

Superstring Amplitudes and Effective Interactions

Eric D'Hoker

Mani L. Bhaumik Institute for Theoretical Physics
University of California, Los Angeles

50 Years of the Veneziano Model:
from dual models to strings, M-theory and beyond
Galileo Galilei Institute, Arcetri, Florence



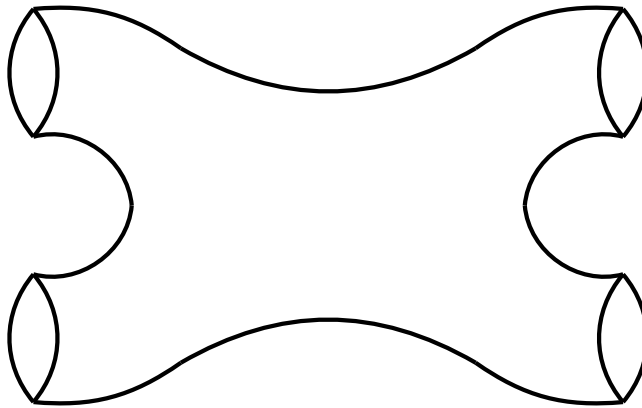
Joining and splitting

Veneziano model 1968

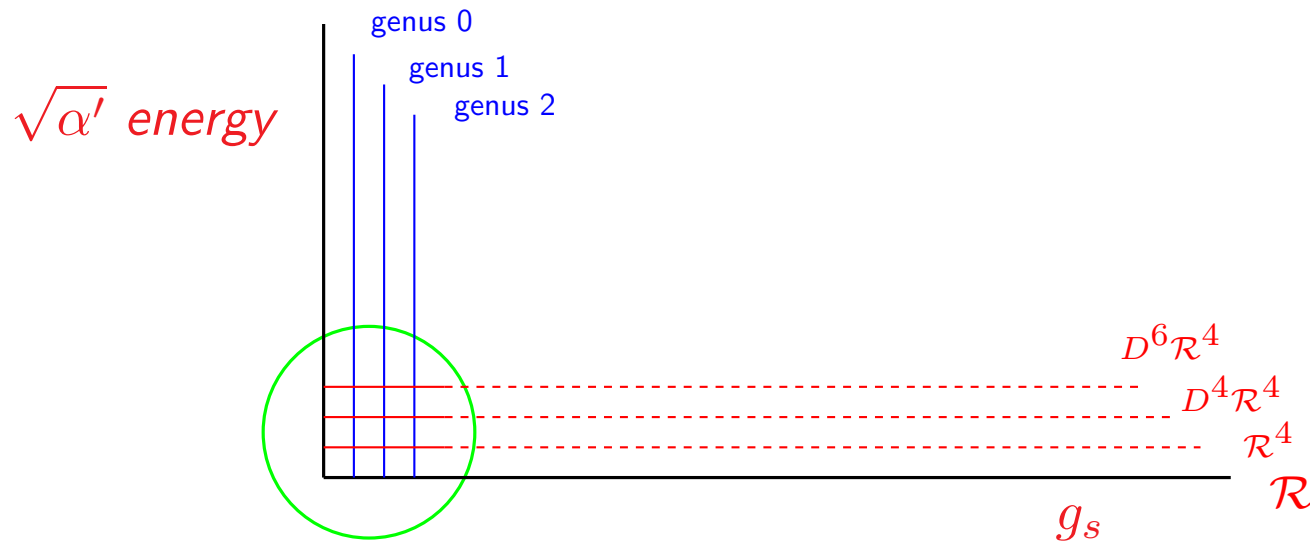


strings

(Nambu, Nielsen, Susskind 1969)



Superstring Perturbation Theory and Supergravity

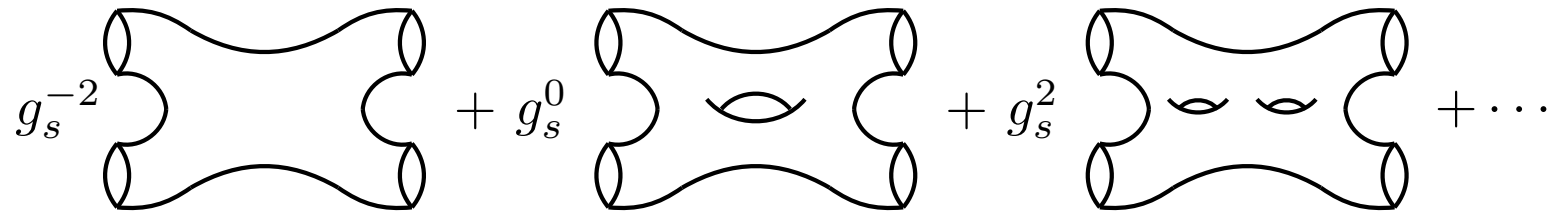


- **Superstring perturbation theory in powers of the string coupling g_s**
 - holds for weak coupling g_s
 - and for all energies
- **Classical supergravity “ \mathcal{R} ”**
 - leading low energy expansion of string theory
 - holds for all couplings g_s
- **String induced effective interactions $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$**
 - Evaluated in perturbation theory for $g_s \ll 1$

Closed Superstring Perturbation Theory

- Theory of fluctuating random surfaces**

– governed by topological expansion in the genus h weighed by g_s^{2h-2}



- Closed superstring tree-level four-graviton amplitude**

– generalization of the Veneziano amplitude ($s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)^2$)

$$\frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \times \mathcal{R}^4$$

- At low energy massive strings produce local effective interactions**

– string-induced corrections to supergravity; eg. in Type II

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2 + t^2 + u^2)^2 + \dots$$

massless

\mathcal{R}^4

$D^4\mathcal{R}^4$

$D^6\mathcal{R}^4$

$D^8\mathcal{R}^4$

Approaches to Superstring Perturbation Theory

- **Goal is to obtain superstring amplitudes at higher genus**
 - Ramond-Neveu-Schwarz formulation of fermionic strings;
 - Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
 - Green-Schwarz space-time supersymmetric formulation;
 - Mandelstam light-cone formulation;
 - String field theory;
 - Topological string theory;
 - Berkovits pure spinor formulation.
- **Different perturbative superstring theories** (in 10 dimensions)
 - Type I open & closed, orientable & non-orientable, D-branes
 - Type IIA,B closed orientable, D-branes
 - Heterotic closed orientable $E_8 \times E_8, Spin(32/\mathbb{Z}_2)$

Outline of this talk

RNS formulation, closed orientable superstrings, critical dimension 10

- Formulation of RNS superstrings in terms of super Riemann surfaces
(Witten 2012-2013)
- Explicit calculations of genus-two amplitudes
(ED & Phong 2001-2007)
- Perturbative supersymmetry breaking and vacuum energy calculations
(Atick, Dixon, Sen; Dine, Seiberg, Witten 1987) . . . (Witten; ED, Phong 2013)
- Perturbative evaluation of effective interactions in Type IIB and S-duality
(Green, Gutperle, Vanhove 1997-2000) . . .
(ED, Green, Pioline, R. Russo, Vanhove 2013-present)

Type II worldsheet fields in the RNS formulation

- $M = \mathbb{R}^{10}$ flat Minkowski space-time with Lorentz group $SO(1, 9)$
 - x^μ scalars on worldsheet Σ , map Σ into M
 - ψ^μ spinors on Σ but Lorentz vector under $SO(1, 9)$
 - ★ Worldsheet supersymmetry $\implies \Sigma$ is a super Riemann surface
 - ★ Two sectors : NS bosons $SO(1, 9)$ -tensors, R fermions $SO(1, 9)$ -spinors
- With Minkowski signature Σ
 - ψ^μ and $\tilde{\psi}^\mu$ are *independent* Majorana-Weyl spinors of opposite chirality
 - Locally, Dirac eq solved by $\psi^\mu(\tau - \sigma)$ and $\tilde{\psi}^\mu(\tau + \sigma)$
- With Euclidean signature Σ
 - ψ^μ and $\tilde{\psi}^\mu$ must be *independent* complex Weyl spinors
 - Locally, Dirac eq solved by $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\tilde{z})$
 - Globally, on a compact Riemann surface of genus h ,
 - ★ All ψ^μ are sections of a the same spin bundle S (and $\tilde{\psi}^\mu$ of \tilde{S})
 - ★ 2^{2h} distinct spin structures for S (and 2^{2h} independently for \tilde{S})
- GSO projection requires independent summation over spin structures

Super Riemann surfaces and their moduli

- **Ordinary Riemann surface** (locally \mathbb{C} with coordinate z)
 - complex manifold: holomorphic transition functions $z \rightarrow z'(z)$;
 - complex structure = conformal structure J
 - Moduli space $\mathcal{M}_h = \{J\}/\text{Diff}(\Sigma)$ of genus h compact Riemann surfaces
- **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
 - holó transition functions $z|\theta \rightarrow z'(z, \theta)|\theta'(z, \theta)$ generate $\mathcal{N} = 2$ super conformal
- **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
 - holó transition functions $z|\theta \rightarrow z'|\theta'$ rescale $D_\theta = \partial_\theta + \theta\partial_z$
 - Transition functions define $\mathcal{N} = 1$ superconformal structure \mathcal{J}
 - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$
- **Moduli space of compact super Riemann surfaces:** $\mathfrak{M}_h = \{\mathcal{J}\}/\text{Diff}(\Sigma)$
 = equivalence classes of superconformal structures \mathcal{J}

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h = 0 \\ 1|0 \text{ or } 1|1 & h = 1 \text{ even or odd spin structure} \\ 3h - 3|2h - 2 & h \geq 2 \end{cases}$$

- odd modulus at $h = 1$ odd spin structure is a book keeping device;
- odd moduli really first appear at genus 2, as curved super spaces.

Superstring worldsheets and moduli spaces

• Heterotic

- Left : RS Σ_L , moduli space \mathcal{M}_L coord resp. \tilde{z} and \tilde{m}^i
- Right : SRS Σ_R , moduli space \mathfrak{M}_R coord resp. (z, θ) and (m^i, ζ^α)
- Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|1$
subject to $\Sigma_{\text{red}} = \text{diag}(\Sigma_{L\text{red}} \times \Sigma_{R\text{red}}) : \tilde{z}^* = z + \text{nilpotent}$
- Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h - 3|2h - 2$ for $h \geq 2$
subject to $\Gamma_{\text{red}} = \text{diag}(\mathcal{M}_{L\text{red}} \times \mathfrak{M}_{R\text{red}}) : (\tilde{m}^i)^* = m^i + \text{nilpotent}$
(reduced space obtained by setting all nilpotent variables to zero)

• Type II

- Left : SRS Σ_L , moduli space \mathfrak{M}_L coord resp. $(\tilde{z}, \tilde{\theta})$ and $(\tilde{m}^i, \tilde{\zeta}^\alpha)$
- Right : SRS Σ_R , moduli space \mathfrak{M}_R coord resp. (z, θ) and (m^i, ζ^α)
- Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|2$
- Moduli space is cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h - 3|4h - 4$ for $h \geq 2$
subject to $\tilde{z}^* = z + \text{nilpotent}$ and $(\tilde{m}^i)^* = m^i + \text{nilpotent}$

• Super-Stokes theorem ensures independence of the choice of cycles

- in amplitudes with BRST invariant vertex operators
- consistent definition of superstring amplitudes to all genera (Witten 2012)

Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
 - Σ_L has superconformal structure $\tilde{\mathcal{J}}$ with local coordinates $\tilde{z}|\tilde{\theta}$
 - Σ_R has superconformal structure \mathcal{J} with local coordinates $z|\theta$
- **Superconformal invariant matter action**
 - worldsheet matter field

$$X^\mu(\tilde{z}, z|\tilde{\theta}, \theta) = x^\mu(\tilde{z}, z) + \theta\psi^\mu(\tilde{z}, z) + \tilde{\theta}\tilde{\psi}^\mu(\tilde{z}, z) + \tilde{\theta}\theta F^\mu(\tilde{z}, z)$$

- Worldsheet action in local coordinates ($D_\theta = \partial_\theta + \theta\partial_z$)

$$I_m[X^\mu, \tilde{\mathcal{J}}, \mathcal{J}] = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta] \tilde{D}_{\tilde{\theta}} X^\mu D_\theta X_\mu$$

- Superconformal algebra on fields generated by

$$\begin{aligned} \mathcal{S}_{z\theta} &= S_{z\theta} + \theta T_{zz} & S_{z\theta} &= \frac{1}{2}\psi^\mu \partial_z x_\mu & T_{zz} &= -\frac{1}{2}\partial_z x^\mu \partial_z x_\mu + \frac{1}{2}\psi^\mu \partial_z \psi_\mu \\ \tilde{\mathcal{S}}_{\tilde{z}\tilde{\theta}} &= \tilde{S}_{\tilde{z}\tilde{\theta}} + \tilde{\theta}\tilde{T}_{\tilde{z}\tilde{z}} & \tilde{S}_{\tilde{z}\tilde{\theta}} &= \frac{1}{2}\tilde{\psi}^\mu \partial_{\tilde{z}} x_\mu & \tilde{T}_{\tilde{z}\tilde{z}} &= -\frac{1}{2}\partial_{\tilde{z}} x^\mu \partial_{\tilde{z}} x_\mu + \frac{1}{2}\tilde{\psi}^\mu \partial_{\tilde{z}} \tilde{\psi}_\mu \end{aligned}$$

Deformations of superconformal structures

- Under deformation of $\tilde{\mathcal{J}}$ for Σ_L and \mathcal{J} for Σ_R

$$\delta I = \int_{\Sigma} [d\tilde{z}dz | d\tilde{\theta}d\theta] \left(H_{\tilde{\theta}^z} S_{z\theta} + \tilde{H}_{\theta^{\tilde{z}}} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

- in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\text{red}}} d\tilde{z}dz \left(\mu_{\tilde{z}^z} T_{zz} + \chi_{\tilde{z}^{\theta}} S_{z\theta} + \tilde{\mu}_z{}^{\tilde{z}} T_{\tilde{z}\tilde{z}} + \tilde{\chi}_z{}^{\tilde{\theta}} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

- recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}^z} = \tilde{\theta}(\mu_{\tilde{z}^z} + \theta\chi_{\tilde{z}^{\theta}}) \quad \tilde{H}_{\theta^{\tilde{z}}} = \theta(\tilde{\mu}_z{}^{\tilde{z}} + \tilde{\theta}\tilde{\chi}_z{}^{\tilde{\theta}})$$

- Finite deformations of the metric with $\tilde{\mu} = \bar{\mu}$ and $\tilde{\chi} = \bar{\chi}$
integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action
(Brink, Di Vecchia, Howe; Deser, Zumino 1976)

- Type II superstring perturbation theory requires $\tilde{\mu} \neq \bar{\mu}$ and $\tilde{\chi} \neq \bar{\chi}$

Type II string amplitude

- Parametrize deformations $\tilde{H}_\theta^{\tilde{z}}, H_{\tilde{\theta}}^z$ by slice $\{\tilde{\mathcal{J}}(\tilde{\mathfrak{m}}), \mathcal{J}(\mathfrak{m})\}$ in $\mathfrak{M}_L \times \mathfrak{M}_R$

$$H_{\tilde{\theta}}^z = \tilde{D}_{\tilde{\theta}} V^z + H_A dm^A \quad H_A = \partial \mathcal{J}_{\tilde{\theta}}^z / \partial m^A \quad m^A = (m^i, \zeta^\alpha)$$

$$\tilde{H}_\theta^{\tilde{z}} = D_\theta \tilde{V}^{\tilde{z}} + \tilde{H}_{\tilde{A}} d\tilde{m}^{\tilde{A}} \quad \tilde{H}_{\tilde{A}} = \partial \mathcal{J}_\theta^{\tilde{z}} / \partial \tilde{m}^{\tilde{A}} \quad \tilde{m}^{\tilde{A}} = (\tilde{m}^i, \tilde{\zeta}^\alpha)$$

$$\text{ghost fields} \quad V^z \rightarrow C^z = c^z + \theta \gamma^\theta \quad H_{\tilde{\theta}}^z \rightarrow B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$$

$$V^{\tilde{z}} \rightarrow \tilde{C}^{\tilde{z}} = \tilde{c}^{\tilde{z}} + \tilde{\theta} \tilde{\gamma}^{\tilde{\theta}} \quad H_\theta^{\tilde{z}} \rightarrow \tilde{B}_{\tilde{z}\tilde{\theta}} = \tilde{\beta}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{b}_{\tilde{z}\tilde{z}}$$

- Super conformal invariant ghost action

$$I_{\text{gh}} = \int_{\Sigma} [d\tilde{z}dz | d\tilde{\theta}d\theta] \left(B_{z\theta} \tilde{D}_{\tilde{\theta}} C^z + \tilde{B}_{\tilde{z}\tilde{\theta}} D_\theta \tilde{C}^{\tilde{z}} + B_{z\theta} H_A dm^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_{\tilde{A}} d\tilde{m}^{\tilde{A}} \right)$$

- The integrand for the full amplitude is given by

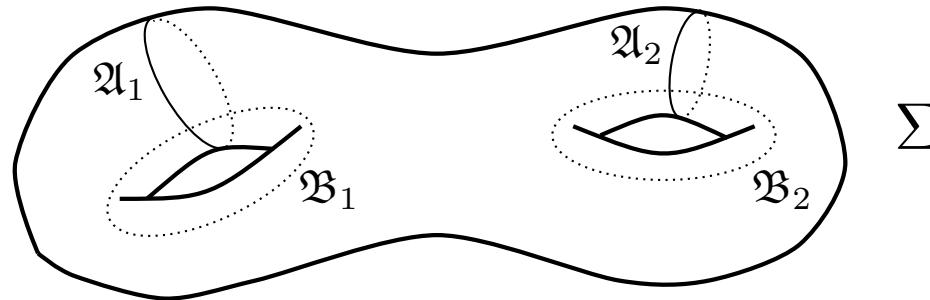
$$\int D(XB\tilde{B}C\tilde{C}) \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A}, A} [d\tilde{m}^{\tilde{A}} dm^A] \delta(\langle \tilde{B}, \tilde{H}_{\tilde{A}} \rangle) \delta(\langle B, H_A \rangle) e^{-I_m - I_{\text{gh}}}$$

- $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.
- Picture Changing Operator formalism (Friedan, Martinec, Shenker 1986)
 - ★ may be obtained as singular limit for χ supported at points
 - ★ globally regular reformulation via “vertical integration” (Sen, Witten 2016)

Loop momenta

- Fix a canonical homology basis of cycles $\mathcal{A}_I, \mathcal{B}_I$ of $H_1(\Sigma, \mathbb{Z})$ $I = 1, \dots, h$
 – with canonical intersection pairing

$$\#(\mathcal{A}_I, \mathcal{A}_J) = \#(\mathcal{B}_I, \mathcal{B}_J) = 0 \text{ and } \#(\mathcal{A}_I, \mathcal{B}_J) = \delta_{IJ}$$



- h independent loop momenta p_I^μ defined to flow across \mathcal{A}_I cycles

$$p_I^\mu = \oint_{\mathcal{A}_I} dz \partial_z x^\mu$$

- At fixed loop momenta, the amplitudes chirally split

Chiral amplitudes

- **Chiral Amplitudes** (ED, Phong 1988)

- For massless NS bosons with factorized polarization tensor $\varepsilon_i^{\mu\tilde{\mu}} = \varepsilon_i^\mu \tilde{\varepsilon}_i^{\tilde{\mu}}$
- Each chiral amplitude at fixed loop momenta is given by

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N e^{p_I^\mu \oint_{\mathfrak{B}_I} dz \partial_z x^\mu} e^{\int_\Sigma H_{\tilde{\theta}}^z S_{z\theta}} \prod_A \delta(\langle B, H_A \rangle) dm^A \right\rangle$$

- Correlation function $\langle \cdots \rangle$ computed with Szegő kernel S_δ and prime for E

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\eta^{\mu\nu} S_\delta(z, w) \quad \langle x_+^\mu(z) x_+^\nu(w) \rangle = -\eta^{\mu\nu} \ln E(z, w)$$

- **Full Superstring Amplitudes**

- obtained by pairing left and right and integrating on a cycle $\Gamma \in \mathfrak{M}_L \times \mathfrak{M}_R$

$$\mathcal{A}^{(h)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_\Gamma \int_{\mathbb{R}^{10}} dp_I^\mu \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde{\varepsilon}_i, k_i, p_I^\mu) \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I^\mu)$$

- cfr “double copy construction” in supergravity calculations (cfr Zvi Bern’s talk)

Parametrization of super moduli

- **Superconformal structure** $\mathcal{J} \in \mathfrak{M}_h$ specified by transition functions
 - Concrete calculations use parametrization by gravitino field $\chi_{\tilde{z}}^\theta$
- **Local parametrization of moduli** (in conformal-invariant theory)
 - Conformal structure J with metric $g = |dz|^2$ in local coordinates (z, \tilde{z})
 - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
 - realized in CFT by inserting $\int_{\Sigma} d\tilde{z} dz \mu_{\tilde{z}}^z T_{zz}$ to all orders in μ
- **Local parametrization of supermoduli** (in superconformal-invariant theory)
 - Start with Σ_{red} with complex structure given by $J \in \mathfrak{M}_{\text{red}}$
 - Deform super conformal structure by inserting T and S

$$\int_{\Sigma_{\text{red}}} d\tilde{z} dz \left(\mu_{\tilde{z}}^z T_{zz} + \chi_{\tilde{z}}^\theta S_{z\theta} \right)$$

- χ and μ parametrized by local odd coordinates on \mathfrak{M}_h
- For $h = 2$, even spin structures, holó projection $\mathfrak{M}_2 \rightarrow \mathcal{M}_2$ exists
 - via the super period matrix (ED, Phong 2001)
- For $h \geq 5$ no holó projection $\mathfrak{M}_h \rightarrow \mathcal{M}_h$ exists (Donagi, Witten 2013)

The super period matrix (even spin structures)

- Start from conformal structure J for Σ_{red} with holó 1-forms ω_I

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \quad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \quad I, J = 1, 2$$

- Deform to superconformal structure \mathcal{J} on Σ with superholó forms $\hat{\omega}_I$

$$\oint_{\mathfrak{A}_I} \hat{\omega}_J = \delta_{IJ} \quad \oint_{\mathfrak{B}_I} \hat{\omega}_J = \hat{\Omega}_{IJ} \quad I, J = 1, 2$$

- Explicit formula for the super period matrix $\hat{\Omega}$ for even spin structure δ

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma_{\text{red}}^2} \omega_I(z) \chi(z) S_\delta(z, w | \Omega) \chi(w) \omega_J(w) + \int_{\Sigma_{\text{red}}} \mu \omega_I \omega_J$$

- $\hat{\Omega}_{IJ}$ is locally supersymmetric; $\hat{\Omega}_{IJ} = \hat{\Omega}_{JI}$; and $\text{Im } \hat{\Omega} > 0$
- Every $\hat{\Omega}$ corresponds to an ordinary Riemann surface
- Szegő kernel $S_\delta(z, w | \Omega)$ is non-singular in the interior of \mathcal{M}_2

\Rightarrow Projection using $\hat{\Omega}$ is holomorphic and natural for genus 2

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on \mathfrak{M}_2**

- Natural parametrization of \mathfrak{M}_2 by $(\hat{\Omega}_{IJ}, \zeta^\alpha)$ (even spin structure δ)
- involves measure $d\kappa[\delta](\hat{\Omega}, \zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i, k_i, p_I | \hat{\Omega}, \zeta)$

- **Projection to chiral amplitudes on \mathcal{M}_2**

- by integrating over ζ and summing over δ at fixed $\hat{\Omega}$

$$\mathcal{R}(\varepsilon_i, k_i, p_I | \hat{\Omega}) = \sum_{\delta} \int_{\zeta} d\kappa[\delta](\hat{\Omega}, \zeta) \mathcal{C}[\delta](\varepsilon_i, k_i, p_I | \hat{\Omega}, \zeta)$$

$$\mathcal{L}(\tilde{\varepsilon}_i, k_i, p_I | \hat{\Omega}) = \sum_{\tilde{\delta}} \int_{\tilde{\zeta}} d\kappa[\tilde{\delta}](\hat{\Omega}, \tilde{\zeta}) \mathcal{C}[\tilde{\delta}](\tilde{\varepsilon}_i, k_i, p_I | \hat{\Omega}, \tilde{\zeta})$$

- for heterotic, \mathcal{L} is chiral half of bosonic string, has no integral in $\tilde{\zeta}$
- phase factors determined by $Sp(4, \mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and $\hat{\Omega}$**

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_{\mathcal{M}_2} d\hat{\Omega} \int dp_I^\mu \mathcal{R}(\varepsilon_i, k_i, p_I | \hat{\Omega}) \overline{\mathcal{L}(\tilde{\varepsilon}_i, k_i, p_I | \hat{\Omega})}$$

- Integral over p_I is Gaussian and can be carried out explicitly.

UV-finiteness and one-loop amplitudes

- Thanks to modular invariance, all string amplitudes are UV-finite
 - shown for the closed bosonic string (Shapiro 1972)
 - holds for all modular invariant superstrings to all loops (i.e. all genera)
 - all chiral amplitudes have a universal loop momentum factor

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu} \times \dots$$

- Modular invariances guarantees $\text{Im}(\Omega)$ bounded from below
 - ⇒ Gaussian suppression at large loop momenta & UV finiteness
- One-loop Type II four-graviton amplitude (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

- Partial amplitude \mathcal{B} is a modular function in $\tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2, \mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^4 \frac{d^2z_i}{\text{Im}\tau} \exp\left(\sum_{i<j} s_{ij} G(z_i - z_j|\tau)\right)$$

- $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ .
- Analogous formulas for Heterotic string (Gross, Harvey, Martinec, Rohm 1985)

Genus two Riemann surfaces and moduli

- Siegel Upper half space \mathcal{H}_h

$$\mathcal{H}_h = \left\{ \Omega = \{\Omega_{IJ}\}_{I,J=1,\dots,h} \text{ with } \Omega^t = \Omega \text{ and } Y = \text{Im}(\Omega) > 0 \right\}$$

- $Sp(2h, \mathbb{R})$ acts by $\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

- \mathcal{H}_h has $Sp(2h, \mathbb{R})$ -invariant metric ds_h^2 and volume form $d\mu_h$

$$ds_h^2 = \sum_{I,J,K,L} Y_{IJ}^{-1} d\bar{\Omega}_{JK} Y_{KL}^{-1} d\Omega_{LI}$$

- Genus-two compact Riemann surfaces Σ

- Canonical homology basis $\mathfrak{A}_I, \mathfrak{B}_I$ with ω_I dual holomorphic (1,0) forms,

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \quad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

- Riemann relations imply $\Omega \in \mathcal{H}_2$; modular group is $Sp(4, \mathbb{Z})$

- Genus-two moduli space $\mathcal{M}_2 = \mathcal{H}_2 / Sp(4, \mathbb{Z})$ (upon removing diagonal Ω)

Genus-two Type II superstring amplitudes

- **Type II four-graviton amplitude** (ED, Phong 2001 – 2005)
 - receives contributions only from even spin structures, as for genus-one
 - use projection by the super period matrix to integral over \mathcal{M}_2

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde{\varepsilon}_I, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

- kinematic factor $\mathcal{R}^4 = \mathcal{K}(\varepsilon_i, k_i) \tilde{\mathcal{K}}(\tilde{\varepsilon}_i, k_i)$ is the same as tree-level and one-loop

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \text{Im } \Omega)^2} \exp \left(\sum_{i < j} s_{ij} G(z_i, z_j|\Omega) \right)$$

- $G(z_i, z_j)$ is the genus-two scalar Green function;
 - $3\mathcal{Y} = (t - u)\Delta(1, 2)\Delta(3, 4) + (s - t)\Delta(1, 3)\Delta(4, 2) + (u - s)\Delta(1, 4)\Delta(2, 3)$
 - $\Delta(i, j) = \omega_1(z_i) \wedge \omega_2(z_j) - \omega_2(z_i) \wedge \omega_1(z_j)$ is a biholomorphic form
 - reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)
- **Heterotic four NS boson amplitude** (ED, Phong 2005)
 - obtained by replacing $\tilde{\mathcal{K}}(\tilde{\varepsilon}_i, k_i) \tilde{\mathcal{Y}} \rightarrow \mathcal{W}(\tilde{\varepsilon}_i, k_i)$
 - \mathcal{W} correlator depends on the NS boson state (graviton versus gauge).

Singularities in the projection $\overline{\mathfrak{M}}_2 \rightarrow \overline{\mathcal{M}}_2$

- Projection $\mathfrak{M}_2 \rightarrow \mathcal{M}_2$ is holó, but integration extends to boundary
 - are there singularities in the projection $\overline{\mathfrak{M}}_2 \rightarrow \overline{\mathcal{M}}_2$?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \quad \begin{array}{ll} u \rightarrow 0 & \text{separating node} \\ \sigma \rightarrow i\infty & \text{non-separating node} \end{array}$$

- Key ingredient in $\hat{\Omega}$ is the Szegő kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z - w|\Omega)}{\vartheta[\delta](0|\Omega) E(z, w)}$$

- As $u \rightarrow 0$ we have $\vartheta[\delta](0|\Omega) \rightarrow \vartheta[\delta_1](0|\tau) \vartheta[\delta_2](0|\tau)$
- Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**

- singularity killed by ψ -zero modes in \mathbb{R}^{10} (space-time susy)
 - contribution when susy is broken by radiative corrections (Witten 2013)
 - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
 - ★ is zero for $E_8 \times E_8 \rightarrow E_6 \times E_8$ with unbroken susy
 - ★ non-zero for $\text{Spin}(32)/\mathbb{Z}_2 \rightarrow SO(26) \times U(1)$ with broken susy
- (Atick, Sen 1988; . . . ; ED, Phong 2013; Berkovits, Witten 2014)

Singularities in the projection $\mathfrak{M}_3 \rightarrow \mathcal{M}_3$

- **Some basic structure theorems**

- A hyper-elliptic surface is a branched double cover of the sphere S^2 ;
- All genus 1 and all genus 2 surfaces are hyper-elliptic;
- Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of \mathcal{M}_3
(referred to as the hyper-elliptic divisor)

- **The genus-three period matrix (for even spin structure)**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \iint \omega_I(z) \chi(z) S_\delta(z, w | \Omega) \chi(w) \omega_J(w) + \mathcal{O}(\chi^4)$$

- For Ω on the hyper-elliptic divisor of \mathfrak{M}_3
there always exists an even spin structure δ such that $\vartheta[\delta](0|\Omega) = 0$
- the presence of the extra Dirac zero modes kills effects of this singularity
- A different even δ does produce a *subtle singularity* in $\hat{\Omega}$ (Witten 2015)
- Rules out earlier beautiful proposals for the genus 3 superstring measure
(Cacciatori, Dalla Piazza, van Geemen 2008)

Effective interactions from Type IIB superstrings

- $SL(2, \mathbb{Z})$ -duality symmetry of Type IIB superstrings

- requires effective interactions to be $SL(2, \mathbb{Z})$ -invariant;
- Einstein frame \mathcal{R}^4 invariant; axion-dilaton $\rho = \chi + ie^{-\Phi}$ by Möbius;
- effective interactions via four-graviton amplitude in Type IIB,

$$\mathcal{E}_0(\rho)\mathcal{R}^4 + \mathcal{E}_4(\rho)D^4\mathcal{R}^4 + \mathcal{E}_6(\rho)D^6\mathcal{R}^4 + \mathcal{E}_8(\rho)D^8\mathcal{R}^4 + \dots$$

- $\mathcal{E}_p(\rho)$ is a real-valued $SL(2, \mathbb{Z})$ -invariant of ρ , or *modular function*

- Non-holomorphic Eisenstein series

- The functions $E_s(\rho)$ are $SL(2, \mathbb{Z})$ -invariant ($\rho = \rho_1 + i\rho_2, \rho_1, \rho_2 \in \mathbb{R}$)

$$\Delta_\rho E_s(\rho) = s(1-s)E_s \quad \Delta = 4\rho_2^2 \partial_\rho \partial_{\bar{\rho}}$$

- For $\text{Re}(s) > 1$ they are given by a Kronecker-Eisenstein sum

$$E_s(\rho) = \sum_{m,n \in \mathbb{Z}}' \frac{\rho_2^s}{\pi^s |m + \rho n|^{2s}}$$

- with asymptotic expansion for $\rho_2 \rightarrow \infty =$ weak string coupling

$$E_s(\rho) = 2\zeta(2s) \frac{\rho_2^s}{\pi^s} + \frac{2\Gamma(s - \frac{1}{2})\zeta(2s - 1)}{\Gamma(s)\pi^{s-\frac{1}{2}}\rho_2^{s-1}} + \mathcal{O}(e^{-2\pi\rho_2})$$

Effective interactions (cont'd)

- **Space-time supersymmetry and S-duality**

- D-instantons and space-time susy (Green, Gutperle, Vanhove; Green, Sethi 1997)
- M-theory perturbation theory on torus (Green, Kwon, Vanhove 1999; GV 2005)

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2 + t^2 + u^2)^2 + \dots$$

$$\mathcal{R}^4$$

$$D^4\mathcal{R}^4$$

$$D^6\mathcal{R}^4$$

$$D^8\mathcal{R}^4$$

$$\mathcal{E}_0 \approx E_{\frac{3}{2}}$$

$$\mathcal{E}_4 \approx E_{\frac{5}{2}}$$

$$(\Delta - 12)\mathcal{E}_6 \approx E_{\frac{3}{2}}^2$$

$$??$$

$$\frac{1}{2}\text{BPS}$$

$$\frac{1}{4}\text{BPS}$$

$$\frac{1}{8}\text{BPS}$$

unprotected

- \mathcal{E}_0 only tree-level $\approx e^{-2\Phi}$ and one-loop $\approx e^{0\Phi}$ (Green, J. Russo, Vanhove 2008)
- \mathcal{E}_4 only tree-level and two-loop $\approx e^{2\Phi}$ (ED, Gutperle, Phong 2005)
- \mathcal{E}_6 only tree-level and two-loops (ED, Green, Pioline, R. Russo 2014)
and three-loops $\approx e^{4\Phi}$ (Gomez, Mafra 2015)

- **Non-renormalization theorems:** no perturbative corrections

- for \mathcal{E}_0 for $h \geq 2$, for \mathcal{E}_4 for $h \geq 3$, for \mathcal{E}_6 for $h \geq 4$

Low energy expansion of the genus-one string integrand

- Type II genus-one four-graviton amplitude is an integral over \mathcal{M}_1 of

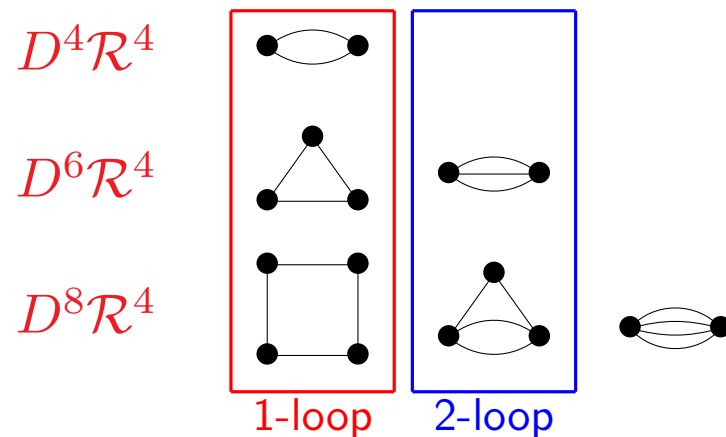
$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^4 \int_{\Sigma} \frac{d^2 z_i}{\tau_2} \exp \left\{ \sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right\}$$

- The scalar Green function G is given by a Fourier sum

$$G(\alpha + \beta\tau|\tau) = \sum'_{m,n \in \mathbb{Z}} \frac{\tau_2}{\pi} \frac{e^{2\pi i(m\beta - n\alpha)}}{|m + \tau n|^2}$$

- For fixed τ the expansion of $\mathcal{B}^{(1)}$ in powers of s_{ij} converges for $|s_{ij}| < 1$

- Graphical expansion of $\mathcal{B}^{(1)}(s_{ij}|\tau) \implies$ Modular Graph Functions of τ




Genus-one modular graph functions

- **One-loop graphs with w Green functions**
 - evaluate to non-holomorphic Eisenstein series $E_w(\tau)$
- **Two-loop graphs evaluate to modular graph functions**
 - given by multiple Kronecker-Eisenstein sums over $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$

$$C_{w_1, w_2, w_3}(\tau) = \sum_{p_1, p_2, p_3 \in \Lambda'_\tau} \delta(p_1 + p_2 + p_3) \prod_{r=1}^3 \left(\frac{\tau_2}{\pi |p_r|^2} \right)^{w_r}$$

- contribute to $D^{2w}\mathcal{R}^4$ with the *weight* given by $w = w_1 + w_2 + w_3$
- satisfy (inhomogeneous) Laplace-eigenvalue equations

$$w = 3 \quad C_{1,1,1} = \text{---} \text{---} \text{---} \quad (\Delta - 0)C_{1,1,1} = 6E_3$$


$$w = 4 \quad C_{2,1,1} = \text{---} \text{---} \text{---} \quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$$


$$w = 5 \quad C_{3,1,1} = \text{---} \text{---} \text{---} \quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$$


Low energy expansion of the genus-two string integrand

- Recall genus-two four-graviton amplitude is an integral over \mathcal{M}_2 of

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \operatorname{Im} \Omega)^2} \exp \sum_{i < j} s_{ij} G(z_i, z_j)$$

– $\mathcal{Y} = (s - t)\Delta(z_1, z_3)\Delta(z_4, z_2) + 2$ permutations.

- Genus-two contributions to local effective interactions

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \underbrace{0}_{\mathcal{R}^4} + \underbrace{32(s^2 + t^2 + u^2)}_{D^4\mathcal{R}^4} + \underbrace{192 stu \varphi(\Omega)}_{D^6\mathcal{R}^4} + \underbrace{(s^2 + t^2 + u^2)^2 \mathcal{B}_8^{(2)}(\Omega)}_{D^8\mathcal{R}^4} + \dots$$

- $\varphi(\Omega)$ = Kawazumi-Zhang invariant (Kawazumi; Zhang 2008)
 - ★ satisfies differential equation on \mathcal{M}_2 (ED, Green, Pioline, R. Russo 2014)

$$(\Delta - 5)\varphi = -2\pi\delta_{SN}$$

- ★ Δ is the Laplace-Beltrami operator on \mathcal{M}_2 with Siegel metric;
- ★ δ_{SN} has support on separating node (into two genus-one surfaces)
- $\mathcal{B}_8^{(2)}(\Omega)$ satisfies remarkably systematic asymptotics (ED, Green, Pioline 2017)
 - ★ system of differential equations on \mathcal{M}_2 (ED, Green, Pioline, in progress)

Outlook

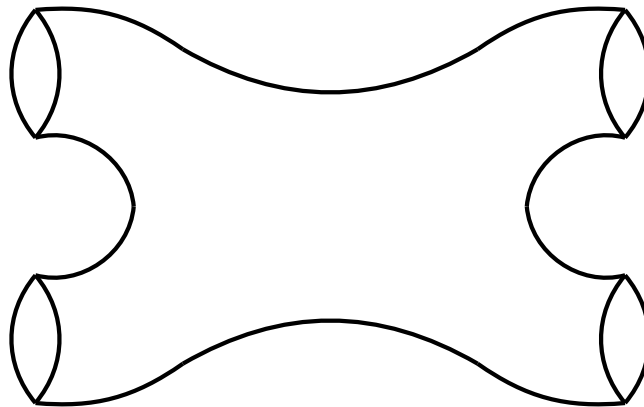
- **Some additional developments**

- Clarification of super Riemann surfaces with R-punctures (Witten 2012)
- There exists a super-period matrix for R-punctures (Witten; ED, Phong 2015)
- New relations between open and closed string amplitudes (Schlotterer's talk)

- **Some outstanding issues**

- Systematic structure of low energy effective interactions
in terms of properties of modular graph functions w/ Green, Pioline
calculation without requiring subtleties of supermoduli space w/ Green
UV divergences in supergravity and effective interactions
- Ambi-twistor strings ? w/ Casali, Tourkine
- string perturbation theory on curved spaces with RR flux, e.g. $AdS_5 \times S^5$

Veneziano model 1968



Thank You