additional terms in the above CKE which are singular but were previously neglected,

$$
\begin{aligned}
& \hbar s\, (\mathbf{E} \cdot \mathbf{B})\, (\nabla_p \cdot \boldsymbol{\Omega}_p)\, f(x, E_p, \mathbf{p}) \\
& - \lim_{\Lambda \to 0} \frac{2\hbar s}{\Lambda} (\mathbf{E} \cdot \mathbf{p})\, (\mathbf{B} \cdot \mathbf{p})\, \delta'(\Lambda^2 - \mathbf{p}^2) f(x, \Lambda, \mathbf{p}) \qquad (42)
\end{aligned}
$$

Derivation of Eq. (41):

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Hidaka, Yang, Pu (2017);

Huang, Shi, Jiang, Liao, Zhuang (2018); Gao, Liang, QW, Wang (2018)

New source of chiral anomaly

- These two terms come from total derivatives in p_0 and are relevant to the anomalous conservation equation

Previous terms (I_1)

New terms (I_2, I_3)

$$\begin{aligned} \partial_t j_0 + \boldsymbol{\nabla}_x \cdot \mathbf{j} &= -\frac{\hbar s Q^2}{2} \int d^3\mathbf{p} \Big[(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_p \cdot \boldsymbol{\nabla}_p f \\ &\quad + (\mathbf{E}\cdot\mathbf{B}) \left(\boldsymbol{\nabla}_p \cdot \boldsymbol{\Omega}_p\right) f \\ &\quad - \lim_{\Lambda\to 0} \frac{2}{\Lambda} (\mathbf{E}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \delta'(\Lambda^2 - \mathbf{p}^2) f \Big], \quad (43) \end{aligned}$$

- Since

$$\begin{aligned} I_1 + I_2 &\sim \int d^3\mathbf{p} \nabla_p (\boldsymbol{\Omega}_p f) = 0 \\ I_1 &= I_3 \end{aligned} \quad (44)$$

so the last term cobtributes to the anomaly.

Summary of main results

- A general formalism for the quantum kinetics of chiral fermions in a background electromagnetic field based on a semiclassical expansion of covariant Wigner functions in $\hbar$.
- Non-equilibrium formalism: without assumptions of equilibrium conditions as in our previous works
- Proof: to any order of $\hbar$ that only the time-component of the Wigner function is independent while other components are explicit derivatives.
- Proof: to any order of $\hbar$ that a system of quantum kinetic equations for multiple-components of Wigner functions can be reduced to one chiral kinetic equation involving only the single-component distribution function.

Summary of main results

- Remarkable properties for chiral fermions: will significantly simplify the description and simulation of chiral effects in heavy ion collisions and Dirac/Weyl semimetals.
- We also present the chiral kinetic equations in four-momentum up to $O(\hbar^2)$.
- We find that the chiral anomaly may come from a new source in CKE, in contrast to the well-known scenario of the Berry phase term.