From the Veneziano Amplitude to the Double Copy

50 Years of Veneziano Model
May 14, 2018

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ZB, John Joseph Carrasco, Wei-Ming Chen, Henrik Johansson, Radu Roiban, arXiv:1701.02519

ZB, John Joseph Carrasco, Wei-Ming Chen, Henrik Johansson, Radu Roiban, Mao Zeng, arXiv:1708.06807

ZB, John Joseph Carrasco, Wei-Ming Chen, Alex Edison, Henrik Johansson, Julio Parra-Martinez, Radu Roiban, Mao Zeng, arXiv:1804.09311

Outline

1. Veneziano and Virasoro amplitudes and the double copy.

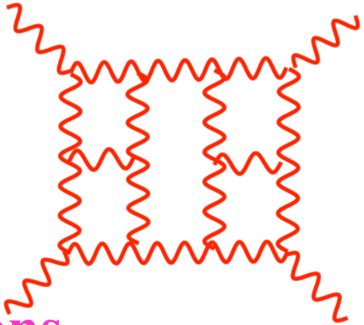
2. Four-point graviton and gluon amplitudes.

3. Duality between color and kinematics and double copy.

4. Applications of double copy to problem of UV divergence in quantum gravity.

5. UV properties at 5 loops in $\mathcal{N} = 8$ supergravity.

6. Towards all loop determination of UV.



See Oliver Schlotterer's talk for string-theory applications of double-copy constructions.

Veneziano and Virasoro

Veneziano Amplitude (open string):

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$
$$u = (k_1 + k_3)^2$$

$$A(s,t) = \frac{\Gamma[-1-\alpha' s]\Gamma[-1-\alpha' t]}{\Gamma[-2-\alpha' s-\alpha' t]}$$

$$u = -s - t - 4/\alpha'$$

Virasoro Amplitude (closed string):

$$A(s,t,u) = \frac{\Gamma[-1-\alpha' s]\Gamma[-1-\alpha' t]\Gamma[-1-\alpha' u]}{\Gamma[-2-\alpha' s-\alpha' t]\Gamma[-2-\alpha' t-\alpha' u]\Gamma[-2-\alpha' u-\alpha' s]}$$

There is a relation between these two amplitudes:

$$A(s,t,u) = \frac{\sin(\pi\alpha' s)}{\pi} A(s,t)A(s,u)$$

Virasoro = Veneziano x Veneziano

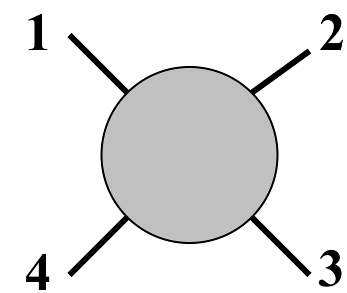
Might seem accidental, but in fact this is the key precursor to a fundamental property of gravity and other types of theories.

Double Copy

Open string four-gluon amplitude:

gluon polarizations

$$A(s,t) = \frac{\Gamma[-\alpha' s]\Gamma[-\alpha' t]}{\Gamma[1-\alpha' s-\alpha' t]} K_{\mu\nu\rho\sigma}\, \zeta_1^\mu \zeta_2^\nu \zeta_3^\rho \zeta_4^\sigma$$



Closed string four-graviton amplitude:

$$A(s,t,u) = \frac{\sin(\pi\alpha' s)}{\pi} A(s,t)A(s,u)$$

graviton polarization

$$\zeta_1^{\mu\alpha} = \zeta_1^\mu \zeta_1^\alpha$$

Closed string = (open string) x (open string)

Same relation as between Veneziano and Virosoro amplitudes

Graviton scattering amplitudes are simply related to gluon ones.
In quantum field theory this is astonishing.

Gravity vs Gauge Theory

Consider the Einstein gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

curvature

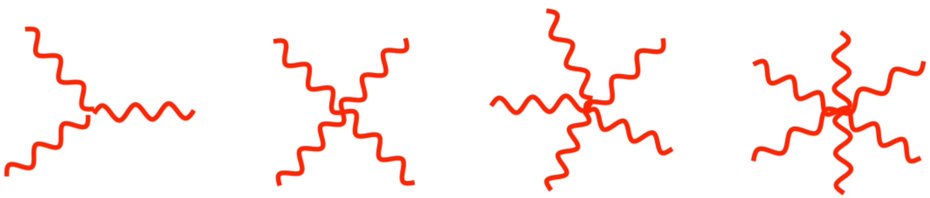
Flat-space metric

graviton field

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of complicated interactions

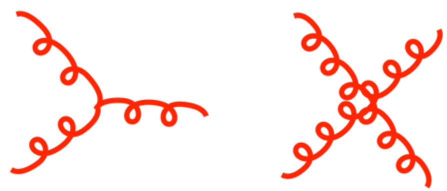


$+ \cdots$

terrible mess

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



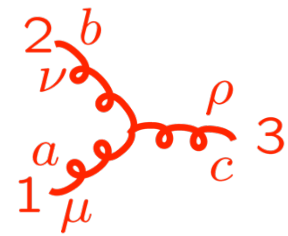
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Does not look related!

Three Vertices

Standard Feynman diagram approach.



Three-gluon vertex:

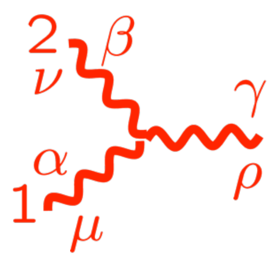
$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}
&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\
&+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\
&+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\
&+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

String theory proves this is not correct way to look at gravity

Kawai-Lewellen-Tye Relations

Kawai-Lewellen-Tye relations in low energy limit: KLT (1985)

gravity — gauge theory color ordered

$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\,A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)\,A_5^{\rm tree}(2,1,4,3,5) + is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\,A_5^{\rm tree}(3,1,4,2,5)$$

Pattern gives explicit all-leg form ZB, Dixon, Rozowsky Perelstein (1998)

1. **Gravity is derivable from gauge theory. Standard QFT offers no hint why this is possible.**
2. **It looked very generally applicable.**
3. **It took people a long time to appreciate the significance of this.**

Gravity From YM

ZB and Grant (1999)

How does the Einstein action reflect KLT factorization?

$$M_4^{\rm tree}(1,2,3,4) = -i s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)$$

$$L^{\rm EH} = \frac{2}{\kappa^2}\sqrt{-g}\,R + \sqrt{-g}\,\partial^\mu\phi\partial_\mu\phi$$

include dilaton

Consider all possible field redefinitions and gauge fixing.

$$g_{\mu\nu} = e^{\sqrt{\frac{2}{D-2}}\kappa\phi} e^{\kappa\, h_{\mu\nu}} \equiv e^{\sqrt{\frac{2}{D-2}}\,\kappa\phi}\Big(\eta_{\mu\nu} + \kappa h_{\mu\nu} + \frac{\kappa^2}{2} h_{\mu\rho}h_{\rho\nu} + \cdots\Big)$$

$$F_\mu = \Big(h_{\mu\nu,\nu} + \phi_{,\mu}\Big) + \kappa\Big(-\frac{1}{4}{\rm Tr}\,(h^2)_{,\mu} - \frac{1}{2}\phi h_{\mu\nu,\nu} - h_{\mu\nu}\phi_{,\nu}\Big) + \kappa^2\Big(\frac{1}{16}{\rm Tr}\,(h^2)h_{\mu\nu,\nu} + \frac{1}{8}{\rm Tr}\,(h^2)_{,\nu}h_{\mu\nu}$$
$$- \frac{1}{12}h_{\mu\nu}h_{\lambda\nu,\rho}h_{\lambda\rho} + \frac{1}{6}h_{\mu\nu,\rho}h_{\lambda\nu}h_{\lambda\rho} + \frac{1}{24}h_{\mu\nu}h_{\lambda\nu}h_{\lambda\rho,\rho} - \frac{1}{8}(\phi\,{\rm Tr}\,(h^2))_{,\mu}\Big) + \cdots\,.$$

Lorentz indices factorize but no KLT in sight. Not satisfactory.

Ugly Lagrangian but relatively pretty amplitudes

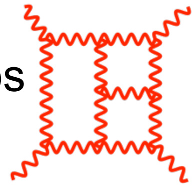
More recent work:

- **Order by order (ugly) “double copy” Lagrangian.**
  ZB, Dennen, Huang, Kiermaier (2004); Mathias Tolotti, Stefan Weinzierl
- **All orders Lorentz index factorization in compact Lagrangian.**
  Cheung and Remmen (2016)

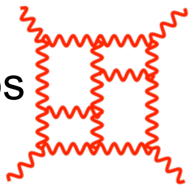
To this day no satisfactory all-order “double-copy Lagrangian”

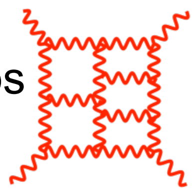
Feynman Diagrams for Gravity

Suppose we want to check UV properties of supergravity theories:

3 loops  $\sim 10^{20}$ TERMS

No surprise it has never been calculated via Feynman diagrams.

4 loops  $\sim 10^{26}$ TERMS

5 loops  $\sim 10^{31}$ TERMS

More terms than atoms in your brain!

- Calculations to settle this seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

Superspace helps, but not enough to make a difference.
Standard techniques utterly hopeless.

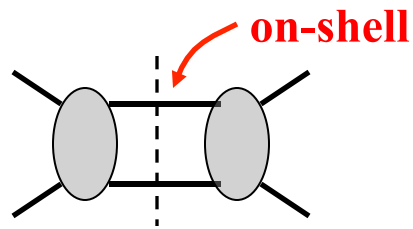
Message from string theory: this is the wrong way to look at it

Bern, Dixon, Dunbar and Kosower (1994)

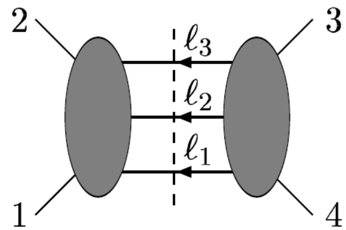
Modern Unitarity Method

To get KLT into loops needed new tools

Two-particle cut:

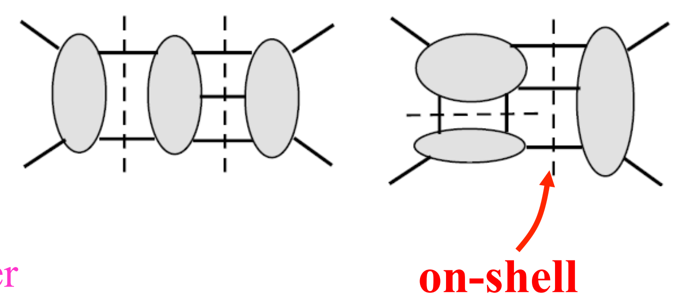


Three-particle cut:



Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool



Different cuts merged to give an expression with correct cuts in all channels.

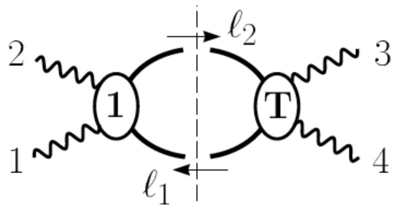
Bern, Dixon and Kosower
Britto, Cachazo and Feng

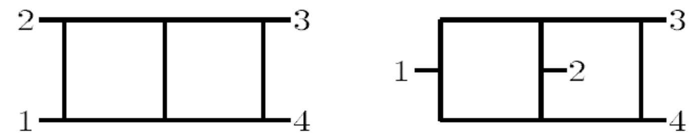
Reproduces Feynman diagrams except intermediate steps of calculation based on physical quantities not unphysical ones.

Gravity

ZB, Dixon, Dunbar, Perelstein and Rozowsky (1998)

Use KLT in cuts to convert $N = 4$ sYM to $N = 8$ supergravity





$$\mathcal{M}_4^{\text{2-loop}}(1,2,3,4) = -i\left(\frac{\kappa}{2}\right)^6 [s_{12}s_{23}\,A_4^{\text{tree}}(1,2,3,4)]^2 \Big(s_{12}^2\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23}) + s_{12}^2\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{24}) + s_{12}^2\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{23}) + s_{12}^2\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{24}) \quad + \text{cyclic}\Big)$$

Result very similar to $N = 4$ sYM. A strong hint of things to come!

A powerful tool to study gravity at loop level.

Still there were some difficulties:

- **Analytic structure of KLT complicated. Spurious singularities.**
- **Crossing symmetry not manifest.**

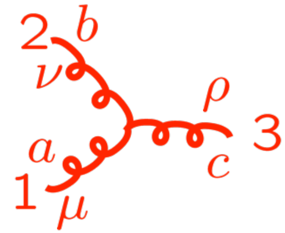
Solution to this came from a unexpected direction.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

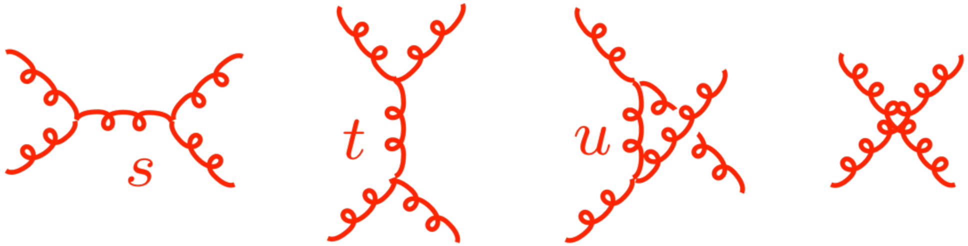
coupling constant → color factor → momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s*/*s* = *t*/*t* = *u*/*u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Proven at tree level and conjectured at loop level.

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Duality Between Color and Kinematics

Consider five-point tree amplitude:

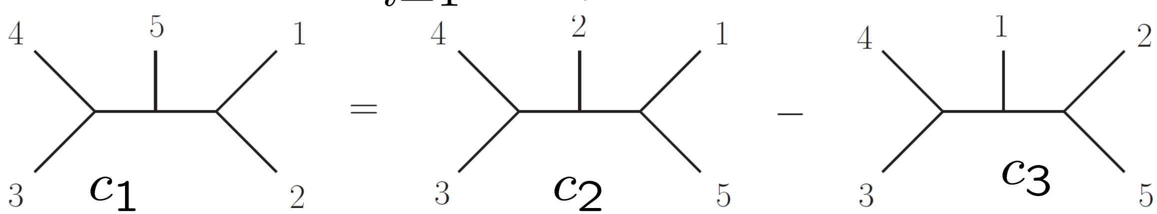
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\rm tree} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

color factor (c_i)

kinematic numerator factor (n_i)

Feynman propagators ($\prod_{\alpha_i} p^2_{\alpha_i}$)



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

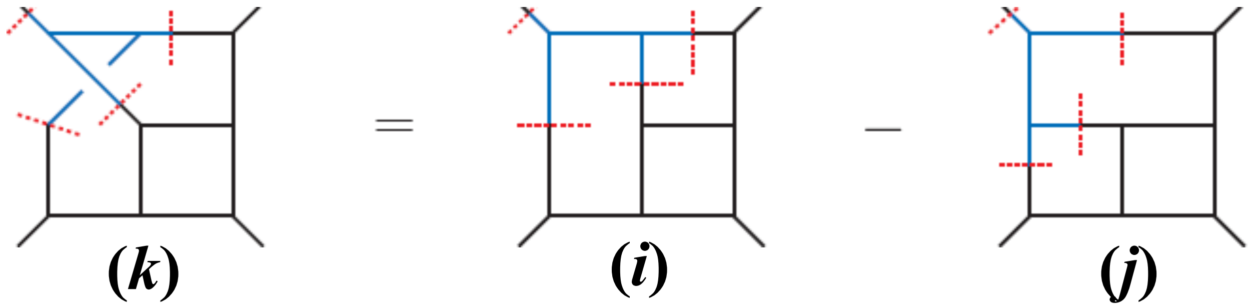
Progress on unraveling relations.

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer;
Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer
O'Connell and Montiero; Bjerrum-Bohr, Damgaard, O'Connell and Montiero; O'Connell, Montiero, White, etc.

Gravity Loop Integrands from YM

BCJ

Ideas conjectured to generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

$$c_k \rightarrow n_k$$

Gravity loop integrands follow from gauge theory!

Gravity From Gauge Theory

Here we consider only simplest constructions:

$$\mathcal{N}=8 \text{ sugra:} \quad (\mathcal{N}=4 \text{ sYM})\times(\mathcal{N}=4 \text{ sYM})$$
$$\mathcal{N}=5 \text{ sugra:} \quad (\mathcal{N}=4 \text{ sYM})\times(\mathcal{N}=1 \text{ sYM})$$
$$\mathcal{N}=4 \text{ sugra:} \quad (\mathcal{N}=4 \text{ sYM})\times(\mathcal{N}=0 \text{ sYM})$$

Spectrum controlled by simple tensor product of gauge theories.

More sophisticated lower-susy cases: QCD, magical supergravities, Einstein-YM with and without Higgsing, twin supergravities.

Anastasiou, Bornsten, Duff; Duff, Hughs, Nagy; Johansson and Ochirov;
Carrasco, Chiodaroli, Günaydin and Roiban; ZB, Davies, Dennen, Huang and Nohle;
Nohle; Chiodaroli, Günaydin, Johansson, Roiban. A. Anastasiou, L. Borsten, M.J. Duff, M.J. Hughes, Marrani, Nagy, Zoccali.

Many other theories in double-copy story, including open and closed string theory, NLSM, Dirac Born Infeld, Galileon and Z theory.

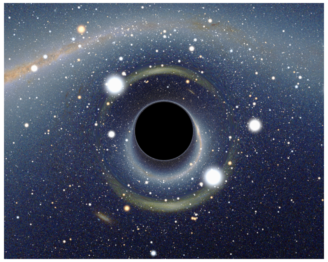
Cachazo, He, Yuan; Chen Du, Broedel, Schlotterer and Stieberger; Carrasco, Mafra, Schlotterer;

Hard to keep track of theories where double copy holds.

Applications to Black Hole Physics

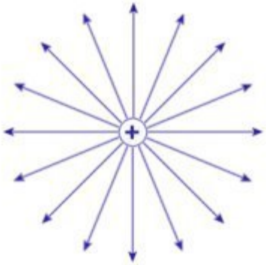
Wouldn't it be really cool if every classical solution in gravity could be mapped to a double copy of classical solutions?

Where to start? **Obviously the coolest place possible: black holes.**



black hole

$\longleftrightarrow$



point charge

Monteiro, O'Connell and White

Special coordinates: Kerr-Schild coordinates:

Schwarzschild black hole $$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \qquad \phi(r) = \frac{2m}{r}$$

***k* is null**

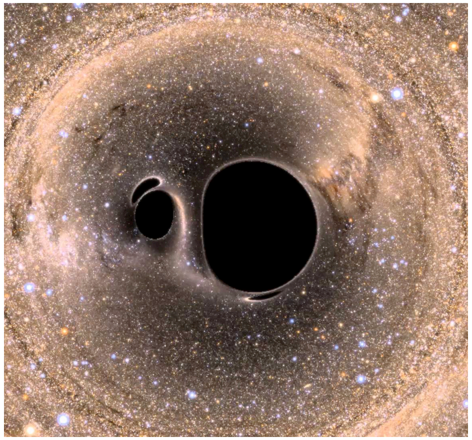
Coulomb point charge $$A_\mu = \phi k_\mu \qquad \phi(r) = \frac{Q}{r}$$

Schwarzschild ~ (Coulomb)^2

Applications to Black Hole Physics

Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White
Ridgway and Wise; Goldberger and Ridgway

A variety of other cases:



- Kerr (rotating) black hole.
- Taub-NUT space.
- Solutions with cosmological constant.
- Radiation from accelerating black hole.

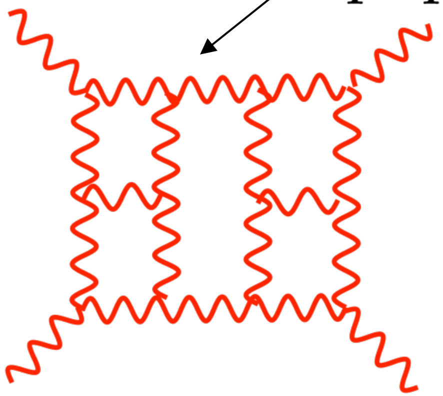
It may be possible to extend this to more general cases.
Recent paper explains possible application to LIGO case.

Goldberger and Ridgway; Luna, Monteiro, Nicholson, Ochirov, O'Connell, Westerberg, White

Application of Double Copy: UV Behavior of Gravity.

UV Behavior of Gravity?

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots \kappa p_j^\mu p_j^\nu \cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots g p_j^\nu \cdots}{\text{propagators}}$$

- **Extra powers of loop momenta in numerator means integrals are badly behaved in the UV and must diverge at some loop order.**
- **Much more sophisticated power counting in supersymmetric theories but this is basic idea.**

- **With more supersymmetry expect better UV properties.**
- **Need to worry about "hidden cancellations".**
- ***N* = 8 supergravity *best* theory to study.**

$N = 8$ supergravity: Where is First $D = 4$ UV Divergence?

3 loops $N = 8$	Green, Schwarz, Brink (1982); Howe and Stelle (1989); Marcus and Sagnotti (1985)	✗
5 loops $N = 8$	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998); Howe and Stelle (2003,2009)	✗
6 loops $N = 8$	Howe and Stelle (2003)	✗
7 loops $N = 8$	Grisaru and Siegel (1982); Bossard, Howe, Stelle (2009);Vanhove; Björnsson, Green (2010); Kiermaier, Elvang, Freedman(2010); Ramond, Kallosh (2010); Biesert et al (2010); Bossard, Howe, Stelle, Vanhove (2011)	?
3 loops $N = 4$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 5$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 4$	Vanhove and Tourkine (2012)	✓
9 loops $N = 8$	Berkovits, Green, Russo, Vanhove (2009)	✗

ZB, Kosower, Carrasco, Dixon, Johansson, Roiban; ZB, Davies, Dennen, A. Smirnov, V. Smirnov; series of calculations.

← **This is what we are most interested in.**

← **Weird structure. Anomaly-like behavior of divergence.**

← **Retracted, but perhaps to be unretracted.**

- **Conventional wisdom holds that it will diverge sooner or later.**
- **Track record of predictions from symmetry not great.**

Supersymmetry and Ultraviolet Divergences

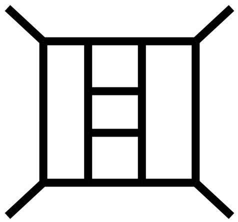
Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Björnsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Bossard, Howe, Stelle, Vanhove, etc

- **First quantized formulation of Berkovits' pure-spinor formalism.** Bjornsson and Green
- **Key point: *all* supersymmetry cancellations are exposed.**

Poor UV behavior, unless new types of cancellations between diagrams exist that are "not consequences of supersymmetry in any conventional sense"

"Since we have not evaluated the precise values of the coefficients the possibility of terms vanishing or cancellations between different contributions to the amplitude cannot be ruled out." Bjornsson and Green

- **$N = 8$ sugra should diverge at 5 loops in $D = 24/5$.**
- **$N = 8$ sugra should diverge at 7 loops in $D = 4$.**



Consensus agreement from all power-counting methods, including our unitarity approaches.

Scorecard on Symmetry Predictions

- $\mathcal{N} = 8$ sugra should diverge at 5 loops in $D = 24/5$. ? ← will answer this here
- $\mathcal{N} = 8$ sugra should diverge at 7 loops in $D = 4$. ?
- $\mathcal{N} = 4$ sugra should diverge at 3 loops in $D = 4$. ✗
- $\mathcal{N} = 5$ sugra should diverge at 4 loops in $D = 4$. ✗
- Half maximal sugra diverges at 2 loops in $D = 5$. ✗

ZB, Davies, Dennen (2012, 2014); ZB, Davies, Dennen, Huang(2012)

String theory arguments against 3 loop divergence in $\mathcal{N} = 4$.
Not symmetry arguments. Calculations coupled with extrapolations

Tourkine and Vanhove (2012); Green and Rudra (2016)

$\mathcal{N} = 4$ sugra has an anomaly that confuses the situation.

Marcus; Carrasco, Kallosh, Roiban, Tseytlin; ZB, Davies, Dennen. Smirnov, Smirnov;
ZB, Parra-Martinez, Roiban

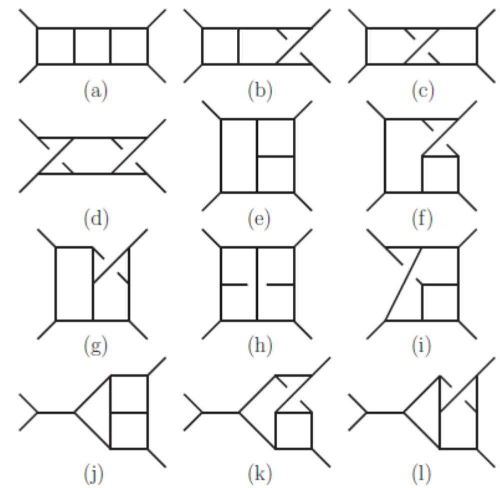
UV cancellation of $\mathcal{N} = 5$ supergravity at 4 loops in $D = 4$ remains a mystery, showing clear problem with standard symmetry arguments.

Our goal is to provide definitive answers.

Enhanced UV Cancellations

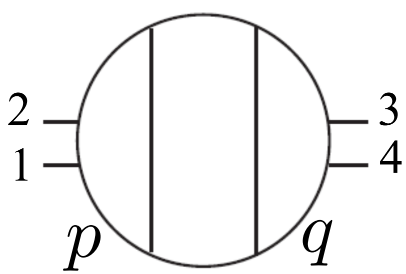
ZB, Davies, Dennen

Suppose diagrams in *all* possible Lorentz covariant representations are UV divergent, but the amplitude is well behaved.



- **By definition this is an enhanced cancellation.**
- **Not the way nonabelian gauge theory works.**

$\mathcal{N} = 4$ sugra



already log divergent

$\mathcal{N} = 4$ sugra: pure YM x $\mathcal{N} = 4$ sYM

$$n_i \sim s^3 t A_4^{\text{tree}} (p\cdot q)^2 \, \varepsilon_1\cdot p \, \varepsilon_2 \cdot p \, \varepsilon_3 \cdot q \, \varepsilon_4 \cdot q + \ldots$$

This diagram is log divergent

- **3 loop UV finiteness of $\mathcal{N} = 4$ supergravity is example of "enhanced cancellation" in supergravity theories.**
- **No known standard symmetry explanation.**

$N = 5$ Supergravity Four Loop Cancellations

ZB, Davies and Dennen

We calculated four-loop divergence in $N = 5$ supergravity.

Industrial strength software needed: FIRE5 and special purpose C++

$N = 5$ sugra: ($N = 4$ sYM) x ($N = 1$ sYM)

Crucial help from (Smirnov)^2

$N = 4$ sYM **$N = 1$ sYM**

Diagrams necessarily UV divergent.

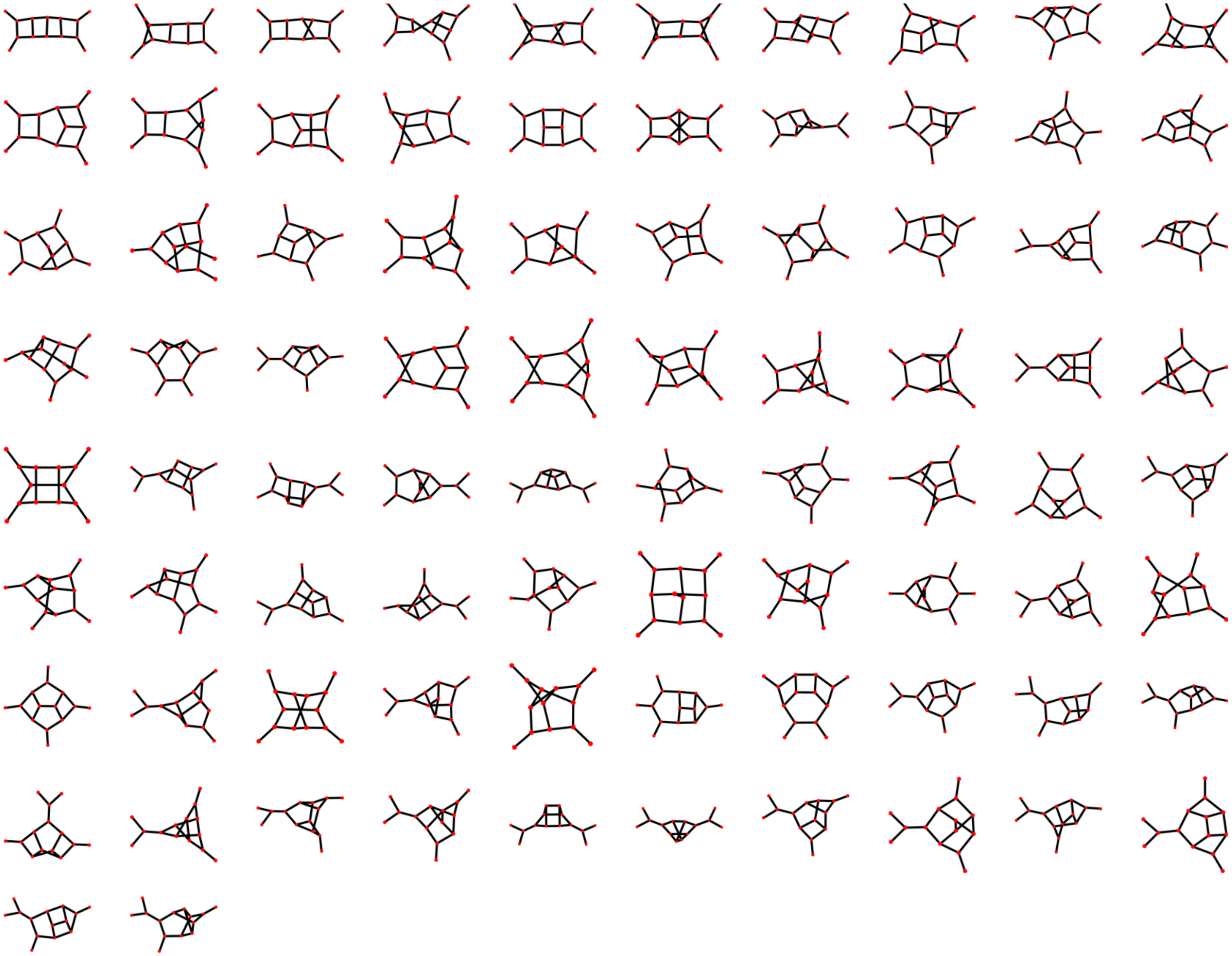
$N = 5$ supergravity has no divergence at four loops in $D = 4$.

Nontrivial example of an "enhanced cancellation".

No standard-symmetry (or other) explanation known!

82 nonvanishing numerators in BCJ representation

ZB, Carrasco, Dixon, Johansson, Roiban ($\mathcal{N} = 4$ sYM)



$N = 5$ supergravity at Four Loops

Special purpose C++ and FIRE5

ZB, Davies and Dennen

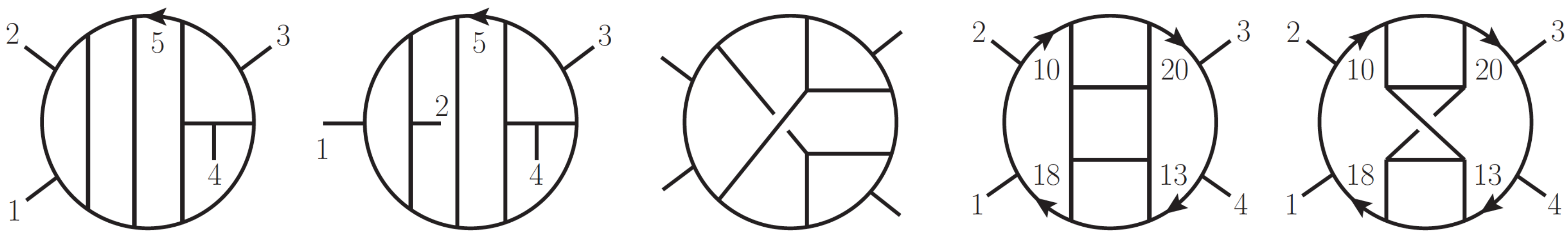
graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\rm tree}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\Big[\frac{7358585}{7962624}s^2+\frac{2561447}{2654208}st-\frac{872683}{1990656}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{75972559}{35389440}s^2+\frac{240984061}{26542080}st+\frac{1302037}{1310720}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{369234283}{11059200}s^2-\frac{257792411}{4915200}st-\frac{101847769}{14745600}t^2\big)+\zeta_2\big(\frac{7358585}{3981312}s^2+\frac{2561447}{1327104}st-\frac{872683}{995328}t^2\big)-\mathrm{S2}\big(\frac{1223621}{49152}s^2+\frac{46816475}{442368}st+\frac{2639903}{221184}t^2\big)+\frac{206093335871}{11466178560}s^2+\frac{320983191023}{3822059520}st+\frac{53309416589}{2866544640}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{84777347}{368640}s^2+\frac{382194721}{1474560}st+\frac{417476581}{1474560}t^2\big)-\zeta_4\big(\frac{3062401}{2457600}s^2+\frac{3881051}{3276800}st-\frac{112081813}{29491200}t^2\big)+\zeta_3\big(\frac{28162691399797}{53747712000}s^2+\frac{19354492750651}{35831808000}st-\frac{22092683352811}{107495424000}t^2\big)-\zeta_2\big(\frac{70861961}{17694720}s^2+\frac{227180689}{13271040}st+\frac{105727243}{53084160}t^2\big)+\mathrm{T1ep}\big(-\frac{1223621}{663552}s^2-\frac{46816475}{5971968}st-\frac{2639903}{2985984}t^2\big)-\mathrm{S2}\big(\frac{11916028151}{5898240}s^2+\frac{72637733971}{13271040}st+\frac{17223563447}{53084160}t^2\big)+\mathrm{D6}\big(-\frac{9001177}{552960}s^2-\frac{264491}{10240}st-\frac{2610157}{552960}t^2\big)+\frac{110945914744727}{1146617856000}s^2+\frac{16989492195991}{127401984000}st-\frac{21362122998269}{573308928000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\Big[-\frac{5502451}{2654208}s^2-\frac{3675877}{884736}st+\frac{11269}{497664}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{38102993}{26542080}s^2-\frac{291607201}{106168320}st-\frac{565798829}{318504960}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(\frac{108955183}{2211840}s^2+\frac{653019571}{8847360}st+\frac{9453043}{1769472}t^2\big)+\zeta_2\big(-\frac{5502451}{1327104}s^2-\frac{3675877}{442368}st+\frac{11269}{248832}t^2\big)+\mathrm{S2}\big(\frac{16797481}{1327104}s^2+\frac{1172969}{16384}st+\frac{978427}{82944}t^2\big)-\frac{304243754383}{19110297600}s^2-\frac{2032063711381}{19110297600}st-\frac{257798086613}{7166361600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(\frac{33327659}{122880}s^2+\frac{13276219}{24576}st+\frac{22251887}{184320}t^2\big)+\zeta_4\big(\frac{12299887}{1474560}s^2+\frac{258056147}{5898240}st+\frac{46913759}{5898240}t^2\big)+\zeta_3\big(-\frac{26846001990157}{42998169600}s^2-\frac{337106527201}{265420800}st-\frac{5298324906787}{42998169600}t^2\big)+\zeta_2\big(\frac{282283789}{39813120}s^2+\frac{975199319}{53084160}st+\frac{60394451}{159252480}t^2\big)+\mathrm{T1ep}\big(\frac{16797481}{17915904}s^2+\frac{1172969}{221184}st+\frac{978427}{1119744}t^2\big)+\mathrm{S2}\big(\frac{10516980893}{4976640}s^2+\frac{389045625329}{53084160}st+\frac{216032337589}{159252480}t^2\big)+\mathrm{D6}\big(\frac{503413}{23040}s^2+\frac{12342607}{552960}st+\frac{3661}{184320}t^2\big)-\frac{166777358259461}{1146617856000}s^2-\frac{565137511429117}{1146617856000}st-\frac{21629055712141}{191102976000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\Big[\frac{285899}{248832}s^2+\frac{1058273}{331776}st+\frac{275869}{663552}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{380329649}{106168320}s^2-\frac{74703227}{11796480}st+\frac{124701919}{159252480}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{1371419}{86400}s^2-\frac{236241539}{11059200}st+\frac{4326077}{2764800}t^2\big)+\zeta_2\big(\frac{285899}{124416}s^2+\frac{1058273}{165888}st+\frac{275869}{331776}t^2\big)+\mathrm{S2}\big(\frac{8120143}{663552}s^2+\frac{1893289}{55296}st+\frac{92293}{663552}t^2\big)-\frac{58867708103}{28665446400}s^2+\frac{71191292711}{3185049600}st+\frac{83016363427}{4777574400}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{1520563}{36864}s^2-\frac{1178767861}{1474560}st-\frac{595491677}{1474560}t^2\big)-\zeta_4\big(\frac{6539029}{921600}s^2+\frac{313837819}{7372800}st+\frac{21665663}{1843200}t^2\big)+\zeta_3\big(\frac{20790944575597}{214990848000}s^2+\frac{6505876281371}{8957952000}st+\frac{70676991239557}{214990848000}t^2\big)+\zeta_2\big(-\frac{491377507}{159252480}s^2-\frac{66476563}{53084160}st+\frac{128393639}{79626240}t^2\big)+\mathrm{T1ep}\big(\frac{8120143}{8957952}s^2+\frac{1893289}{746496}st+\frac{92293}{8957952}t^2\big)+\mathrm{S2}\big(-\frac{14810628499}{159252480}s^2-\frac{19698937889}{10616832}st-\frac{10272602953}{9953280}t^2\big)+\mathrm{D6}\big(-\frac{616147}{110592}s^2+\frac{1939907}{552960}st+\frac{1299587}{276480}t^2\big)+\frac{9307894793789}{191102976000}s^2+\frac{206124003456599}{573308928000}st+\frac{21562322533673}{143327232000}t^2\Big]$

graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\rm tree}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\Big[\frac{1052159}{995328}s^2+\frac{509789}{331776}st-\frac{121001}{497664}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{9042569}{1474560}s^2+\frac{34360945}{1327104}st+\frac{73518401}{13271040}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{11443919}{2764800}s^2+\frac{32520079}{552960}st+\frac{5836531}{230400}t^2\big)+\zeta_2\big(\frac{1052159}{497664}s^2+\frac{509789}{165888}st-\frac{121001}{248832}t^2\big)-\mathrm{S2}\big(\frac{637991}{6144}s^2+\frac{10978729}{27648}st+\frac{5080825}{55296}t^2\big)+\big(\frac{270806866183}{7166361600}s^2+\frac{89848068067}{597196800}st+\frac{218093645149}{7166361600}t^2\big)\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(\frac{100843}{360}s^2+\frac{17118043}{30720}st-\frac{30266471}{92160}t^2\big)+\zeta_4\big(\frac{11435323}{614400}s^2+\frac{232002227}{1843200}st+\frac{22211783}{460800}t^2\big)+\zeta_3\big(\frac{223300432349}{3359232000}s^2-\frac{178732984847}{716636160}st+\frac{951659436383}{53747712000}t^2\big)-\zeta_2\big(\frac{5492357}{245760}s^2+\frac{53468887}{663552}st+\frac{129714599}{6635520}t^2\big)+\mathrm{T1ep}\big(-\frac{637991}{82944}s^2-\frac{10978729}{373248}st-\frac{5080825}{746496}t^2\big)+\mathrm{S2}\big(-\frac{5700088747}{3686400}s^2-\frac{69470348491}{16588800}st-\frac{713512871}{6635520}t^2\big)+\mathrm{D6}\big(-\frac{357421}{43200}s^2-\frac{2891743}{230400}st-\frac{470219}{138240}t^2\big)-\frac{3571506237341}{28665446400}s^2-\frac{1611591325291}{5971968000}st+\frac{2301084608777}{143327232000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\Big[-\frac{150715}{82944}s^2-\frac{668333}{221184}st-\frac{7213}{1990656}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{68021833}{13271040}s^2-\frac{36852103}{1327104}st-\frac{298377299}{39813120}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{36448033}{2764800}s^2-\frac{455889533}{2764800}st-\frac{82059281}{1382400}t^2\big)+\zeta_2\big(-\frac{150715}{41472}s^2-\frac{668333}{110592}st-\frac{7213}{995328}t^2\big)+\mathrm{S2}\big(\frac{13910839}{165888}s^2+\frac{1340033}{4096}st+\frac{26303855}{331776}t^2\big)-\frac{68286245653}{2388787200}s^2-\frac{20649690431}{119439360}st-\frac{351701043553}{7166361600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{2362679}{9216}s^2-\frac{178668311}{92160}st-\frac{1268313}{10240}t^2\big)+\zeta_4\big(-\frac{124344121}{1843200}s^2-\frac{491722333}{1843200}st-\frac{68141309}{921600}t^2\big)-\zeta_3\big(\frac{630084012997}{53747712000}s^2-\frac{1250670277213}{663552000}st-\frac{6913218302303}{13436928000}t^2\big)+\zeta_2\big(\frac{352368061}{19906560}s^2+\frac{35509679}{663552}st+\frac{227699801}{19906560}t^2\big)+\mathrm{T1ep}\big(\frac{13910839}{2239488}s^2+\frac{1340033}{55296}st+\frac{26303855}{4478976}t^2\big)+\mathrm{S2}\big(\frac{188312318729}{99532800}s^2+\frac{110749829741}{16588800}st+\frac{5056299197}{3981312}t^2\big)+\mathrm{D6}\big(\frac{1220779}{76800}s^2+\frac{44791}{6912}st-\frac{1159831}{230400}t^2\big)+\frac{2755666297013}{28665446400}s^2+\frac{5622513975899}{35831808000}st-\frac{196197363193}{1769472000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\Big[\frac{756421}{995328}s^2+\frac{985421}{663552}st+\frac{163739}{663552}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{1670161}{1658880}s^2+\frac{415193}{221184}st+\frac{4863881}{2488320}t^2\Big]+\frac{1}{\epsilon^2}\Big[\zeta_3\big(\frac{110861}{6400}s^2+\frac{16293841}{153600}st+\frac{9408019}{276480}t^2\big)+\zeta_2\big(\frac{756421}{497664}s^2+\frac{985421}{331776}st+\frac{163739}{331776}t^2\big)+\mathrm{S2}\big(\frac{1657459}{82944}s^2+\frac{7734025}{110592}st+\frac{4181095}{331776}t^2\big)-\frac{8243516153}{895795200}s^2+\frac{558349337}{24883200}st+\frac{11133949867}{597196800}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{1094509}{46080}s^2+\frac{63657091}{46080}st+\frac{5210161}{11520}t^2\big)+\zeta_4\big(\frac{11254769}{230400}s^2+\frac{129860053}{921600}st+\frac{23717743}{921600}t^2\big)-\zeta_3\big(\frac{2745647960587}{53747712000}s^2+\frac{3654260151947}{2239488000}st+\frac{5720906529119}{10749542400}t^2\big)+\zeta_2\big(\frac{11564107}{2488320}s^2+\frac{2244901}{82944}st+\frac{40360999}{4976640}t^2\big)+\mathrm{T1ep}\big(\frac{1657459}{1119744}s^2+\frac{7734025}{1492992}st+\frac{4181095}{4478976}t^2\big)+\mathrm{S2}\big(-\frac{420043}{1215}s^2-\frac{825589625}{331776}st-\frac{5785239343}{4976640}t^2\big)+\mathrm{D6}\big(-\frac{210731}{27648}s^2+\frac{4196129}{691200}st+\frac{1457647}{172800}t^2\big)+\frac{33976742047}{1194393600}s^2+\frac{4046536311847}{35831808000}st+\frac{212357840779}{2239488000}t^2\Big]$

Adds up to zero: no divergence. Enhanced cancellations!
No standard (super)symmetry explanation exists.
Must be a better way. Enhanced cancellations nontrivial.

$N = 8$ Sugra 5 Loop Calculation

What is the true UV behavior of $N = 8$ sugra.



Place your bets:

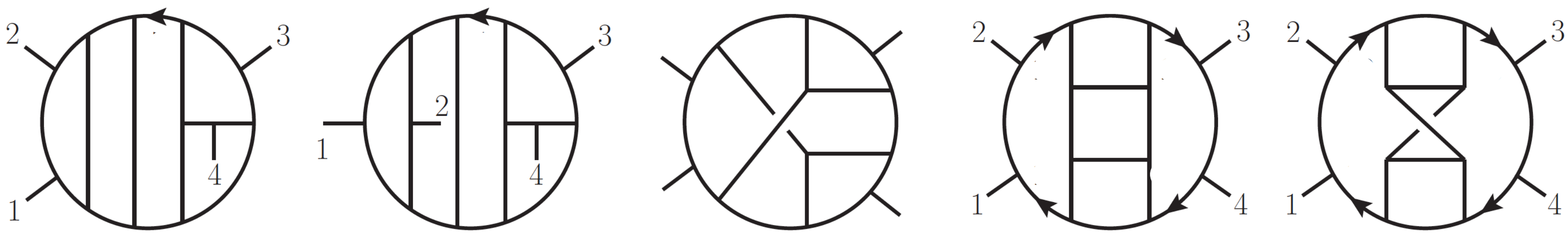
- **At 5 loops in $D = 24/5$ does $N = 8$ supergravity diverge?**
- **At 7 loops in $D = 4$ does $N = 8$ supergravity diverge?**

5 loops

Kelly Stelle:	**Zvi Bern:**
English wine	**California wine**
"It will diverge"	**"It won't diverge"**

$\mathcal{N} = 8$ Sugra 5 Loop Calculation

What is the true UV behavior of $\mathcal{N} = 8$ sugra.



Place your bets:

- **At 5 loops in $D = 24/5$ does $\mathcal{N} = 8$ supergravity diverge?**
- **At 7 loops in $D = 4$ does $\mathcal{N} = 8$ supergravity diverge?**

7 loops

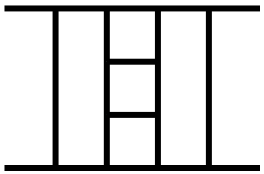
David Gross:
California wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

Finding BCJ Forms Nontrivial

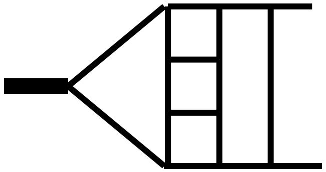
Gravity integrands might be "free", but gauge-theory ones are not. Trouble beyond four loops.

5-loop 4-pt $\mathcal{N} = 4$ sYM amplitude:



Despite considerable effort no one has succeeded in finding a BCJ form.

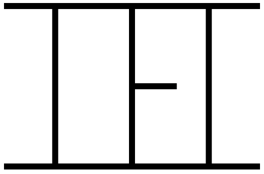
$\mathcal{N} = 4$ sYM 5 loop form factor:



On other hand, no trouble with form factor.

Gang Yang (2016)

Two-loop five-point QCD identical helicity:



This required an ansatz with curiously high power counting.

O'Connell and Mogull (2015)

It can be difficult to find BCJ representations.

New Contact Term Method

ZB, Carrasco, Chen, Johansson, Roiban, Zeng (2017)

Task is to convert $\mathcal{N} = 4$ sYM 5-loop integrand into $\mathcal{N} = 8$ sugra.

BCJ representation hard to find and KLT on cuts too complicated.

New method:

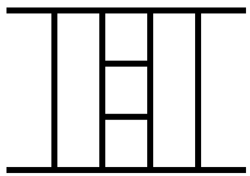
Start with "naïve double copy" of *any* correct sYM integrand:

$c_i \to n_i$ ***Not* a BCJ representation**

Without BCJ duality, *not* the correct $\mathcal{N} = 8$ integrand

$\mathcal{N} = 8$ cuts:

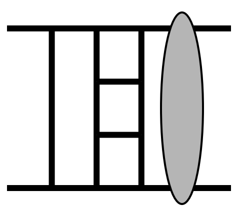
Max cuts:



Automatic

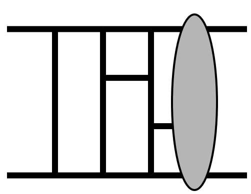
Generalized Unitarity
All exposed legs on shell

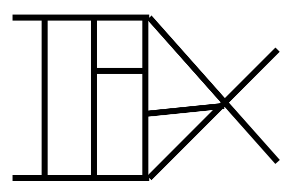
Nmax cuts:



Automatic via BCJ, 4pt trees always work

N^2max cuts:

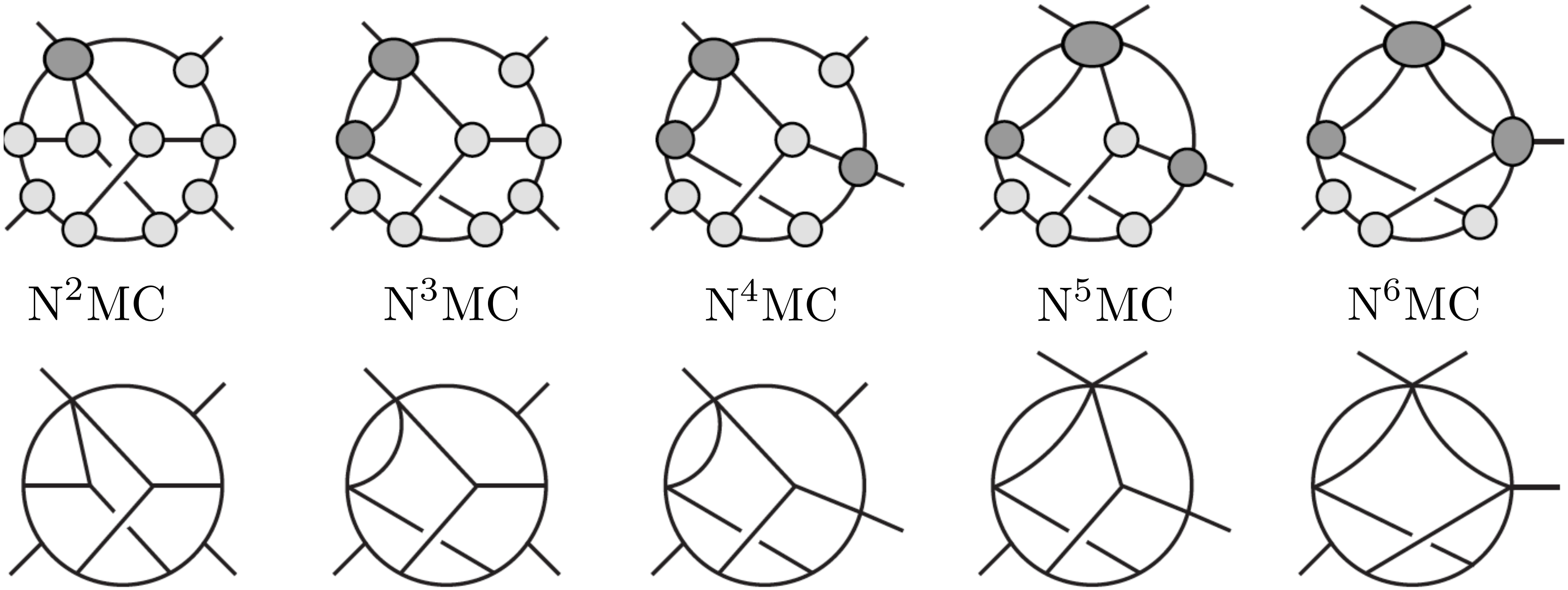




Add contact term to make it work

Contact Term Method

contact = (gravity cut) – (cut of incomplete amplitude)



- **Contact each associated with each cut directly giving missing piece of amplitude.**
- **75K cuts need to be evaluated.**
- **Sounds daunting. Not for faint of heart!**

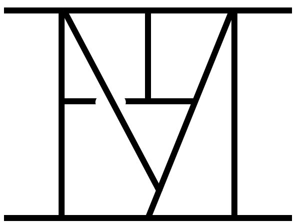
Game for optimists: "Simplifying miracle is around the corner"

A Simplifying Miracle

ZB, Carrasco, Chen, Johansson, Roiban, Zeng (2017)

contact = (gravity cut) – (cut of incomplete amplitude)

1. **Most contact terms vanish!**
2. **Gravity contacts far simpler than expected.**
3. **Four-point double-contacts factorize. Extremely striking.**



double 4pt contact

$$\left[2s^3 - s^2u + 4s^2(2k_1\cdot l_6) + \cdots\right]$$
$$\times\left[s^2u + 2su^2 - s^2(2k_1\cdot l_6) + \cdots\right]$$

Each factor looks like gauge theory

Reminds us of KLT factorization:

$$M^{\rm tree}(1,2,3,4) = s_{12}A^{\rm tree}(1,2,3,4)\times A^{\rm tree}(1,2,4,3)$$

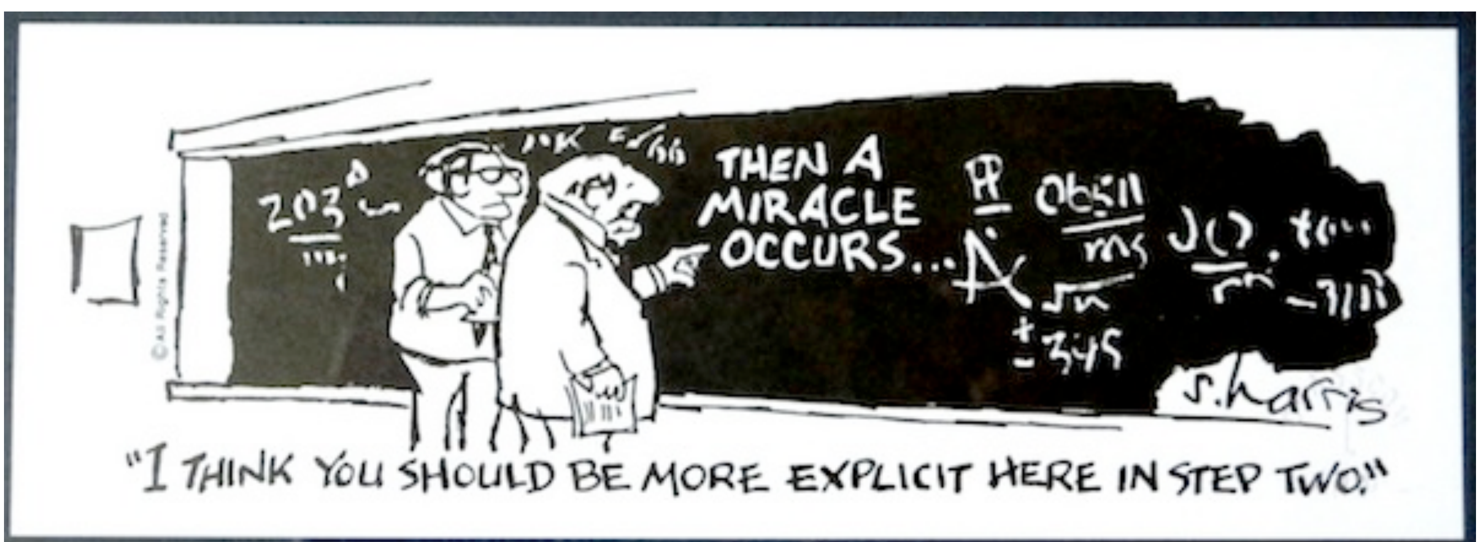
For 5 or higher-point contacts no overall factorization, as with KLT.

Can we write down formulas that give missing gravity pieces directly from gauge theory, bypassing complicated gravity cuts?

A Miracle

1. Start from gauge-theory loop amplitude.
2. Construct naïve double copy.
3. Compute cut of naïve double copy.
4. Compute gravity cut from gauge-theory cuts via KLT.
5. Subtract and shake hard (nontrivial).
6. Extract surprisingly simple gravity contact.

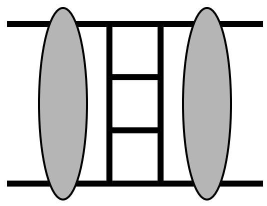
Miracle: The contact terms are so simple we should be able write down missing gravity contacts directly from gauge theory.

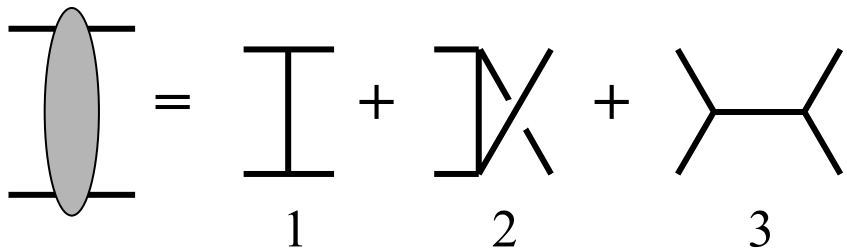


BCJ Discrepancy Functions

Need a function defined purely in gauge theory as building block for missing gravity pieces.

Inside multiloop diagram





kinematic numerators

BCJ discrepancy function: $$J \equiv \sum_{i=1}^{3} n_i$$

Vanishes if we have BCJ form of gauge theory.

Obvious guess is these are building blocks for missing gravity pieces.

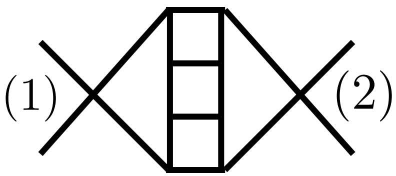
Missing pieces: $$\sim \sum J \times J$$

Gravity from Gauge Theory

ZB, Carrasco, Chen, Johansson, Roiban

Missing gravity from any gauge theory representation

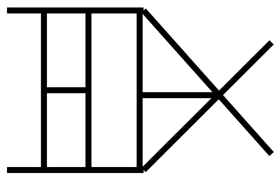
BCJ discrepancy functions



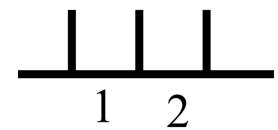
$$\mathcal{E}^{4\times 4}_{\rm GR} = -\frac{1}{d_1^{(1)} d_1^{(2)}}\Big(J_{\bullet,1}\tilde{J}_{1,\bullet} + J_{1,\bullet}\tilde{J}_{\bullet,1}\Big)$$

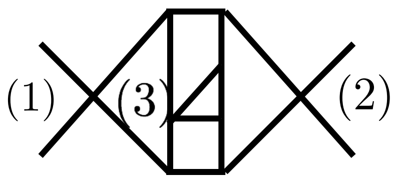
propagators cancel trivially

Expand into 15 diagrams



$$\mathcal{E}^{5}_{\rm GR} = -\frac{1}{6}\sum_{i=1}^{15}\frac{J_{\{i,1\}}\tilde{J}_{\{i,2\}} + J_{\{i,2\}}\tilde{J}_{\{i,1\}}}{d_i^{(1)} d_i^{(2)}}$$



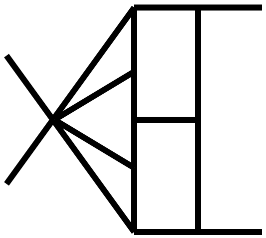


$$\mathcal{E}^{4\times4\times4}_{\rm GR} = -\sum_{i_3=1}^{3}\frac{J_{\bullet,1,i_3}\tilde{J}_{1,\bullet,i_3}}{d_1^{(1)}d_1^{(2)}d_{i_3}^{(3)}} - \sum_{i_2=1}^{3}\frac{J_{\bullet,i_2,1}\tilde{J}_{1,i_2,\bullet}}{d_1^{(1)}d_{i_2}^{(2)}d_1^{(3)}} - \sum_{i_1=1}^{3}\frac{J_{i_1,\bullet,1}\tilde{J}_{i_1,1,\bullet}}{d_{i_1}^{(1)}d_1^{(2)}d_1^{(3)}} + \frac{J_{\bullet,1,1}\tilde{J}_{1,\bullet,\bullet}}{d_1^{(1)}d_1^{(2)}d_1^{(3)}} + \frac{J_{1,\bullet,1}\tilde{J}_{\bullet,1,\bullet}}{d_1^{(1)}d_1^{(2)}d_1^{(3)}} + \frac{J_{1,1,\bullet}\tilde{J}_{\bullet,\bullet,1}}{d_1^{(1)}d_1^{(2)}d_1^{(3)}} + \{J \leftrightarrow \tilde{J}\}$$

Etc.

- **Applies to *any* adjoint gauge theory, not just $N = 4$ sYM.**
- **Simple generalization for asymmetric double copies.**
- **Same constructions work at tree level! Five-point formula similar to known tree formula.** Bjerrum-Bohr, Damgaard, Søndergaad, Vanhove

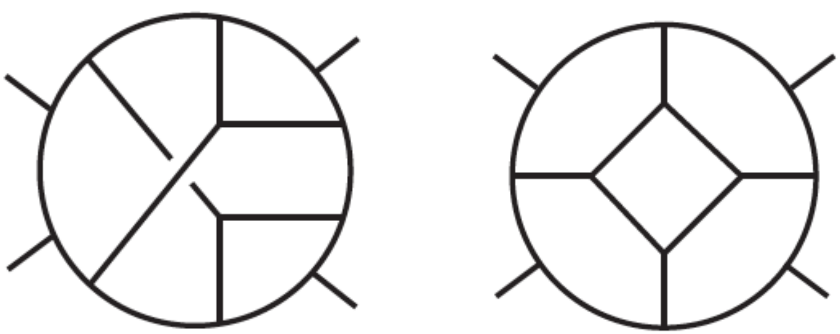
5 Loop $N = 8$ supergravity



Generalized double copy enormously simplifies the computation of missing gravity contact terms. The impossible becomes doable!

We have constructed five-loop integrand!

ZB, Carrasco, Chen, Johansson, Roiban, Zeng (2017)

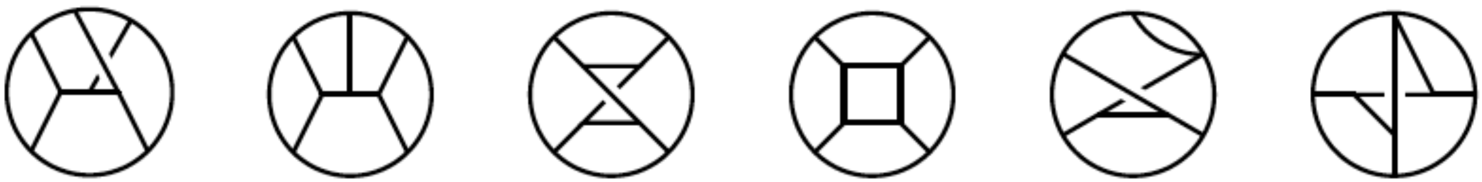


See mathematica attachment of paper for integrand.

Large Loop Momentum Expansion

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

- **Series expand and large loop momentum to extract log divergent terms.**
- **Obtain diagrams with no external momenta, analogous to vacuum diagrams.**



Originally constructed integrand poor: log divergent terms underneath spurious quartic divergence. Billions of terms. Not the way to evaluate it.

To deal with this:

- **Built a much better integrand without this difficulty.**
- **Applied efficient algorithms to integrate.**

Integrating $N = 8$ supergravity

Cheterkin and Tkachov

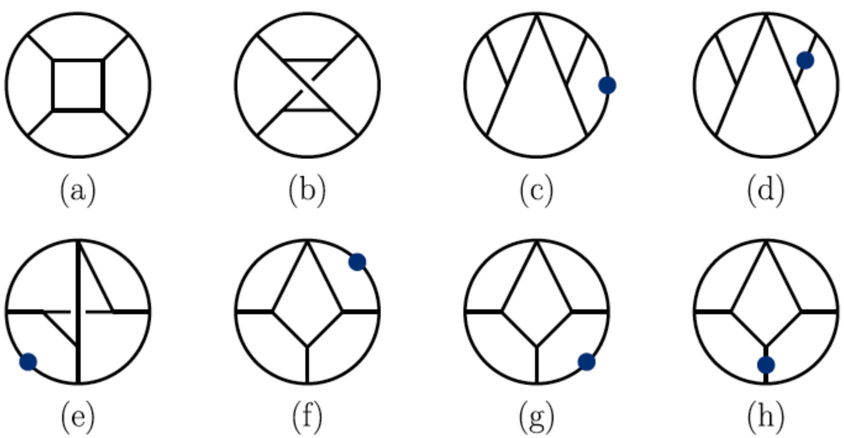
$$\int \prod_{k=1}^{5} \frac{d^D \ell_k}{(2\pi)^D} \frac{\partial}{\partial \ell_i^\mu} \frac{v_i^\mu}{\prod_j \ell_j^2} = 0 \qquad v_i^\mu \quad \textbf{ibp vector}$$

Smart choices make huge difference (don't mix UV and IR).

- **Lorentz invariance.**
- **Generators of SL(5) relabeling symmetry.**
- **Automorphism symmetries to generate identities.**

Integrals reduce to 8 master integrals.

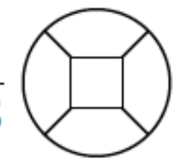
ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)



The result!

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

In $D = 24/5$ we obtain a divergence:

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16 \times 629}{25}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2 stu M_4^{\text{tree}} \left(\frac{1}{48}\right.$$

$$+\ \frac{1}{16}$$

$$\left.\right)$$

Integrals are positive definite
No "enhanced cancellations"

- **$N = 8$ sugra at $L = 5$ in $D = 24/5$ has no enhanced cancellation**
- **$N = 5$ sugra at $L = 4$ in $D = 4$ has enhanced cancellation.**

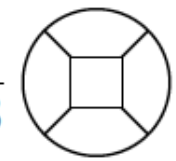
What is the difference? $D = 4$?

Need to get to at least 7 loops to study $N = 8$ sugra in $D = 4$.

$N = 8$ UV at Five loops

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

In $D = 24/5$ we obtain a divergence:

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16 \times 629}{25}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2 stu M_4^{\text{tree}} \left(\frac{1}{48}\right.$$  $$+ \frac{1}{16}$$  $$\left.\vphantom{\frac{1}{48}}\right)$$

With hindsight it is *very easy* to foresee this result, as I will show you.

But our purpose:

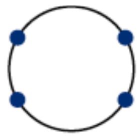
- **Determine the answer with complete certainty.**
  No "arguments". Only proven facts and calculations.
- **Understand the structures so we can get to 7 loops and beyond in $D = 4$.**
- **To build a firm foundation to be able to get to 7 and higher loops.**

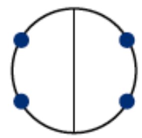
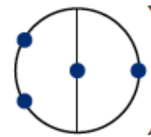
Higher-loop Structure.

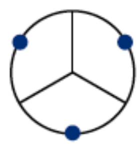
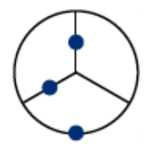
Green Schwarz, Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky;Carrasco, Dixon, Johansson, Roiban; ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng

Over the years we've obtained results for $N = 8$ sugra through five loops.

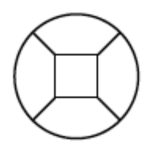
dots represent extra propagators

$\mathcal{M}_4^{(1)}\Big|_{\text{leading}} = -3\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^4$  , $D_c = 8$

$\mathcal{M}_4^{(2)}\Big|_{\text{leading}} = -8\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^6 (s^2+t^2+u^2) \Big(\frac{1}{4}$  $+ \frac{1}{4}$  $\Big)$, $D_c = 7$

$\mathcal{M}_4^{(3)}\Big|_{\text{leading}} = -60\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^8 stu \Big(\frac{1}{6}$  $+ \frac{1}{2}$  $\Big)$, $D_c = 6$

$\mathcal{M}_4^{(4)}\Big|_{\text{leading}} = -\frac{23}{2}\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^{10} (s^2+t^2+u^2)^2 \Big(\frac{1}{4}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $\Big)$, $D_c = \frac{11}{2}$

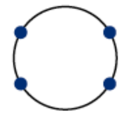
$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16 \times 629}{25}\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^{12} (s^2+t^2+u^2)^2 \Big(\frac{1}{48}$  $+ \frac{1}{16}$  $\Big)$, $D_c = \frac{24}{5}$

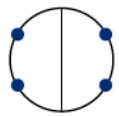
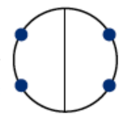
We now have a lot of theoretical "data" to guide us.

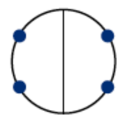
Higher-loop Structure.

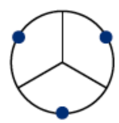
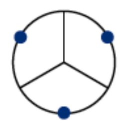
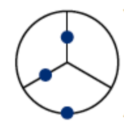
Green Schwarz, Brink; ZB, Rozowsky, Yan; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Carrasco, Dixon, Johansson, Roiban; ZB, Carraco, Dixon, Douglas, von Hippel, Johansson

Have up to six loop results for $N = 4$ sYM UV behavior:

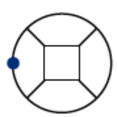
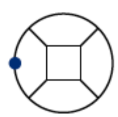
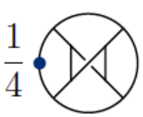
$\mathcal{A}_4^{(1)}\Big|_{\rm leading} = g^4 \mathcal{K}_{\rm YM}\left(N_c(\tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4} + \tilde{f}^{a_2a_3b}\tilde{f}^{ba_4a_1}) - 3B^{a_1a_2a_3a_4}\right)$  , $\qquad D_c = 8$

$\mathcal{A}_4^{(2)}\Big|_{\rm leading} = -\,g^6\,\mathcal{K}_{\rm YM}\Bigg[F^{a_1a_2a_3a_4}\Bigg(N_c^2$  $+\,48\left(\frac{1}{4}\right.$  $+\,\frac{1}{4}$  $\left.\right)\Bigg)$

$+\,48\,N_c\,G^{a_1a_2a_3a_4}\left(\frac{1}{4}\right.$  $+\,\frac{1}{4}$  $\left.\right)\Bigg]$, $\qquad D_c = 7$

$\mathcal{A}_4^{(3)}\Big|_{\rm leading} = 2\,g^8\,\mathcal{K}_{\rm YM}\,N_cF^{a_1a_2a_3a_4}\Bigg(N_c^2$  $+\,72\left(\frac{1}{6}\right.$  $+\,\frac{1}{2}$  $\left.\right)\Bigg)$, $\qquad D_c = 6$

$\mathcal{A}_4^{(4)}\Big|_{\rm leading} = -6\,g^{10}\,\mathcal{K}_{\rm YM}\,N_c^2F^{a_1a_2a_3a_4}\Bigg(N_c^2$  $+\,48\left(\frac{1}{4}\right.$  $+\,\frac{1}{2}$  $+\,\frac{1}{4}$  $\left.\right)\Bigg)$, $\qquad D_c = \frac{11}{2}$

$\mathcal{A}_4^{(5)}\Big|_{\rm leading} = \frac{144}{5}g^{12}\mathcal{K}_{\rm YM}\,N_c^3\,F^{a_1a_2a_3a_4}\Bigg(N_c^2$  $+\,48\left(\frac{1}{4}\right.$  $+\,\frac{1}{2}$  $+\,\frac{1}{4}$  $\left.\right)\Bigg)$, $\qquad D_c = \frac{24}{5}$

$A_4^{(6)}\Big|_{\rm leading} = -120g^{14}\mathcal{K}_{\rm YM}\,F^{a_1a_2a_3a_4}N_c^6\left(\frac{1}{2}\right.$  $+\,\frac{1}{4}(\ell_1+\ell_2)^2$  $-\,\frac{1}{20}$  $\left.\right)$ $\qquad D_c = 5$

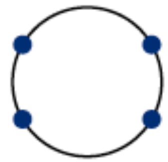
$+\,\mathcal{O}(N_c^4)\,,$

We now have a lot of theoretical "data" to guide us.

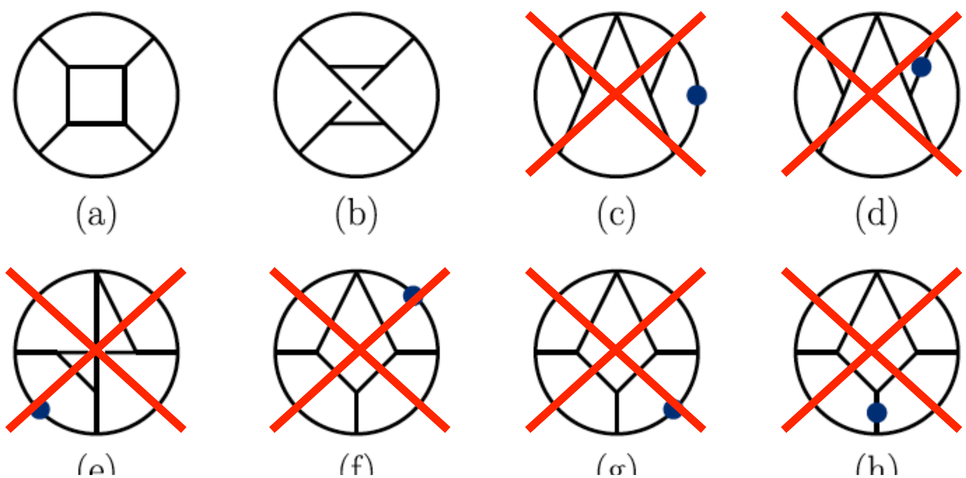
Simple consistency conditions

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

$$\mathcal{M}_4^{(1)}\Big|_{\text{leading}} = -3\,\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^4$$

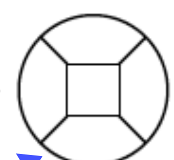


Might expect all one-loop subdiagram to have 4 propagators.



No one-loop triangle diagrams. Similar structure well known to hold for amplitudes.

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16\times 629}{25}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2 stu M_4^{\text{tree}}\left(\frac{1}{48}\right.$$



$$+\frac{1}{16}$$

$$\left.\right)$$

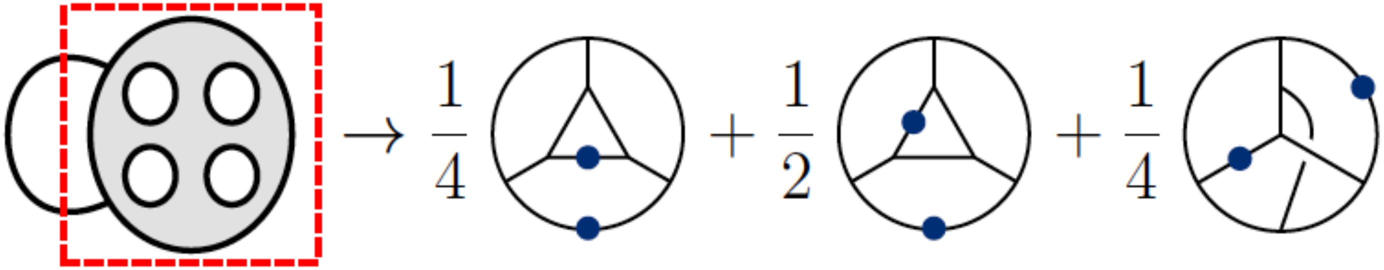
Relative coefficients symmetry factors.

symmetry factors

Simple consistency conditions

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

In fact, the different loop orders are all consistent with each other even after all the nontrvial processing!



Get correct 4-loop vacuum diagrams starting from 5-loop vacuum diagrams, even though in different dimenions!

Nontrivial processing makes it surprising that it is this simple! Should be possible to develop proof of structure.

Vacuum diagram consistency

Helps in two key ways:

1. **By demanding lower-loop consistency we should be able to figure out relative coefficients of vacuum integrals.**

2. **By limiting focus to certain integrals only need small part of 6 or 7 loop integrand. Apply unitarity compatible IBP methods.**

Gluze, Kajda, Kosower; Kosower & Larsen; Caron-Huot & Larsen; Johansson, Kosower, Larsen; Sogaard and Zhang; Schabinger; Ita; Zhang; etc.

Overwhelmingly more powerful new ways to Analyze higher loops.
$D = 4$ and 7 loops within reach!

Summary

Happy Birthday to Veneziano Amplitude!

The double copy properties first exposed in Veneziano and Virasoro amplitudes should continue to guide us for many years to come.

gravity ~ (gauge theory) x (gauge theory)

Summary

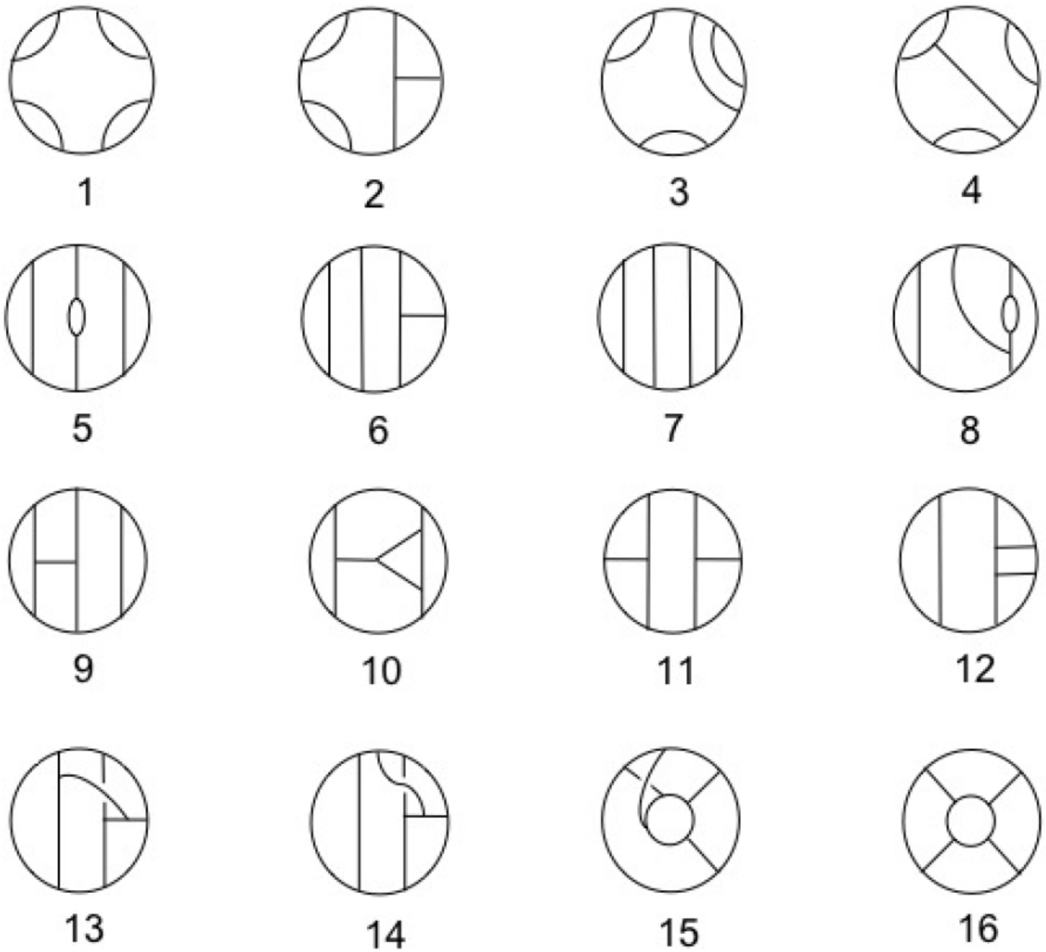
1. Double copy: origins in Veneziano vs Virasoro amplitudes.
2. KLT relations between open and closed strings.
3. Duality between color and kinematics clarifies double copy.
4. Double-copy offers remarkable insight into gravity:
   — Gravity loops from gauge theory loops.
   — Classical solutions. Gravitational radiation.
5. Generalized double copy: convert any representation of gauge-theory amplitude to gravity one.
6. 5-loop 4-point integrand of $\mathcal{N} = 8$ supergravity constructed.
7. $\mathcal{N} = 8$ sugra in $D = 24/5$ at $L = 5$ has no enhanced cancellations.
8. Simple pattern for higher loops uncovered.
9. Even $D = 4$, $L = 7$, $\mathcal{N} = 8$ now looks within reach.

Want to answer why $\mathcal{N} = 5$ sugra in $D = 4$ has enhanced cancellations but $\mathcal{N} = 8$ supergravity in 24/5 does not.

Extra Slides

First Quantized Approach

Bjornsson and Green

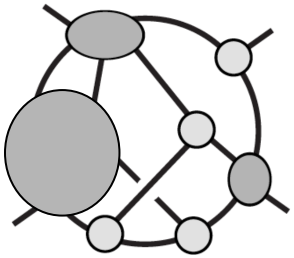


- **Contributions 15 and 16 are the crucial ones.**
- **Pure spinors have regularization issues at 5 loops and beyond**

"Since we have not evaluated the precise values of the coefficients the possibility of terms vanishing or cancellations between different contributions to the amplitude cannot be ruled out."

Bjornsson and Green

Apply KLT to Unitarity Cuts



ZB, Dixon, Dunbar, Perelstein, Rozowsky

$$\text{cut} = \sum_{\substack{\text{gravity}\\ \text{states}}} = M_1^{\text{tree}} M_2^{\text{tree}} \cdots M_m^{\text{tree}}$$

Gravity tree ampl — **KLT kernel** — **Color order YM tree**

KLT Relation:

$$M_n^{\text{tree}} = \sum_{i,j} K_{i,j} (A_n^{\text{tree}})_i (\tilde{A}_n^{\text{tree}})_j$$

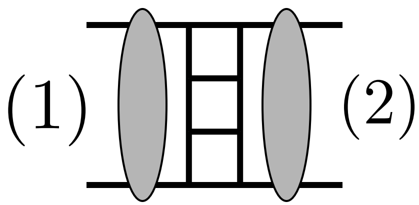
Kawai, Lewellen and Tye

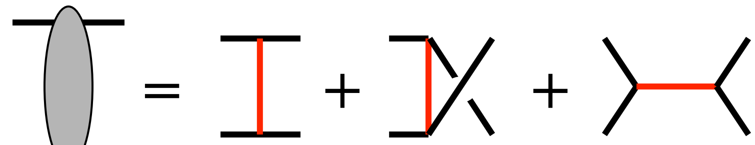
(gravity cut) = (gauge theory cut) x (gauge theory cut)

Gravity cut generated directly from known gauge theory result.

Apply KLT to cuts of known $\mathcal{N} = 4$ sYM loop amplitudes.
Fast numerically, but complicated analytic structure at high loops.

Deriving Gravity Contact Formulas





color → numerator → propagator →

$$\mathcal{C}^{4\times4}_{\rm YM} = \sum_{i_1,i_2} \frac{c_{i_1 i_2} n_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

Generalized gauge transformation

$$\delta_{i1,i2} \equiv n_{i1i2} - n^{\rm BCJ}_{i1,i2} = d^{(1)}_{i1} k^{(2)}(i_2) + d^{(2)}_{i_2} k^{(1)}(i1)$$

Generalized gauge invariance:

$$\sum_{i_1,i_2} \frac{c_{i_1 i_2}\delta_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} = 0 = \sum_{i_1,i_2} \frac{n^{\rm BCJ}_{i_1 i_2}\delta_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

BCJ discrepancy function:

$$J^{(1)}_{i_2} \equiv \sum_{i_1}^{3} n_{i_1 i_2} = d^{(1)}_{i_2} \sum_{i_1}^{3} k^{(1)}(i_1) \qquad J^{(2)}_{i_1} \equiv \sum_{i_2}^{3} n_{i_1 i_2} = d^{(2)}_{i_1} \sum_{i_2}^{3} k^{(2)}(i_2)$$

$$\mathcal{C}^{4\times4}_{\rm SG} = \sum_{i_1,i_2} \frac{n^{\rm BCJ}_{i_1 i_2} n^{\rm BCJ}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

cross term between numerators and discrepancy vanishes.

Formula for missing contact:

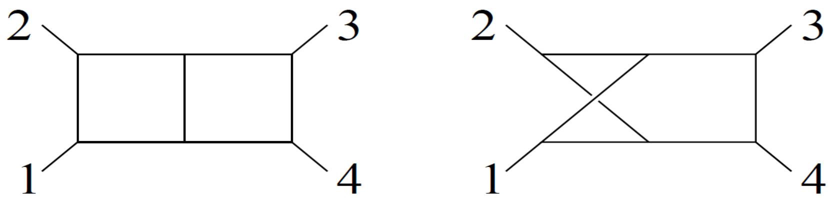
$$\mathcal{C}^{4\times4}_{\rm SG} = \sum_{i_1,i_2} \frac{n_{i_1 i_2} n_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} - \frac{2}{d^{(1)}_1 d^{(2)}_1} J^{(1)}_1 J^{(2)}_1$$

Where do enhanced cancellations come from?

ZB, Davies, Dennen, Huang; Bossard, Howe, Stelle

To analyze we need a simpler example: Half-maximal supergravity in *D* = 5 at 2 loops. No known symmetry explanation in this case.

Similar to *N* = 4, *D* = 4 sugra at 3 loops, except much simpler.



***D* = 5 half max sugra**
***N* = 4 sYM x *N* = 0 YM**

Quick summary:

— **Finiteness in *D* = 5 tied to double-copy structure.**

— **Cancellations in certain forbidden gauge-theory color structures imply hidden UV cancellations in supergravity.**

Double-copy structure implies extra cancellations!

Unfortunately, argument relies on special two-loop property: integrals of *N* = 4 sugra are identical to those of QCD.

Need a more general approach

Some Related Recent Activities

- **Examples of exact classical solutions, including black holes.**
  Monteiro, O'Connell, White; Luna,Monteiro, O'Connell, White (2015); Bahjat-Abbas, Luna, White (2017)
- **Perturbative constructions of general classical solutions, including gravitational radiation problems (LIGO)**
  Goldberger, Ridgway (2016); Luna, Monterio, Nicholson, O'Connell, Ochirov, Westerberg, White (2016)
- **Loop level KLT and BCJ: using CHY, ambitwistor string, Q-cuts**
  Song He, Oliver Schlotterer (2016), Tourkine, Vanhove (2016,2017); Hohenegger, S. Stieberger (2017); Y. Geyer, L. Mason, R. Monteiro, P. Tourkine (2016) K. A. Roehrig, D. Skinner (2017)
- **Analytic properties of gravity integrands.** Herrmann and Trnka (2016)
- **Simplified gravity Lagrangian.**
  Cheung and Remmen (2016,2017)
- **Double copy as consequence of gauge invariance.**
  Chiodaroli; Boels, Medina (2016), Arkani-Hamed, Rodina, Trnka (2016), Feng et al (2016)
- **Applications in string theory.** Steiberger; Vahhove
  Carrasco, Mafra, Schlotterer, (2016); Mafra and Schlotterer (2015, 2016)