From Veneziano to Yang-Mills

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Simon Caron-Huot, Amit Sever, and Alexander Zhiboedov
Hellerman, Swanson
Yu-tin Huang et al.
Veneziano, Yankielowicz, Onofri
...

$$A(s,t) = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)}$$

- This amplitude has some miraculous mathematical properties which we will now review.
- We will ask if there are other amplitudes with the same miraculous properties and try to constrain such putative amplitudes.

$$A(s,t) = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)}$$

This amplitude has equally spaced poles $s = -1, 0, 1...$, $t = -1, 0, 1....$ Let us consider the reside at some pole in s:

$$A(s \to n, t) = \frac{Pol_{n+1}(t)}{s-n} \ , \qquad n \in \{-1, 0, 1...\} \ ,$$

and

$$Pol_{n+1}(t) = (-1)^{n+1} \frac{\Gamma(-t-1)}{\Gamma(-n-t-2)}$$

is just a polynomial of degree $n+1$.

The scattering angle is given by $\cos(\theta) = 1 + \frac{2t}{s+4}$ and we should be able to decompose $Pol_{n+1}(t)$ in terms of $P_l(1 + \frac{2t}{s+4})$ where P_l are the usual partial waves

$$P_l(z) = {}_2F_1\left(-l, l+D-3, \frac{D-2}{2}, \frac{1-l}{2}\right)$$

and D is the number of space-time dimensions. For $D = 4$ these are the familiar Legendre polynomials.

$$Pol_{n+1}(t) = \sum_{l=0}^{n+1} a^l_{n+1} P_l \left(1 + \frac{2t}{n+4}\right)$$

Now comes the key requirement of unitarity. Typically unitarity means that the amplitude is smaller than 1, $|A| < 1$. But in the present case this is not constraining because we can take the amplitude and multiply it by an arbitrarily small number. Yet there is an additional very nontrivial constraint

$$a^l_{n+1} \geq 0 \ .$$

This positivity constraint is extremely hard to satisfy in general. What happens if we check it for the Veneziano amplitude? It is satisfied for Pol_0, Pol_1 always. The first interesting case is

$$Pol_2(\cos(\theta)) = \frac{25}{4}\cos^2\theta - \frac{1}{4}$$

and now one is supposed to expand it in terms of the basis

$$\{1, 2\alpha\cos(\theta), 2\cos^2(\theta)(\alpha^2+\alpha)-\alpha\}~, \qquad \alpha = \frac{D-3}{2}~.$$

The coefficients are all non-negative for $D \leq 26$.

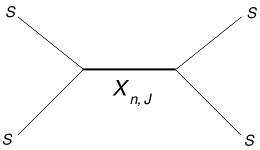
I do not know of any **elementary** proof that the amplitude is indeed unitary for $D \leq 26$ though it is easy to check on the computer up to very high levels.

So let us consider the problem in more generality. The scattering amplitude is a function of two complex variables s, t

$$A(s,t)$$

At the center of mass

$$s = E^2_{c.m.} \ , \qquad 1 + \frac{2t}{s - 4M_S^2} = \cos(\theta) \ .$$



The general properties of $A(s,t)$:

- Polynomial residues (and no other singularities):

$$\lim_{s\to M^2} A(s,t) = \frac{\sum_J f_J^2 P_J\left(1+\frac{2t}{M^2-4M_S^2}\right)}{s-M^2}$$

- Duality

$$A(s,t) = A(t,s)$$

These conditions by themselves are not sufficiently interesting. For example, in tree-level ϕ^3 theory we get

$$A(s,t) = \lambda \left[\frac{1}{s - M^2} + \frac{1}{t - M^2} \right] ,$$

which satisfies all the axioms above.

There is a very natural way to eliminate such "uninteresting" solutions.

We impose that there is some t_0 and spin J_0 particle such that

- Boundedness

$$\lim_{s\to\infty} A(s,t_0) < s^{J_0}$$

Equivalently,

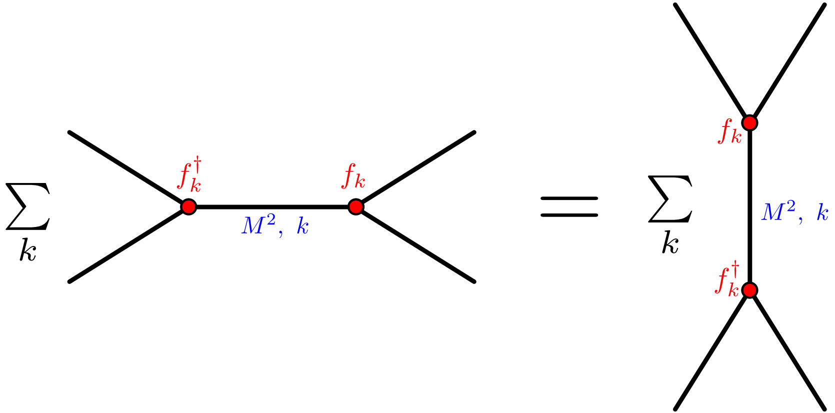
$$\lim_{s\to\infty} s^{-J_0} A(s,t_0) = 0 \ .$$

This 'Boundedness' condition eliminates all classical field theories. Moreover, if there is any particle with spin>2 in the spectrum then this condition must be satisfied if the theory makes sense in the ultraviolet [Camanho et al.].

This condition holds in planar Yang-Mills theory and String Theory. It allows to write the amplitude as *essentially* a sum over s-channel poles only (up to finitely many subtractions which would be unimportant for us)

$$A(s,t)=\sum_{n,J} f_{n,J}^2 \frac{P_J\left(1+\frac{2t}{M_{n,J}^2-4M_S^2}\right)}{s-M_{n,J}^2}$$

The property $A(s,t)=A(t,s)$ is highly nontrivial in this presentation. Also the boundedness is nontrivial.



$\sum_k$
$f_k^\dagger$
f_k
$M^2,\ k$
=
$\sum_k$
f_k
$M^2,\ k$
$f_k^\dagger$

So we now have a system of axioms which are satisfied in planar gauge theories, string theory, etc.

- Tree-Level String Theory: the spins are populated to ∞, large degeneracies and Hagedorn density.
- planar Yang-Mills Theory + Matter: Expect the spins to be populated to ∞, don't expect exact degeneracies and expect Hagedron density of states.

With only these assumptions about the properties of the S-matrix, it is possible to prove a theorem that if the trajectories are exactly linear then the Veneziano amplitude is unique (up to linear combinations).

It is possible that if we further assume a massless spin 2 particle then the Veneziano amplitude follows. [cf. N. Arkani-Hamed's talk]

Historically, the investigation of the resonances of Yang-Mills theory led to String Theory and later String Theory was re-connected with Yang-Mills theory via Holography.

Here we will try to understand in what precise limit String Theory **in flat space** and Yang-Mills theory are connected.

Simple corollaries:

- Such theories must have infinitely many particles.
- Such theories must have particles of unbounded spin.

Both conditions follow because otherwise there won't be appropriate poles in t. Therefore we refer to theories satisfying these conditions as **"Massive Higher Spin Theories."**

It is very interesting to consider in these theories the large s asymptotics with $t \ll s$

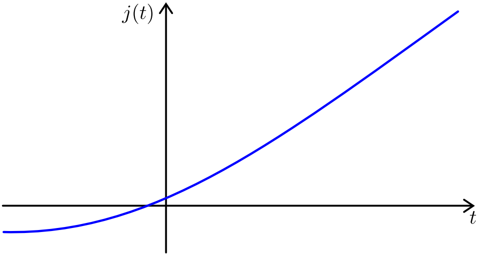
$$\lim_{s\to\infty} A(s,t) = F(t)s^{j(t)} \ .$$

importantly,

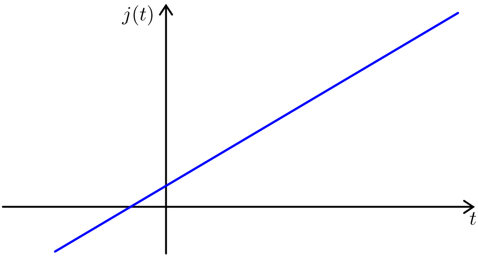
$$j(t_n) = n \ , \qquad n \geq 0$$

describe the fastest spinning particles (i.e. the leading Regge trajectory). If $t \leq 0$ and $s > 0$ then we are describing physical small angle scattering.

Confining Gauge Theory



Free String Theory



We see that at negative t Yang-Mills theory and String Theory are quite different. The Veneziano amplitude does not correctly describe the qualitative behaviour of small angle physical scattering. However, they seem rather similar at positive, large t.

Conjecture: Any Massive Higher Spin Theory behaves like

$$A(s,t) \sim e^{(s+t)\log(s+t) - s\log s - t\log t}$$

for large, positive, s, t (and arbitrary fixed s/t).

This form is, of course, also correct in Tree-Level String Theory [Veneziano...]. The consequences of this claim are

- Infinitely many *asymptotically linear and parallel* trajectories (e.g. in any planar gauge theory!).
- In impact parameter space, if we take $b >> \Lambda_{QCD}^{-1}$ and if $s >> \Lambda_{QCD}^2$ the inelastic part of the amplitude is dominated by a saddle point off the contour of integration and we find

$$Im_s A(s,t) = e^{-\Lambda_{QCD}^2 b^2/\log(s)} \ .$$

  This signifies the existence of strings, because $\langle X_\perp^2 \rangle \sim \log(s)$ in free string theory.

Therefore, any such theory must contain strings and agree with string theory in flat space in the high-energy imaginary-angle limit.

For $x > 1$, $P_J(x) > 0$. So it follows from

$$\lim_{s \to M^2} A(s,t) = \frac{\sum_k f_k^2 P_k \left(1 + \frac{2t}{M^2 - 4M_S^2}\right)}{s - M^2}$$

that for $t > 0$ all the residues are positive. Therefore, there is at least one zero between any two poles.

There may also be "excess" zeroes. But how many?!

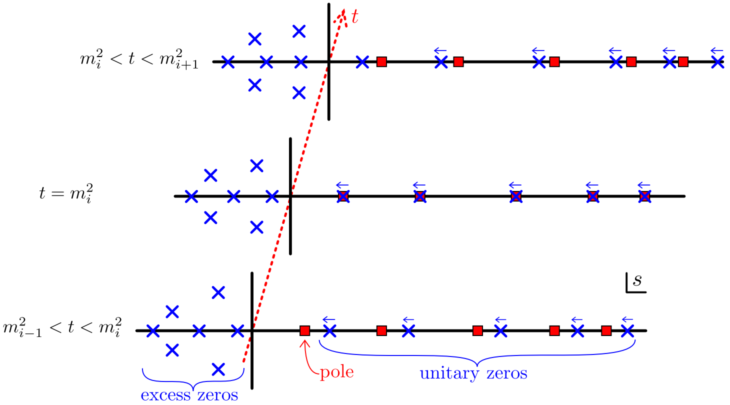
We can count them (reminds of Levinson's theorem from Quantum Mechanics) as follows:

$$\log A \sim j(t) \log(s)$$

implies that the discontinuity in s is given by $j(t)$. On the other hand, we can write the discontinuity as a sum over the number of zeroes minus poles

$$j(t) = \sum (zeroes - poles) \ .$$

So the number of excess zeroes is given by the number of fast-spinning bound states, i.e. $j(t)$!



$m_i^2 < t < m_{i+1}^2$
t
$t = m_i^2$
$m_{i-1}^2 < t < m_i^2$
s
excess zeros
pole
unitary zeros

The next key point is that for large *positive* s,t the amplitude is dominated by the excess zeroes. This is because the unitarity zeroes and poles give a contribution that is bounded by a constant. Therefore we denote the distribution of zeroes by $\rho(z,\bar{z};t)$ and we write for the amplitude

$$\log A = \int d^2 z \ \rho(t;z,\bar{z}) \log(z-s) \ .$$

For very large t, s we can use dimensional analysis

$$\rho(t;z,\bar{z}) = \frac{j(t)}{t^2} \rho(z/t,\bar{z}/t)$$

$(\int d^2 z \rho(z,\bar{z}) = 1, \ \rho \geq 0)$

And thus we obtain (take $j(t) = t^k$)

$$\log A = t^k \int d^2 z \rho(z, \bar{z}) \log \left(1 - \frac{\beta}{z}\right)$$

with $\beta = s/t$.

This looks like the electric potential due to positive charges at $(z, \bar{z})$.

Duality and unitarity thus place nontrivial constraints on the allowed distributions $\rho(z, \bar{z})$. It would be convenient to define the "electric field" $F(\beta)$ as

$$F(\beta) = t^{1-k} \partial_s \log A = \int d^2 z \frac{\rho(z, \bar{z})}{\beta - z}$$

Unitarity 1

$$\partial_\theta^2 \log\left(\sum_{n=0}^{j(t)} C_n^2(t)\cosh(n\theta)\right) > 0$$

After some algebra one can see that this implies an inequality on the dipole moment

$$M_1 \equiv -\int d^2z\; z\rho(z,\bar{z}) \geq \frac{1}{2}$$

Veneziano: $\rho = \delta(Im(z))$ for $-1 \leq Re(z) \leq 0$ and thus $M_1 = 1/2$.

Crossing 1

At $s \gg t$ we have large β and we have the standard multipole expansion from electrostatics

$$F(\beta) = \frac{1}{\beta} - \frac{M_1}{\beta^2} + \cdots$$

but by Duality this implies the small β expansion (Duality takes $\beta \to \beta^{-1}$).

$$F(\beta) = -k\log(\beta)\beta^{k-1} + (k+1)M_1\beta^k + ...$$

which is consistent with the positivity of the second derivative only if

$$k > \frac{1}{2}$$

Furthermore, since we certainly have some positive charges away from the imaginary axis (as $M_1 \geq \frac{1}{2}$), the electric field cannot vanish at $\beta = 0$ and thus $F(\beta) = -k \log(\beta)\beta^{k-1} + ...$ is only consistent with unitarity if

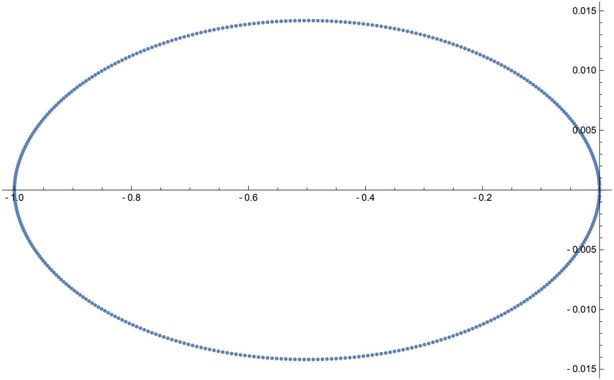
$$k \leq 1$$

Unitarity 2

We have a positive sum of partial waves

$$\sum_{n=0}^{j(t)} C_n^2(t) P_n\left(1+\frac{2s}{t}\right)$$

and the coefficients are not allowed to decrease too fast for otherwise the sum won't Reggeize. The zeroes in $\beta = s/t$ obviously all lie at $Re(z) \leq 0$. If the coefficients do not decay fast then the distribution is supported within the unit circle.


0.015
0.010
0.005
-1.0
-0.8
-0.6
-0.4
-0.2
-0.005
-0.010
-0.015

Crossing 2

Crossing $\beta \leftrightarrow \beta^{-1}$ is now very powerful as we are looking for a function that transforms nicely under $\beta \leftrightarrow \beta^{-1}$ and has branch points only at $0, -1$. This is because the electric field is analytic away from the charge distribution.
The solution to this electrostatics problem is unique:

$$F_k(\beta) =_2 F_1(k, k, k+1, \frac{-1}{\beta}) \ .$$

The dipole moment can be read from the large β expansion. It is $k^2/(k+1)$. Hence we only remain with

$$k = 1$$

Thus we remain with $F_1(\beta) = \log\left(\frac{1+\beta}{\beta}\right)$ and uniform density between $[-1, 0]$. This fixes the amplitude uniquely to be, for large positive s, t,

$$\log A = (t+s)\log(t+s) - s\log(s) - t\log(t)\ .$$

Hence, every theory with spin>2 resonances, including Yang-Mills, must have strings and it is described by the Veneziano amplitude at large positive s, t.

Therefore under this assumption that the zeroes form a sensible distribution it is possible to derive that any planar gauge theory must have infinitely many asymptotically linear parallel trajectories. Also, there is a huge degeneracy asymptotically. We cannot derive yet a Hagedorn density.

Can we continue and determine the corrections to asymptotic linearity? Lifting the asymptotic degeneracy?

The study of the next nontrivial order was done by [Sever,Zhiboedov]. We remind that the leading order term was

$$\log A = (t+s)\log(t+s) - s\log(s) - t\log(t) \ .$$

and it has no parameters. The next correction has one parameter m and it scales like $s^{1/4}$:

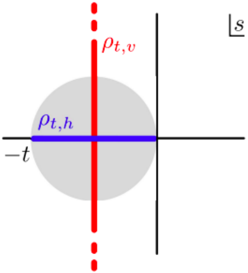
$$\delta \log A = -m^{3/2}\left(\frac{st}{s+t}\right)\left[K\left(\frac{s}{s+t}\right) + K\left(\frac{t}{s+t}\right)\right]$$

where K is an elliptic integral of the first kind. This corresponds to

$$j(t) = \left(t - \frac{1}{2}m^{3/2}t^{1/4} + \mathcal{O}(1)\right)$$

There cannot be corrections like $\log(t)$ or $\sqrt{t}$.
In a different regime [Armoni et al] obtained a logarithmic correction. It would be nice to see how they fit together.

We therefore see very clearly bending of the trajectory, which should arise in any theory with massive endpoints. This is also consistent with AdS models of QCD-like theories. In terms of the dynamics of zeroes, the distribution now looks like it has a vertical line with zeroes escaping the disc.



An Interloper from the Past

To my knowledge there is only one additional explicitly known amplitude which obeys unitarity and crossing. It is also quite an amazing construction [Coon]. The evidence for unitarity is numerical but convincing. The physical particles are at

$$M_n^2 = m_*^2 + \frac{\sigma^n - 1}{\sigma - 1} \ .$$

- $\sigma = 1$: This is the Veneziano Amplitude.
- $\sigma > 1$: $A \sim s^{C \log t}$ and $C > 0$. The amplitude is non-unitary.
- $0 \leq \sigma < 1$: An accumulation point at $m^2 = m_*^2 + \frac{1}{1-\sigma}$. Amazingly the amplitude is unitary. The case of $\sigma = 0$ is just

$$A(s,t) = \frac{1}{(s - m_*^2)(t - m_*^2)} \ .$$

What is the significance of this amplitude?!

- Show that the Veneziano amplitude is the only amplitude with exactly linear trajectories [arguments aren't so nice...]... And what if we assume massless spin 2 particles?
- Can we write interesting toy models for amplitudes reproducing the kind of $j(t)$ that we expect in QCD? (See Veneziano, Yankielowicz, Onofri) looks almost like gluing two Veneziano amplitudes with different slopes.
- The asymptotic, positive s, t regime (i.e. string theory) is separated from the physical small-angle high-energy scattering regime (the "AF" regime) by a phase transition in t. (Analytic continuations and asymptotic limits do not generally commute.) Can we characterise it?
- Can we set up a systematic expansion in $1/s$ for the distribution of zeroes?
- Why does the Coon amplitude exist?

Happy Birthday to $A(s, t)$!