# The Emergence of String Theory from the Dual Resonance Model


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50 years of the Veneziano Model

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# General Overview

- The factorization of the N-point Veneziano amplitude defined the spectrum of excited states of the corresponding DRM.
- These states could be described within the space of states of a quantized vibrating material string.
- This suggested that the DRM should be pictured as describing rubber bands, threads or strings.
- This material string description was a metaphor rather than a detailed analogy, because the energy (rest mass) of the excited states of such a material string are equally spaced, while the squared masses of the states in the DRM have equal spacing.

# General Overview

- As a result, the string interpretation of the DRM initially had little impact on the development of the theory.

- Once the subtleties of the states of the DRM had been understood, the geometric action principle of Nambu and Goto, was shown to lead to a quantum theory that describes precisely the DRM's physical states and dynamics, including interactions.

- Then the conceptualization of the DRM as string theory gained acceptance.

# A Quite Simple Expression

## Veneziano

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx\,.$$

$$\alpha(s) = \alpha_0 + \alpha' s\,, \qquad s = -(p_1+p_2)^2\,, \qquad t = -(p_2+p_3)^2\,,$$

$$x \cdot y \equiv x^\mu \eta_{\mu\nu} y^\nu = -x^0 y^0 + x^j y^j\,.$$

## Koba-Nielsen

$$A_N = \int \prod_{1\le i<j\le N} (z_i - z_j)^{p_i\cdot p_j} \prod_{i=1}^{N} (z_i - z_{i+1})^{\alpha_0 - 1} dz_i \Big/ d\omega,$$

$$d\omega = \frac{dz_a dz_b dz_c}{(z_a - z_b)(z_a - z_c)(z_b - z_c)}\,, \qquad z_i \mapsto \frac{az_i + b}{cz_i + d},$$

# Factorization

$$A_N = \int \prod_{1\le i<j\le N} (z_i - z_j)^{p_i\cdot p_j} \prod_{i=1}^{N} (z_i - z_{i+1})^{\alpha_0 - 1} dz_i \Big/ d\omega,$$

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n}\,; \quad a_n^{\mu\dagger} = a_{-n}^\mu\,; \quad a_n^\mu|0\rangle = 0, \quad \text{for } n > 0,$$

$$Q^\mu(z) = q^\mu - ip^\mu \log z + i\sum_{n\neq 0} \frac{1}{n} a_n^\mu z^{-n},$$

$$\begin{aligned} V(k,z) &= :\exp\{ik\cdot Q(z)\}: \\ &= e^{ik\cdot q}\exp\left\{-k\cdot\sum_{n<0}\frac{1}{n}a_n^\mu z^{-n}\right\}\exp\left\{-k\cdot\sum_{n>0}\frac{1}{n}a_n^\mu z^{-n}\right\} z^{k\cdot p}. \end{aligned}$$

$$\langle 0|V(k_1,z_1)V(k_2,z_2)\dots V(k_N,z_N)|0\rangle = \prod_{1\le i<j\le N} (z_i - z_j)^{k_i\cdot k_j}\,,$$

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n}\,; \quad a_n^{\mu\dagger} = a_{-n}^\mu\,; \quad a_n^\mu|0\rangle = 0, \quad \text{for } n > 0,$$

$$R|\psi\rangle = l|\psi\rangle, \quad \text{where} \quad R = \sum_{n>0} a_{-n} \cdot a_n.$$

$$\tfrac{1}{2}p^2 + R = \alpha_0, \qquad M_l^2 = -p^2$$

$$M_l = \sqrt{2(l - \alpha_0)}$$

$A_N$ has poles evenly spaced in $s$ i.e. $M_l^2 = 2(l - \alpha_0)$

rest energy levels spaced like $\sqrt{l - \alpha_0}$.

Simple Harmonic Oscillator energy levels evenly spaced in $l$.

# Metaphors and Analogies

## Nambu, Wayne Univ. Conf., 18-20 June 1969



ing amplitude seems to be free of ghosts.[8] But the general answer to this question is not known yet.

The appearance of harmonic oscillators in our problem is intriguing since the simple bound-state picture of quarks with a harmonic oscillator potential would naturally give rise to linear trajectories and the $U(3, 1)$ level scheme. We can bring out this analogy more clearly in the following way. Let us introduce a Bose field $\phi_\alpha(\xi)$ and its canonical conjugate $\pi_\alpha(\xi)$, which are even and periodic with periodicity $2\pi$ in $\xi$. In analogy to the ordinary field theory, we decompose it into plane waves:

$$\phi_\alpha(\xi) = \sum_{r=1}^{\infty} \frac{1}{\sqrt{2r}} (a_\alpha^{(r)} + a_\alpha^{+(r)}) \cos r\xi$$

$$\pi_\alpha(\xi) = \sum_{r=1}^{\infty} i \sqrt{\frac{r}{2}} (a_\alpha^{(r)} - a_\alpha^{+(r)}) \cos r\xi \qquad (16)$$

where we have excluded the constant mode $r = 0$. The $a$'s and $a^+$'s are the operators we have defined above. In view of Eq. (13), the quantum number $N$ which determines the resonance energy can be written

$$N = -\sum_r r a^{+(r)} \cdot a^{(r)}$$

$$= \frac{-1}{\pi} \int_0^{2\pi} :(\partial_\xi \phi(\xi) \cdot \partial_\xi \phi(\xi) + \pi(\xi) \cdot \pi(\xi)): d\xi. \qquad (17)$$

# Metaphors and Analogies

## Nambu, Wayne Univ. Conf., 18–20 June 1969

$$\phi^\mu(\sigma,\tau) = \tfrac{1}{2}Q^\mu(e^{i\tau+i\sigma}) + \tfrac{1}{2}Q^\mu(e^{i\tau-i\sigma})$$
$$= q^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{a_n^\mu}{n}e^{-in\tau}\cos n\sigma\,.$$

$$\frac{\partial^2\phi^\mu}{\partial\sigma^2} = \frac{\partial^2\phi^\mu}{\partial\tau^2}\,, \qquad \mathcal{L}_O = \frac{1}{2}\left(\frac{\partial\phi}{\partial\sigma}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2.$$

$$\frac{1}{2\pi}\int_0^\pi\left(\frac{\partial\phi^\mu}{\partial\tau}\frac{\partial\phi_\mu}{\partial\tau} + \frac{\partial\phi^\mu}{\partial\sigma}\frac{\partial\phi_\mu}{\partial\sigma}\right)d\sigma = \tfrac{1}{2}p^2 + \sum_{n>0}a_{-n}\cdot a_n$$

$$\frac{1}{2\pi}\int_0^{\pi}\left(\frac{\partial\phi^\mu}{\partial\tau}\frac{\partial\phi_\mu}{\partial\tau}+\frac{\partial\phi^\mu}{\partial\sigma}\frac{\partial\phi_\mu}{\partial\sigma}\right)d\sigma=\tfrac{1}{2}p^2+\sum_{n>0}a_{-n}\cdot a_n$$

Nambu: this "suggests that the internal energy of a meson is analogous to that of a quantized string of finite length (or a cavity resonator for that matter) whose displacements are described by the field $\phi^\mu$",

... but the energy of a DRM meson $\sim\sqrt{l-\alpha_0}$.

$\mathcal{L}_O$ describes the transverse vibrations of a material string,

- moving in a non-physical space, $\mathbb{R}^{1,1}\times\mathbb{R}^{1,D-1}$,
- coordinates $\sigma, \tau,$ and $\phi^\mu$
- stretched between $\sigma=0$ and $\sigma=\pi$ on the $\sigma$-axis,
- with $\tau$ (rather than $\phi^0$) being the time variable.
- energy of material string $l$ not analogous to energy in DRM

L. Susskind: *Dual symmetric theory of hadrons I* [1970]

"a meson is described by the degrees of freedom of a four-dimensional rubber band with a quark pair [at the ends]"

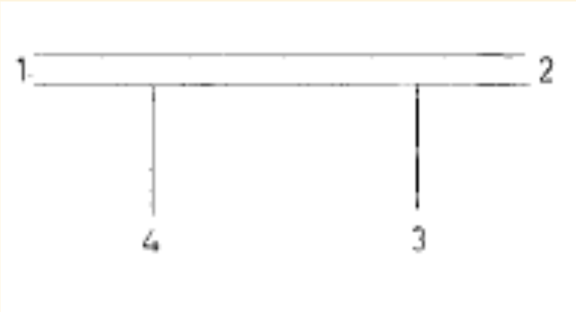
"the level spacing separating rotational excitations of hadrons is very nearly a universal quantity of order 1(GeV)$^{2}$. The only systems which are known from quantum mechanics to possess this property are harmonic systems such as a harmonic oscillator."

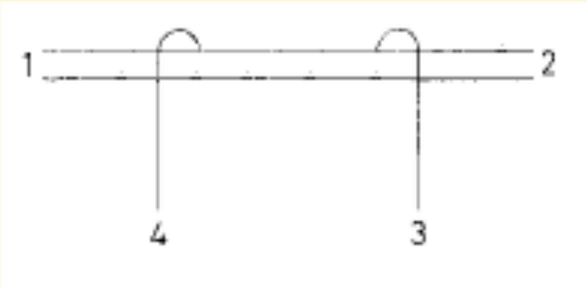
- but hadron level spacing is in energy squared whereas harmonic oscillator spacing is in energy

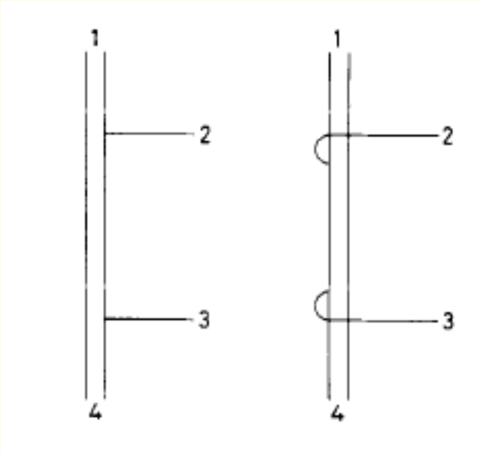
L. Susskind: *Dual symmetric theory of hadrons I* [1970]

"a meson is composed of a quark-antiquark pair at the ends of an elastic string" which generates a two-dimensional strip or "world sheet" as it moves through space-time.

Interactions are pictured as a single "elastic string" interacting with quanta through the quarks at its ends. The various ways of singling out the "string" are equal as a result of "dual symmetry".



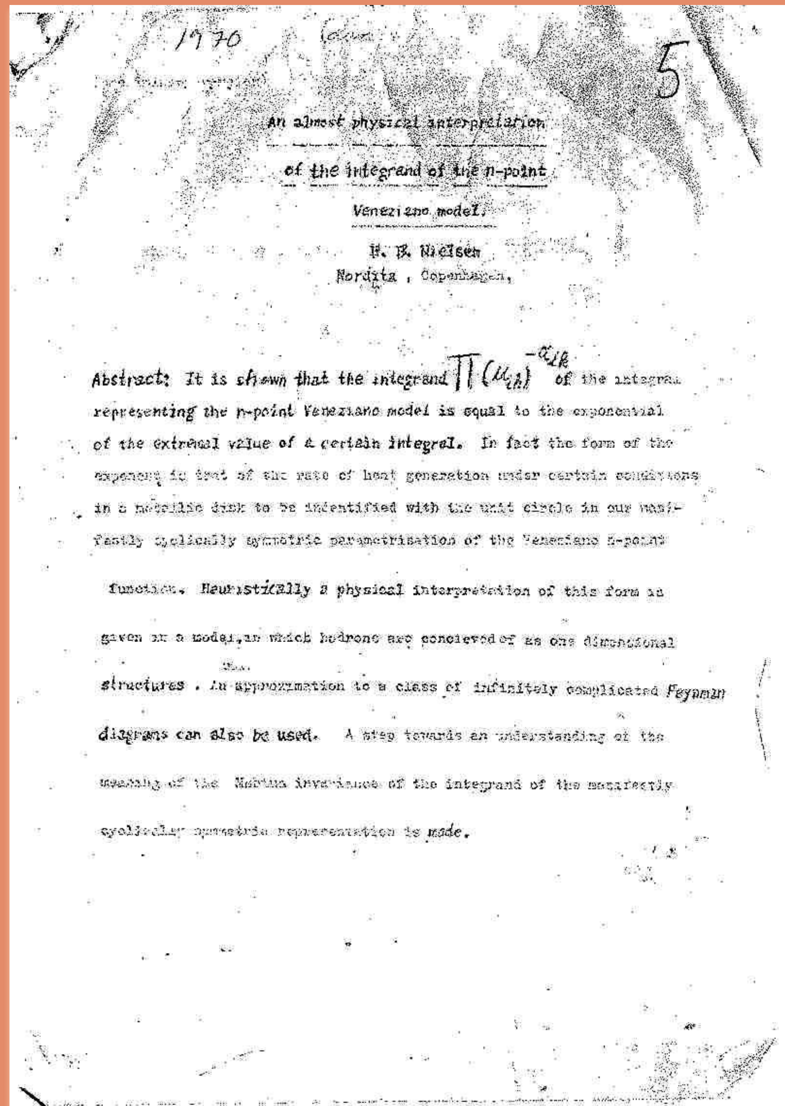



Nambu: “a quantized string of finite length”

Susskind: “the degrees of freedom of the internal state of a hadron are equivalent to those of a violin string or an organ pipe”

- states of DRM constructed within space of states of material string
- quotient space of subspace
- potentially there are ghost states
- string is in an unphysical space: $\mathbb{R}^{1,1} \times \mathbb{R}^{1,D-1}$ with two time variables $\tau$, $\phi^0$
- energies of DRM and string do not agree



1970

5

An almost physical interpretation of the integrand of the n-point Veneziano model.

H. B. Nielsen
Nordita, Copenhagen,

Abstract: It is shown that the integrand $\prod (u_{ik})^{-\alpha_{ik}}$ of the integral representing the n-point Veneziano model is equal to the exponential of the extremal value of a certain integral. In fact the form of the exponent is that of the rate of heat generation under certain conditions in a metallic disk to be identified with the unit circle in our manifestly cyclically symmetric parametrisation of the Veneziano n-point function. Heuristically a physical interpretation of this form is given in a model, in which hadrons are concieved of as one dimensional structures. An approximation to a class of infinitely complicated Feynman diagrams can also be used. A step towards an understanding of the meaning of the Moebius invariance of the integrand of the manifestly cyclically symmetric representation is made.

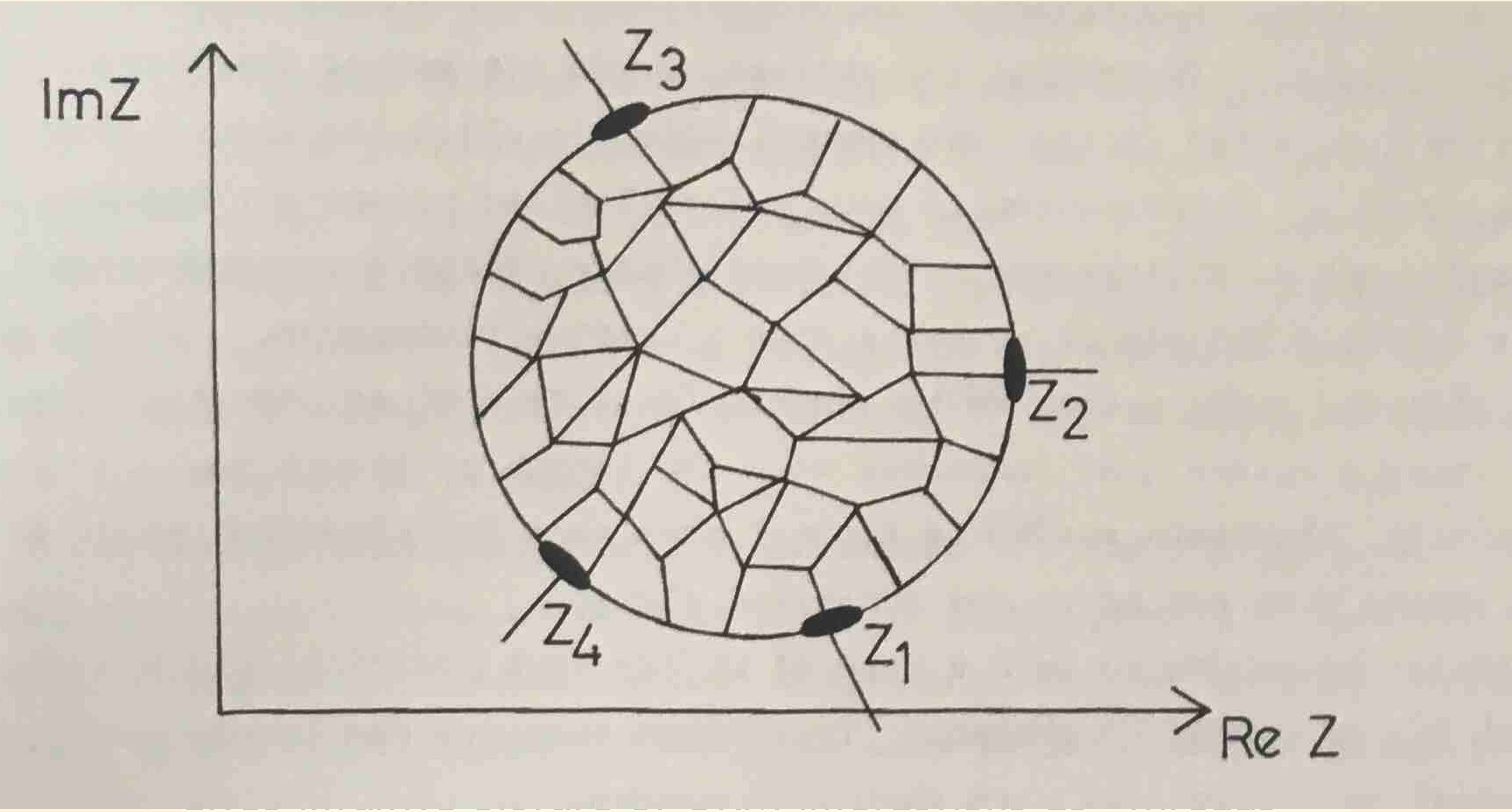
*An almost physical interpretation of the integrand of the n-point Veneziano Model*

H. B. Nielsen

# *An almost physical interpretation of the integrand of the n-point Veneziano Model*

## H. B. Nielsen

- Fairlie-Nielsen analogue model: Koba-Nielsen DRM integrand = heat generated by steady current flow in a disc
- can be used by analogy to calculate M loop integrands, agreeing with operator formalism
- DRM is approximation to contribution of very complicated fishnet Feynman diagrams — argued to be two-dimensional
- gives intuitive picture of DRM interactions as “threads” joining and splitting

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# *An almost physical interpretation of the integrand of the n-point Veneziano Model*

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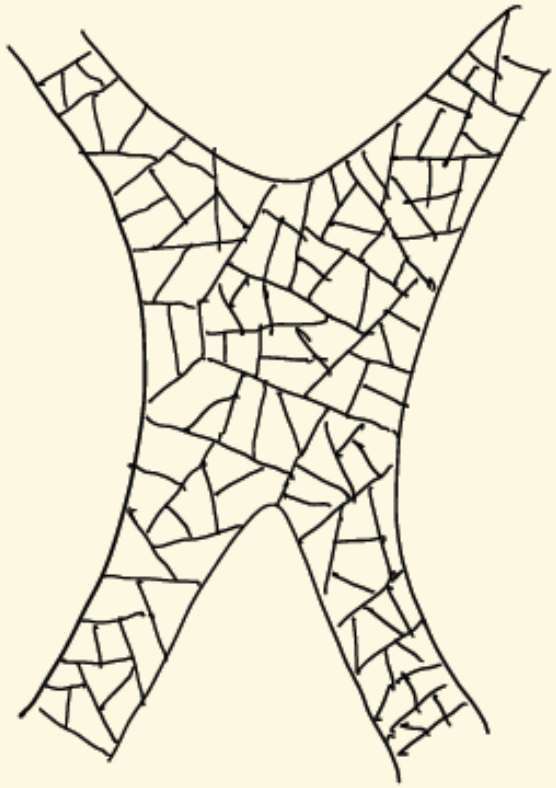
Section II

## Physical interpretation

Using our formula (11) it is possible to interprente the generalized Veneziano within a model in which the mesons are thread like structures ( allthough we are not able to say yet which satelite terms might follow from such a model ),

Hadronic interactions are conceived of then as proces - ses in which threads are connected at the end points into ( at first) longer threads which are then again split up into ( at first)shorter threads. In fact the mapping $V^{\mu}: \mathcal{P} \rightarrow$ "Minchowski space" des-

"it is possible to interpret the generalized Veneziano [amplitude as] a model in which the mesons are thread like structures ... Hadronic interactions are conceived of then as processes in which threads [join] at the end points into ... longer threads which are then split up into ... shorter threads."



“it is possible to interpret the generalized Veneziano [amplitude as] a model in which the mesons are thread like structures ... Hadronic interactions are conceived of then as processes in which threads [join] at the end points into ... longer threads which are then split up into ... shorter threads.”

| | | |
|---|---|---|
| Nambu | 18 Jun 69 | quantized string of finite length; cavity |
| Susskind | 23 Jun 69 | spring; continuum limit of chain of springs. |
| Susskind | 11 Jul 69 | violin string; organ pipe; |
| | | continuum limit of chain of springs. |
| Nielsen | 69-70 | one-dimensional structure; |
| | | infinitely complicated Feynman diagrams; |
| | | infinitely-long chain of molecules; |
| | | thread-like structure; thread or stick. |
| Susskind | Jul/Aug | |
| Susskind | 3 Jan 70 | rubber band; violin string; elastic string; |
| Nambu | Aug 70 | elastic string of finite intrinsic length; |
| | | elastic string; rubber string; rubber band. |

Terms used to interpret the Veneziano model in 1969-70

# Ghosts

## Physical States

$$L_0|\psi\rangle = |\psi\rangle; \qquad L_n|\psi\rangle = 0, \quad n > 0,$$

$$L_n = \frac{1}{2}\sum_m : a_m \cdot a_{n-m} :$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m,-n},$$

## Null Physical States

$$L_{-1}|\phi_0\rangle\,; \qquad \text{if } D = 26,\ \left(L_{-2} + \tfrac{3}{2}L_{-1}^2\right)|\phi_{-1}\rangle\,,$$

$$L_0|\phi_l\rangle = l|\phi_l\rangle; \qquad L_n|\phi_l\rangle = 0, \quad n > 0,$$

$$Q^\mu(z) = q^\mu - ip^\mu \log z + i \sum_{n \neq 0} \frac{1}{n} a_n^\mu z^{-n},$$

$$[L_n, Q^\mu(z)] = z^{n+1} \frac{dQ^\mu(z)}{dz},$$

$$\begin{aligned} V(k,z) &=: \exp\{ik \cdot Q(z)\} : \\ &= e^{ik\cdot q} \exp\left\{-k \cdot \sum_{n<0} \frac{1}{n} a_n^\mu z^{-n}\right\} \exp\left\{-k \cdot \sum_{n>0} \frac{1}{n} a_n^\mu z^{-n}\right\} z^{k\cdot p}. \end{aligned}$$

$$[L_n, V(k,z)] = z^{n+1} \frac{dV(k,z)}{dz} + \frac{n}{2} z^n k^2 V(k,z),$$

# DDF states

$$P^{\mu}(z) = i\frac{dQ^{\mu}(z)}{dz},$$

$$k^2 = 0, \qquad \epsilon^i \cdot \epsilon^j = \delta^{ij}, \qquad k \cdot \epsilon^i = 0, \quad 1 \leq i, j \leq D-2,$$

$$A^i_n = \epsilon^i_{\mu} A^{\mu}_n, \qquad A^{\mu}_n = \frac{1}{2\pi i}\oint P^{\mu}(z) V(nk, z) dz$$

$$[L_m, A^i_n] = 0.$$

$$[A^i_m, A^j_n] = m\delta^{ij}\delta_{m,-n}, \qquad 1 \leq i, j \leq D-2.$$

The $A^i_n$ generate the space of DDF states, manifestly positive definite.

When $D = 26$ these are essentially all the physical states, the rest are null.

Characterization of DDF states $|f\rangle$ :

$$L_n|f\rangle = K_n|f\rangle = 0, \quad n > 0, \qquad L_0|f\rangle = K_0|f\rangle = |\psi\rangle,$$

Use algebra :

$$[L_m, K_n] = -nK_{m+n}\,, \qquad [K_m, K_n] = 0\,.$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m,-n},$$

No Ghost Theorem

If $|\psi\rangle$ is a physical state and $D = 26$, then

$$|\psi\rangle = |f\rangle + |n\rangle,$$

where $|f\rangle$ is a DDF state and $|n\rangle$ is a null physical state.

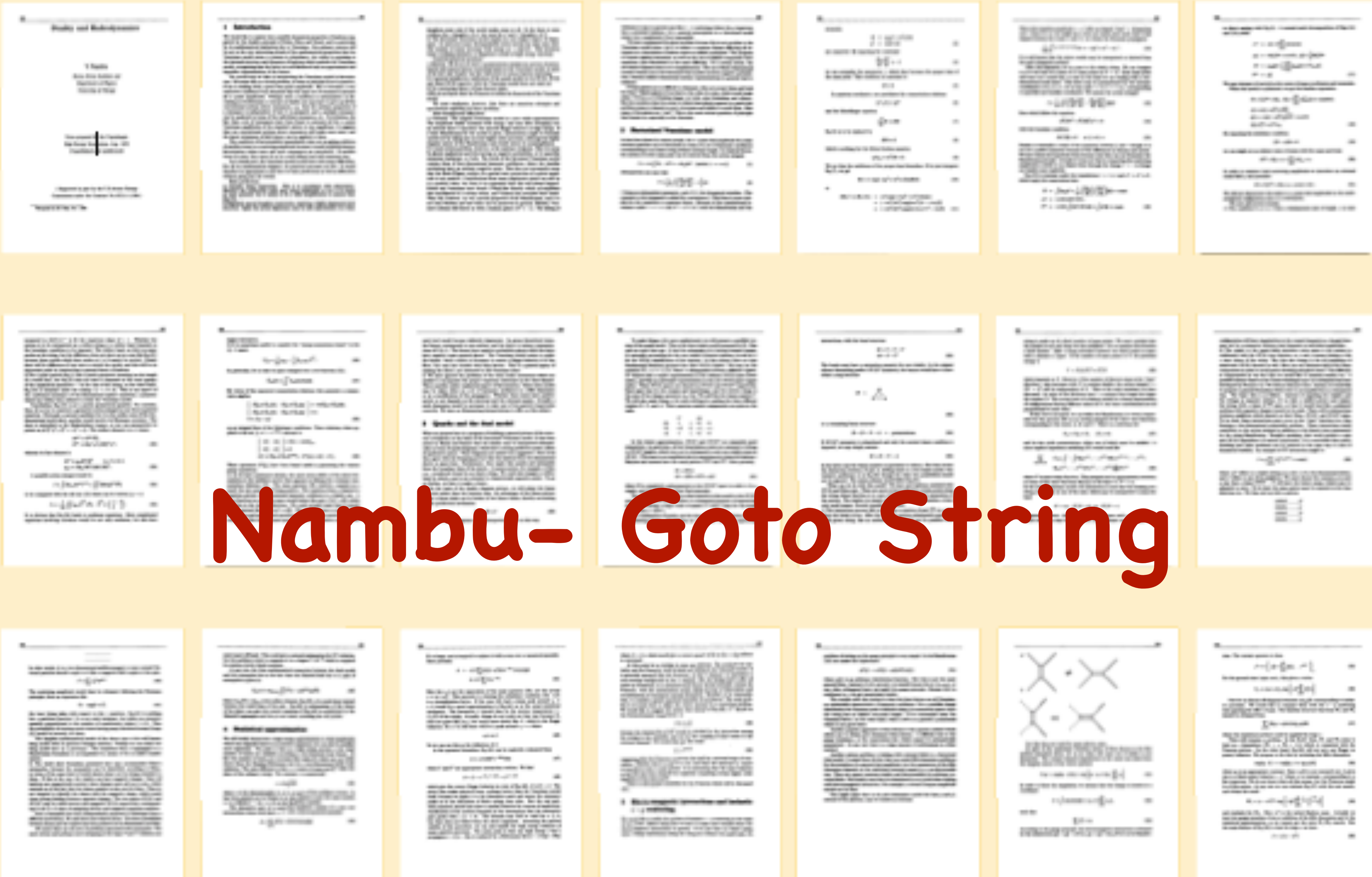
Lovelace [1971]:

nonplanar loop needs D = 26
and for 2 dimensions of oscillators
to be effectively removed
to avoid a unitarity violating cut

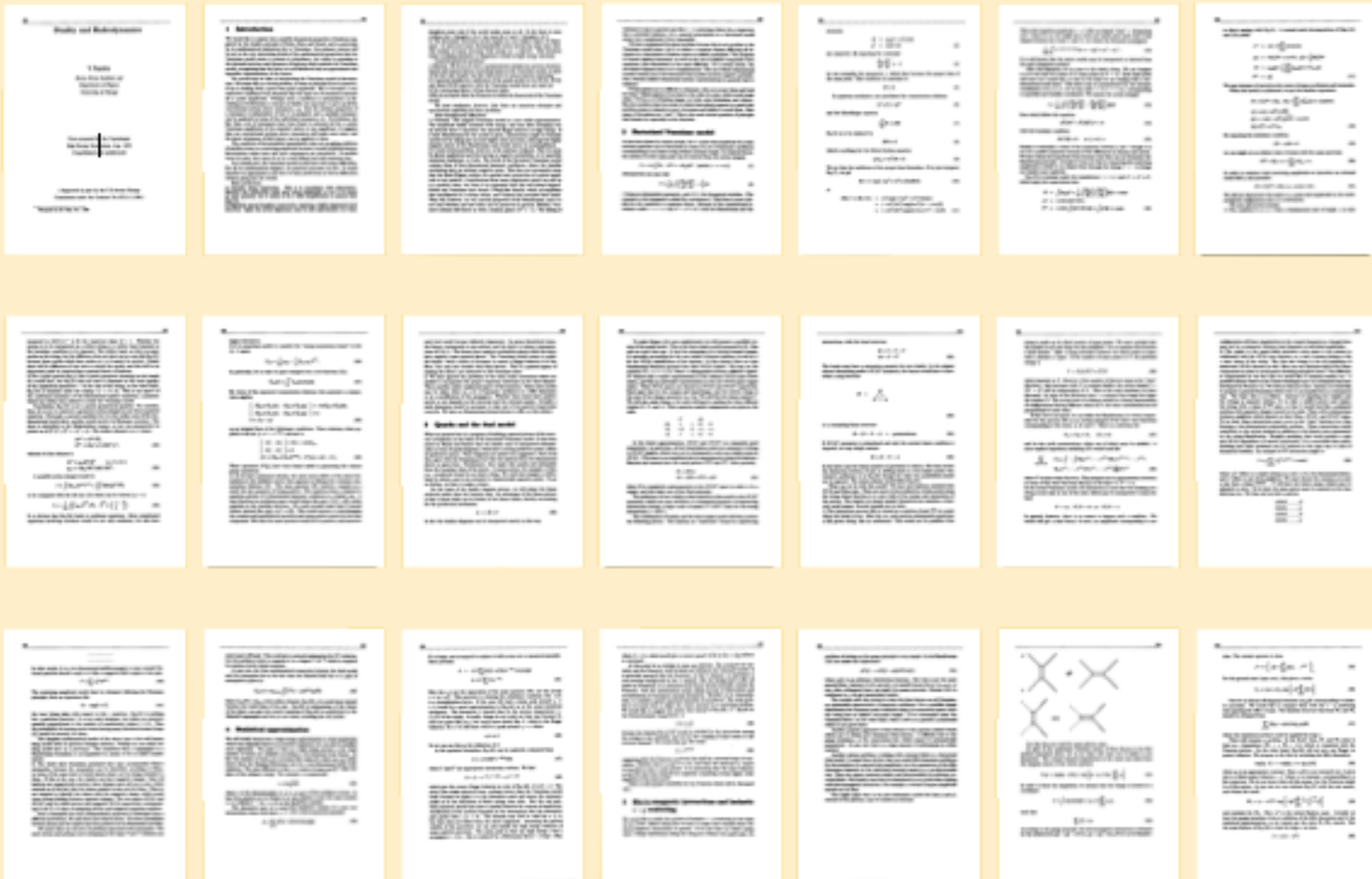
achieved by null states
corresponding to one dimension of
oscillators at D = 26

at D = 26 cut becomes a pole
corresponding to closed string
states

# Nambu- Goto String



Nambu undelivered Copenhagen Talk Aug 1970



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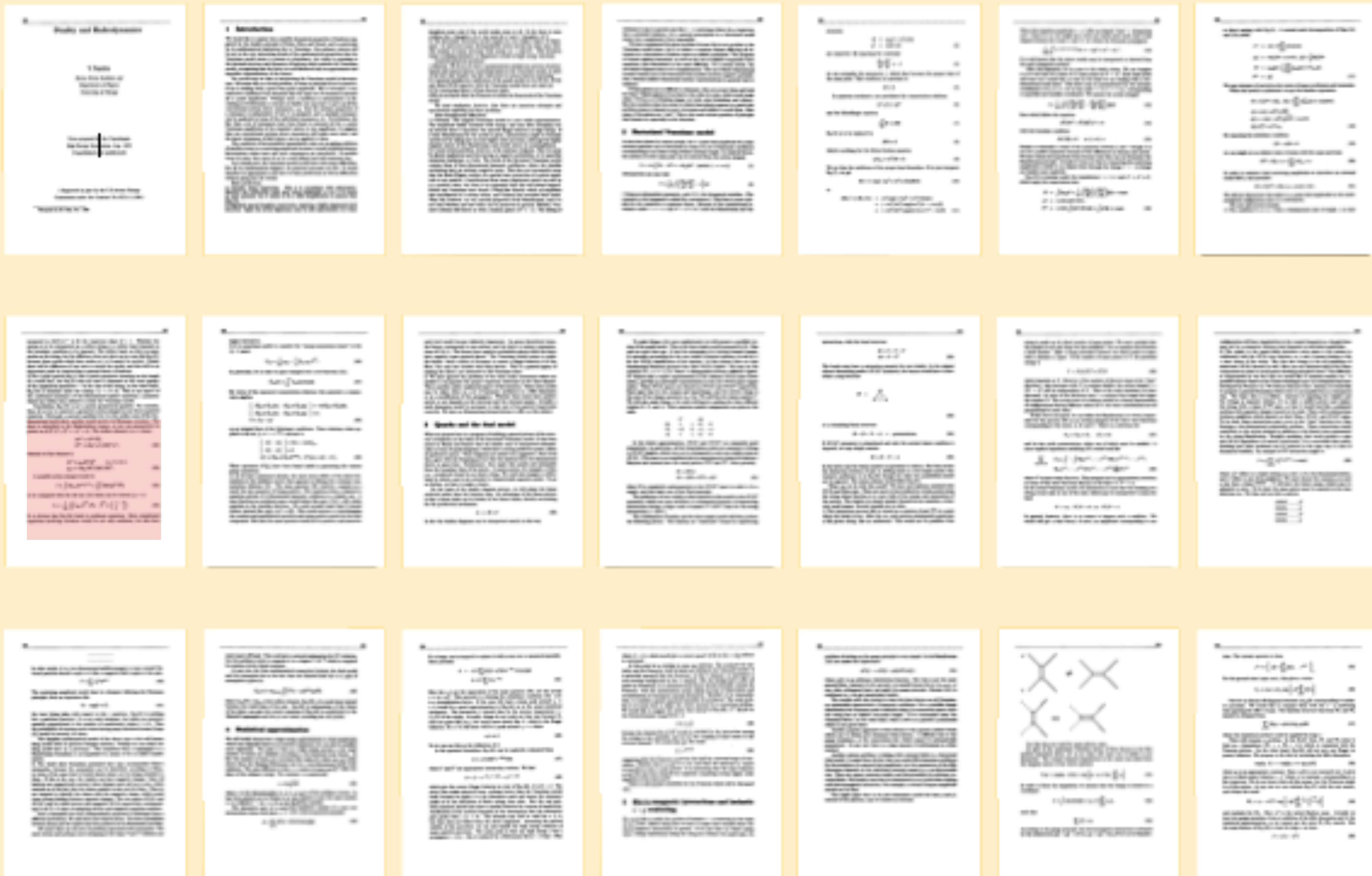

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Nonetheless, Eq.(13) is not a purely geometrical quantity. For curiosity, then, let us try to construct a geometric action integral as one does in general relativity. Obviously a natural candidate for it is the surface area of the two-dimensional world sheet; another would involve its Riemann curvature. The sheet is imbedded in the Minkowskian 4-space, so one can parametrize its points as $y^\mu(\xi^0, \xi^1)$, $(\xi^0 \sim \tau, \xi^1 \sim \xi)$. The surface element is a $\sigma-$tensor

$$\begin{aligned} d\sigma^{\mu\nu} &= G^{\mu\nu} d^2\xi, \\ G^{\mu\nu} &= \partial(y^\mu, y^\nu)/\partial(\xi^0, \xi^1) \end{aligned} \tag{22}$$

whereas its line element is

$$\begin{aligned} ds^2 &= g_{\alpha\beta} d\xi^\alpha d\xi^\beta \qquad (\alpha, \beta = 0, 1) \\ g_{\alpha\beta} &= (\partial y_\mu/\partial\xi^\alpha)(\partial y^\mu/\partial\xi^\beta) \end{aligned} \tag{23}$$

A possible action integral would be

$$I = \int |d\sigma_{\mu\nu} d\sigma^{\mu\nu}|^{1/2} = \iint |2\det g|^{1/2} d^2\xi \tag{24}$$

to be compared with the old one (13) which can be written $(y \to x)$

$$I = -\frac{1}{4\pi} \iint g_{\alpha\beta} \overset{\circ}{g}{}^{\alpha\beta} d^2\xi, \quad \overset{\circ}{g}{}^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

It is obvious that Eq.(24) leads to nonlinear equations. More complicated equations involving curvature would be not only nonlinear, but also have higher derivatives.

Nambu undelivered Copenhagen Talk Aug 1970

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$$d\sigma^{\mu\nu} = G^{\mu\nu}\, d^2\sigma\,, \qquad G^{\mu\nu} = \frac{\partial(x^\mu, x^\nu)}{\partial(\sigma^0, \sigma^1)}$$

Nambu undelivered Copenhagen Talk Aug 1970

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$$\mathcal{A}_{NG} = -\frac{T_0}{c} \int \sqrt{\left(\frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau}\right)^2 - \left(\frac{\partial x}{\partial \sigma}\right)^2 \left(\frac{\partial x}{\partial \tau}\right)^2}\, d\sigma d\tau$$

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Geometric Reparametrization invariant

Nambu-Goto ‘string’

$$\mathcal{A}_{NG} = -\frac{T_0}{c}\int \mathcal{L}_{NG}\, d\sigma d\tau, \qquad \mathcal{L}_{NG} = \sqrt{\left(\frac{\partial x}{\partial \sigma}\cdot\frac{\partial x}{\partial \tau}\right)^2 - \left(\frac{\partial x}{\partial \sigma}\right)^2\left(\frac{\partial x}{\partial \tau}\right)^2}\,,$$

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}_{NG}}{\partial x_{\mu\sigma}}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}_{NG}}{\partial x_{\mu\tau}}\right) = \frac{\partial \Pi^{\sigma\mu}}{\partial \sigma} + \frac{\partial \Pi^{\tau\mu}}{\partial \tau} = 0\,,$$

$$\Pi^{\sigma\mu} = \frac{x_\tau^\mu (x_\sigma \cdot x_\tau) - x_\sigma^\mu x_\tau^2}{\left[(x_\sigma \cdot x_\tau)^2 - x_\sigma^2 x_\tau^2\right]^{\frac{1}{2}}}\,, \qquad \Pi^{\tau\mu} = \frac{x_\sigma^\mu (x_\sigma \cdot x_\tau) - x_\tau^\mu x_\sigma^2}{\left[(x_\sigma \cdot x_\tau)^2 - x_\sigma^2 x_\tau^2\right]^{\frac{1}{2}}}\,.$$

$$\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau}\right)^2 = 0, \qquad \frac{\partial x}{\partial \sigma}\cdot\frac{\partial x}{\partial \tau} = 0\,,$$

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}_{NG}}{\partial x_{\mu\sigma}}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}_{NG}}{\partial x_{\mu\tau}}\right) = \frac{\partial \Pi^{\sigma\mu}}{\partial \sigma} + \frac{\partial \Pi^{\tau\mu}}{\partial \tau} = 0\,,$$

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$$\frac{\partial^2 x^\mu}{\partial \sigma^2} = \frac{\partial^2 x^\mu}{\partial \tau^2}\,.$$

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$$\frac{1}{\ell} x^\mu(\sigma, \tau) = q^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{a_n^\mu}{n} e^{-in\tau} \cos n\sigma \, ,$$

$$\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau}\right)^2 = 0, \qquad \frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau} = 0 \, ,$$

$$L_n = \frac{1}{2} \sum_m : a_m \cdot a_{n-m} : = \alpha_0 \, \delta_{n,0} \, ,$$

$\tau \propto \kappa \cdot x \, ,$ $\kappa$ **timelike**; $\kappa \cdot a_n = 0, \quad n \neq 0.$

# Canonical Quantization

$$k^2 = \tilde{k}^2 = k \cdot \epsilon^i = \tilde{k} \cdot \epsilon^i = 0, \qquad \epsilon^i \cdot \epsilon^j = \delta^{ij}, \qquad k \cdot \tilde{k} = -1$$

$$\tau \propto k \cdot x\,, \qquad k \cdot a_n = 0, \quad n \neq 0.$$

$$a_n^i = \epsilon^i \cdot a_n\,, \qquad \tilde{k} \cdot a_n = \frac{1}{k \cdot p} L_n^a, \qquad n \neq 0,$$

$$[a_m^i, a_n^j] = m\delta^{ij}\delta_{m,-n}, \qquad 1 \leq i, j \leq D-2.$$

$$M^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n}(a_{-n}^\mu a_n^\nu - a_{-n}^\nu a_n^\mu)$$

closure of Lorentz algebra requires $D = 26, \quad \alpha_0 = 1.$

# Covariant Quantization

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n}, \qquad 0 \le \mu, \nu \le D-1.$$

$$\langle\psi|L_n|\psi\rangle = \alpha_0\,\delta_{n,0}$$

impose physical state conditions

$$L_n|\psi\rangle = 0, \quad n > 0; \qquad L_0|\psi\rangle = \alpha_0|\psi\rangle\,.$$

absence of ghosts requires

$D = 26, \quad \alpha_0 = 1,$ or $D < 26, \quad \alpha_0 \le 1.$

If $D < 26$, anomalous longitudinal modes are present.

# Summary

- The states of DRM constructed using harmonic oscillators.
- Suggests metaphor or analogy of a quantized material string.
- Problems: space-time unphysical; ghosts; energies different; space of states too big.
- Operator formalism, analogue model provide means of calculating.
- Virasoro constraints, algebra, when $\alpha_0 = 1$.
- Lovelace: $D = 26$, $D - 2$ dimensions of physical oscillators for consistency of loops
- Nambu-Goto geometric action: only transverse modes
- No Ghost Theorem: only transverse modes if $D = 26$, $\alpha_0 = 1$.

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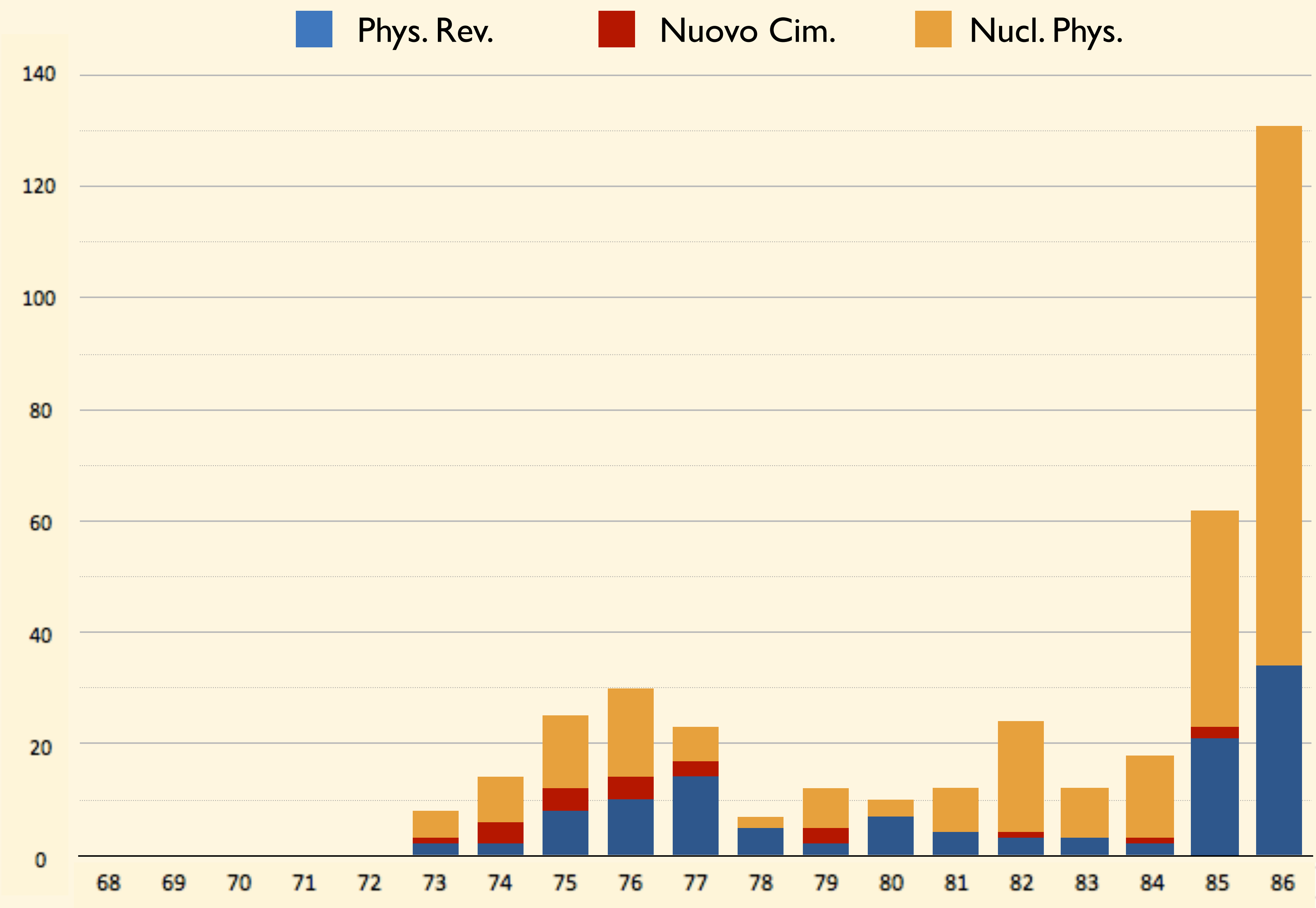
- No Ghost Theorem: only transverse modes if $D = 26$, $\alpha_0 = 1$.
- Canonical quantization of Nambu-Goto string needs $D = 26$, $\alpha_0 = 1$.
- String interactions by splitting, joining give DRM [Mandelstam]



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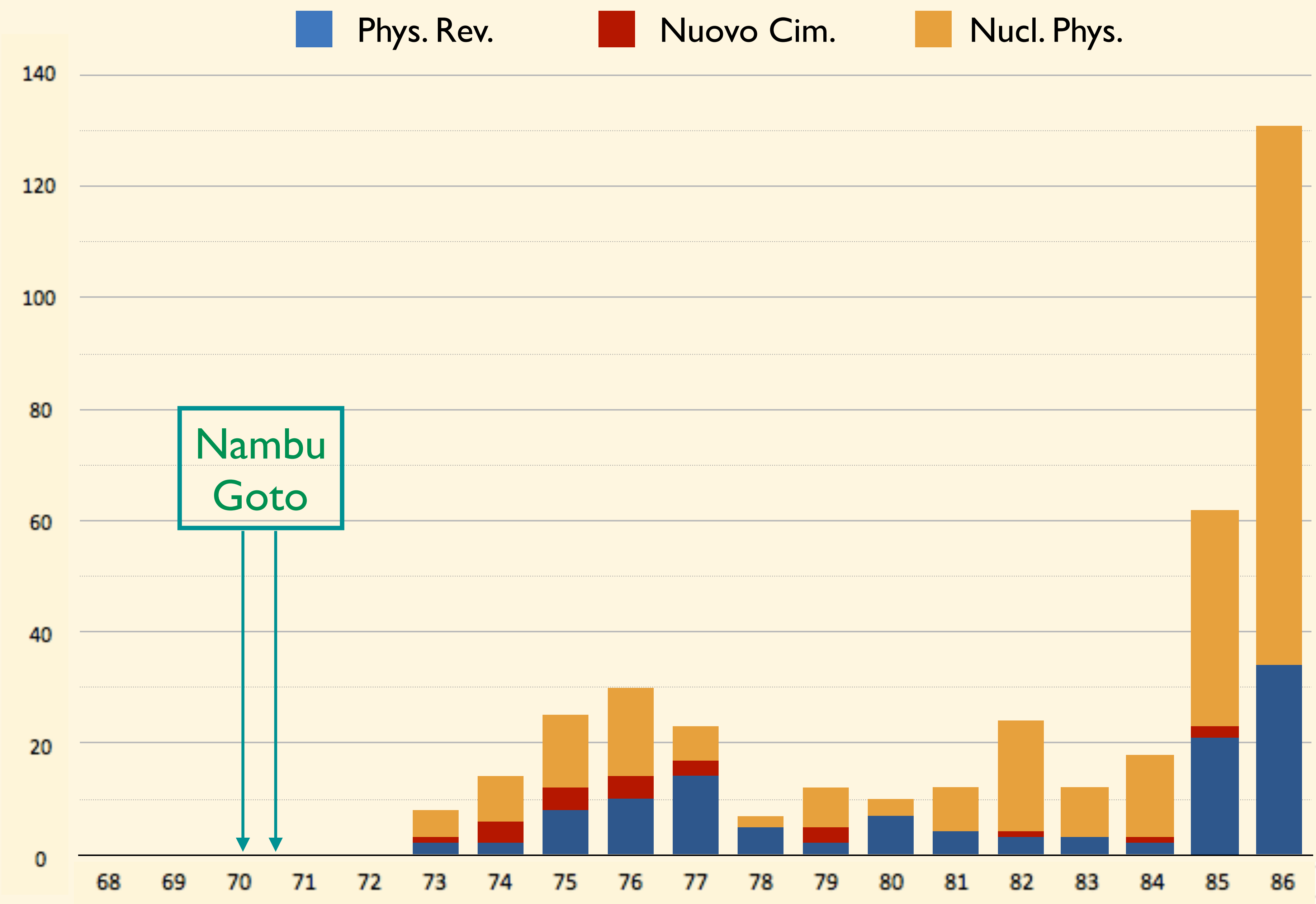
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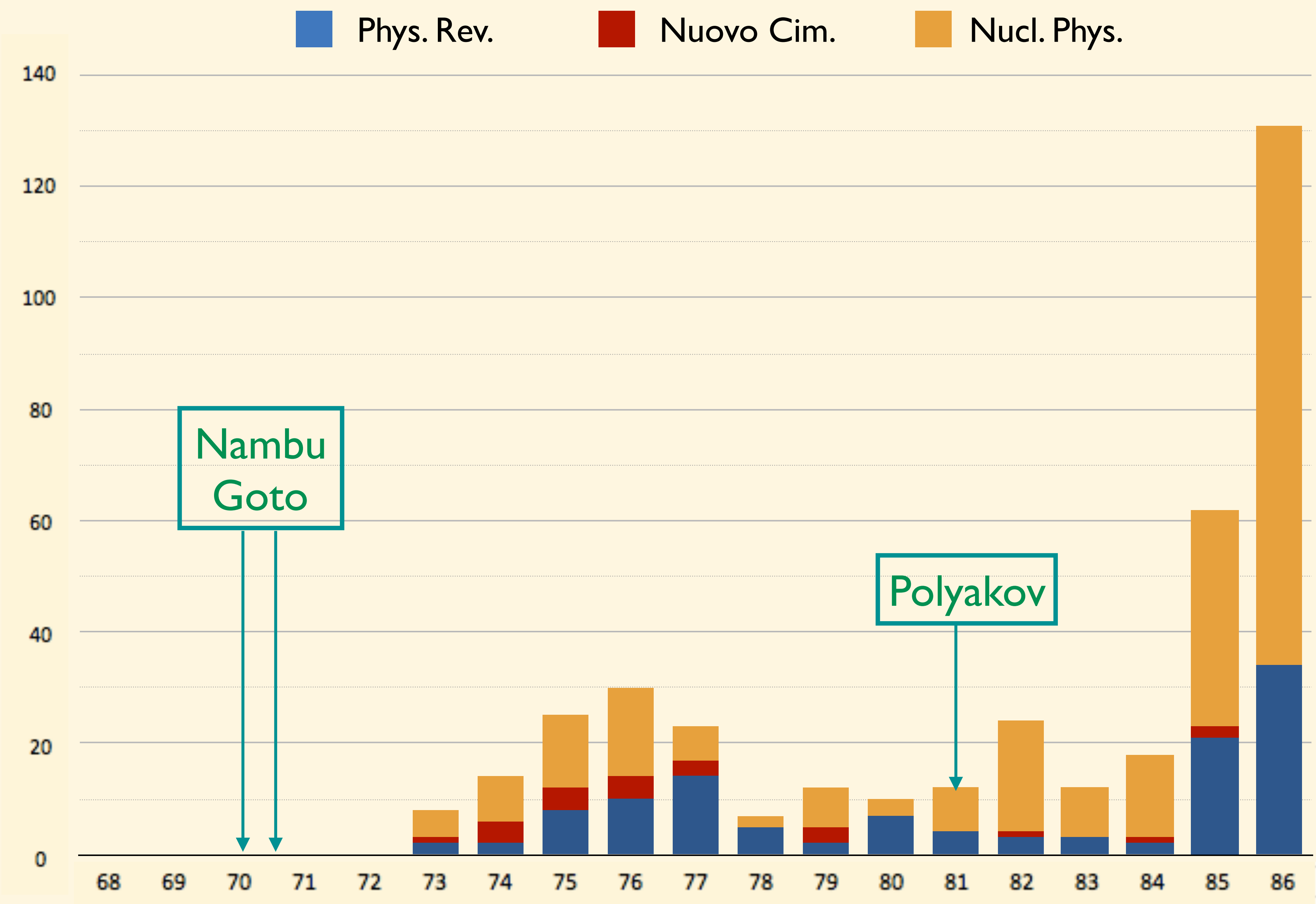

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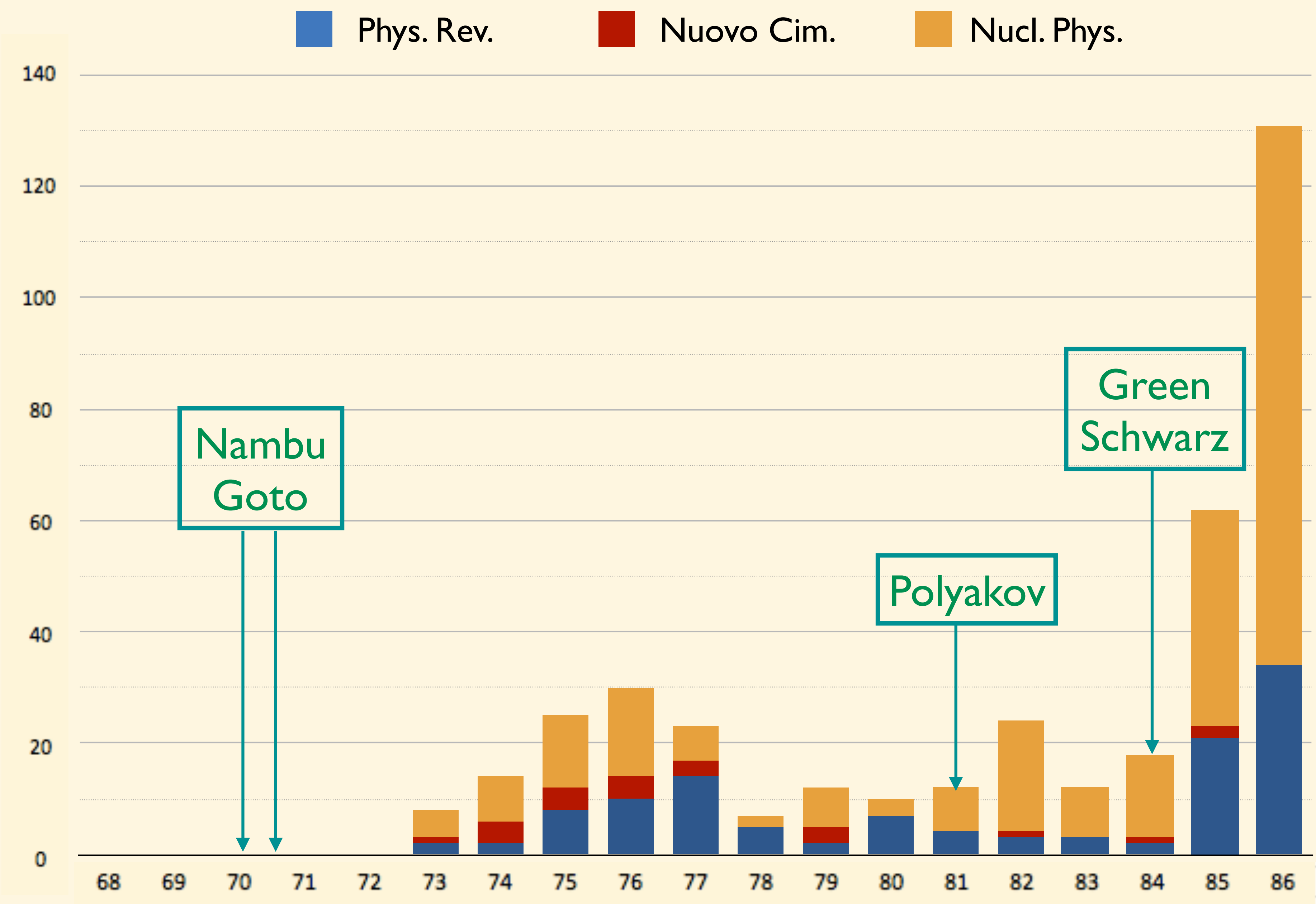

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