Finite Energy Sum Rules in CFTs

Alexander Zhiboedov, Harvard U

50 Years of The Veneziano Model

with Baur Mukhametzhanov, (to appear)

Veneziano Amplitude

In the original paper by G. Veneziano there is an interesting and slightly confusing formula (FESR)

[Veneziano 68′]

$$\int_0^{\bar{\nu}} \nu \, \mathrm{Im}\, A(\nu, t)\, \mathrm{d}\nu = \frac{\beta(t)(2\alpha'\bar{\nu})^{\alpha(t)-1}\bar{\nu}^2}{\alpha(t)+1} .$$

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The density of states is given by a sum of delta-functions

$$\mathrm{Im}\, A(\nu, t) = -\sum_J \frac{\bar{\beta}\Gamma(1-\alpha(t))}{\alpha'\Gamma(J)\Gamma(2-J-\alpha(t))}\,\delta(s-s_J)(-1)^J + (t \leftrightarrow u) .$$

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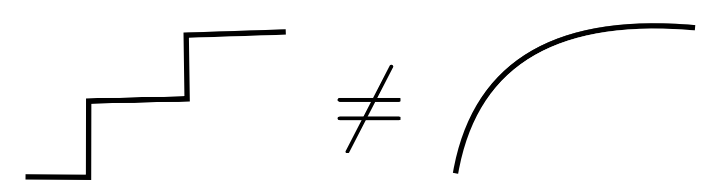
What is the actual meaning of this equation?



Veneziano Amplitude

$$\int_0^{\bar{\nu}} \nu \operatorname{Im} A(\nu, t)\, d\nu = \frac{\beta(t)(2\alpha'\bar{\nu})^{\alpha(t)-1}\bar{\nu}^2}{\alpha(t)+1} .$$

The two are **asymptotically the same**:

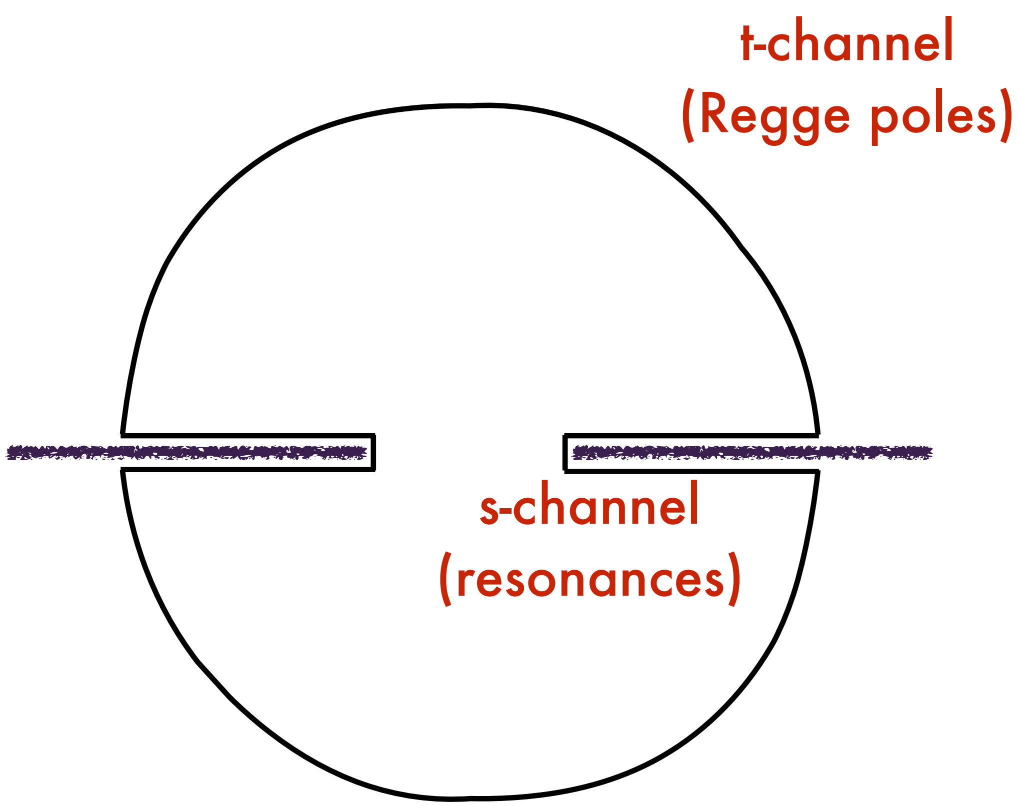
$$a(\bar{\nu}) \sim b(\bar{\nu}), \qquad (\bar{\nu} \to \infty)$$

$$\lim_{\bar{\nu}\to\infty} \frac{a(\bar{\nu})}{b(\bar{\nu})} = 1$$

Finite Energy Sum Rules

The usual derivation of the finite energy sum rules is based on analyticity and Regge theory

[Igi 62′], [Igi, Matsuda 67′]
[Soloviev, Logunov, Tavhelidze 67′]
[Dolen, Horn, Schmid 67′]



$$\oint ds (s-u)^n A(s,t) = 0$$

$$\int_0^{s_0} ds (s-u)^n \mathrm{Im} A(s,t) = f(t) s_0^j(t) + \ldots$$

This is an example of crossing.

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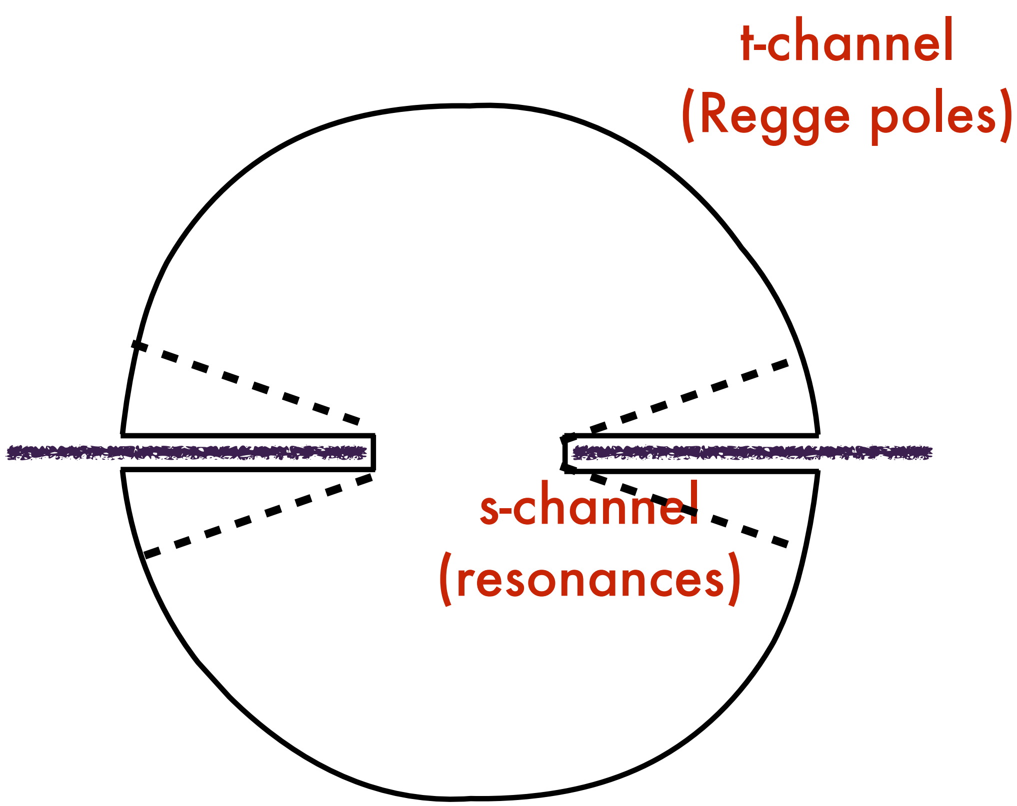

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This is an example of crossing.

Unjustified for meromorphic amplitudes.

Asymptotic Energy Sum Rules

Another way to derive it: dispersion relations, unitarity, tauberian theorem.

The result is formula from the paper by Veneziano

$$\int_0^{\bar{\nu}} \nu \, \mathrm{Im}\, A(\nu, t)\, \mathrm{d}\nu = \frac{\beta(t)(2\alpha'\bar{\nu})^{\alpha(t)-1}\bar{\nu}^2}{\alpha(t)+1} \, .$$

asymptotically

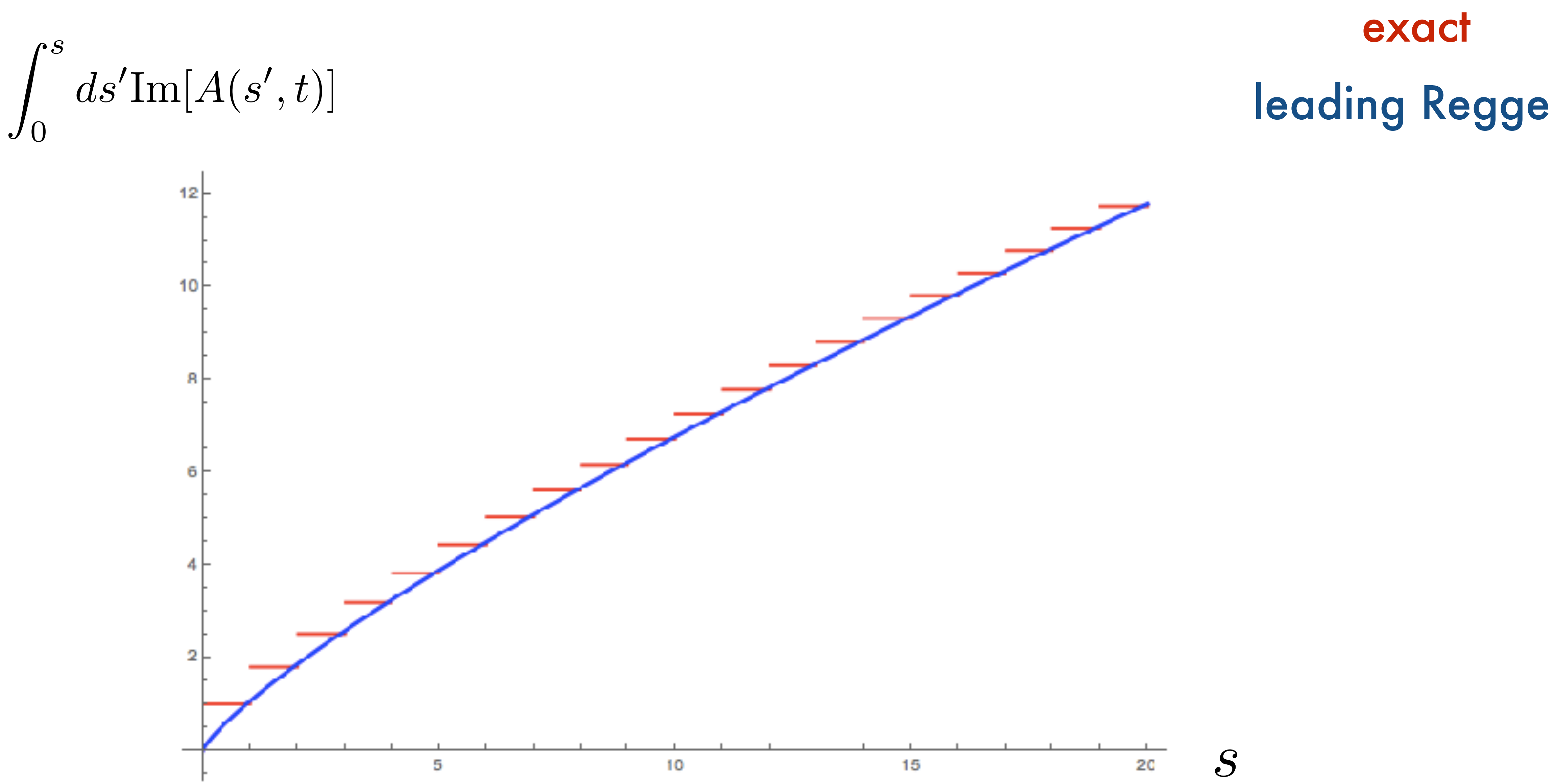
$$(\bar{\nu} \to \infty)$$
$$t > 0$$

Asymptotic is rigorous!

(large N QCD)

Finite Energy Sum Rules: Veneziano

It works great for finite energy as well.



(Similarly, one can see the subleading corrections)

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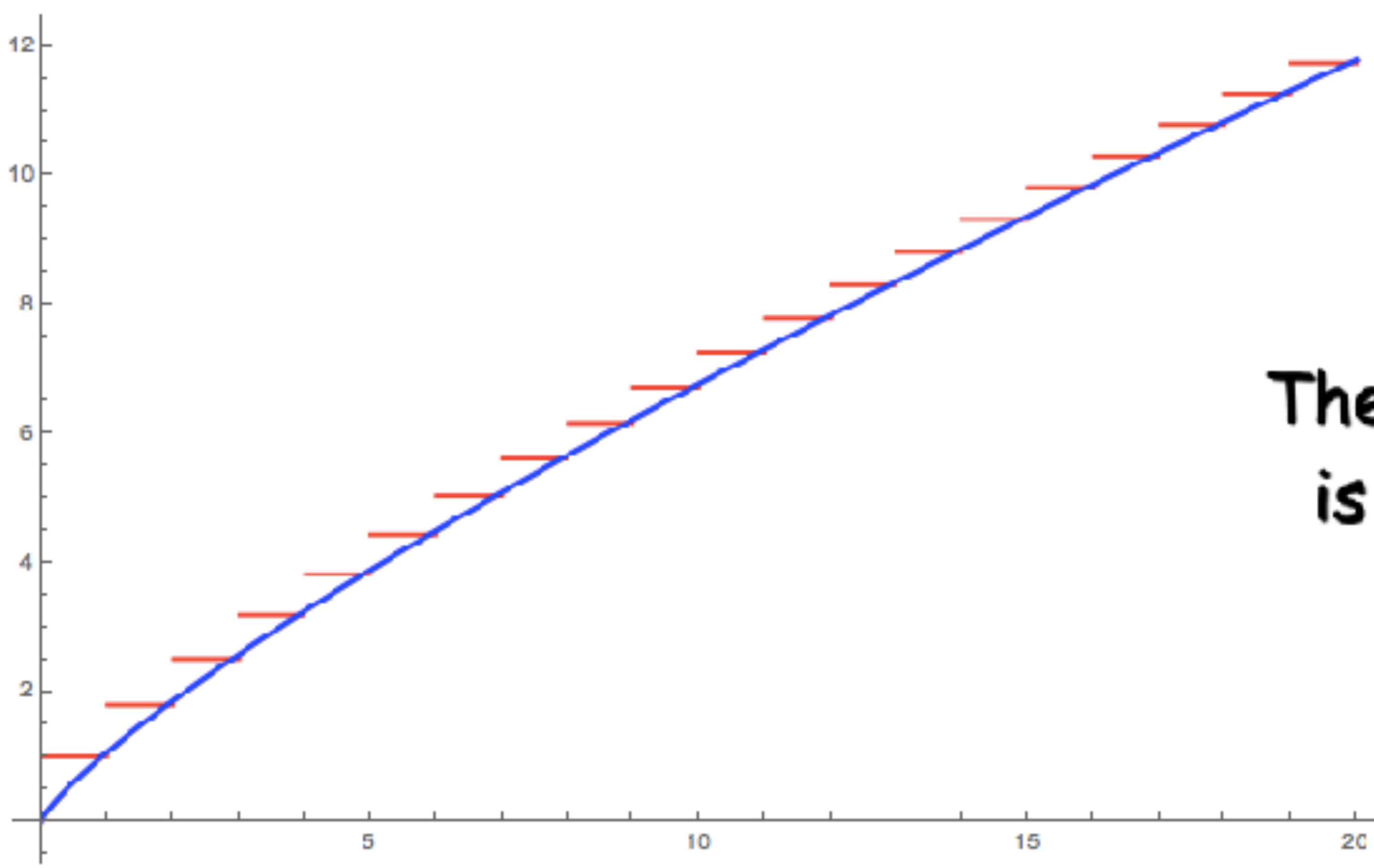

The cheapest bootstrap is working (too?) well

[Veneziano's talk]

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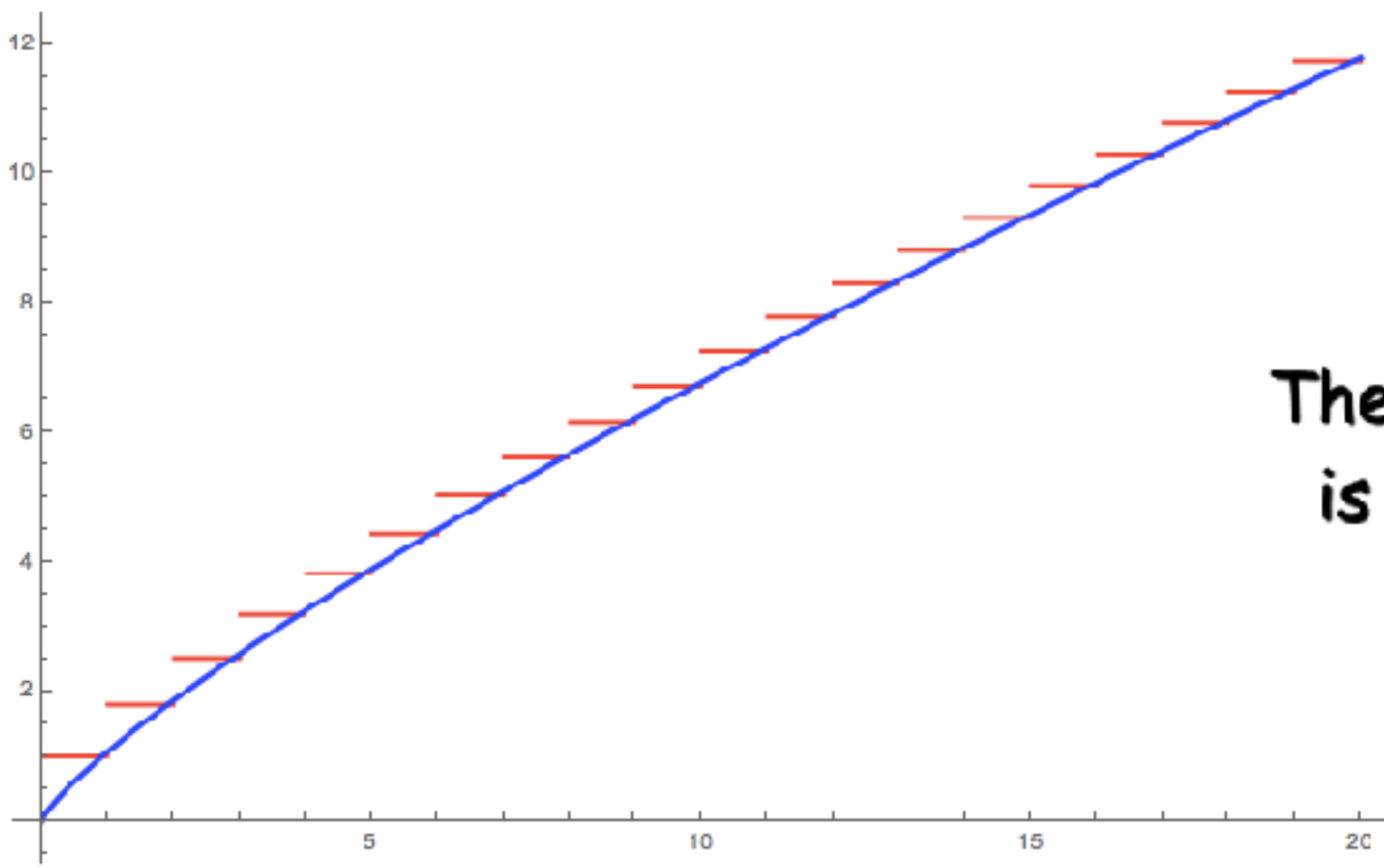
exact

leading Regge

$$\int_0^s ds' \mathrm{Im}[A(s',t)]$$



s

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How general it is?

Plan

The goal of my talk is to describe a similar construction for unitary CFTs.

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$$s-\text{plane} \longleftrightarrow \Delta-\text{plane}$$

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$A(s,t)$ (scattering amplitude)	$\longleftrightarrow$	$c(J,\Delta)$ (conformal partial wave)
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dispersion relations

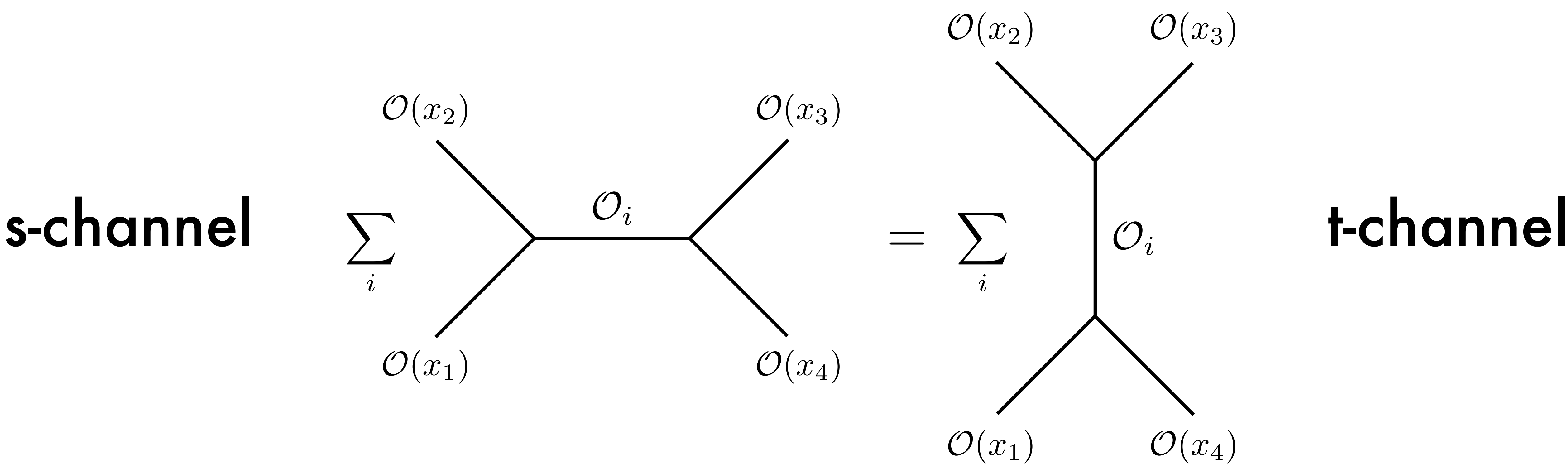
Result: solving crossing in $1/\Delta$ expansion
(analytic Euclidean bootstrap)

Conformal Bootstrap

Consider the four-point function of identical operators:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta}}$$

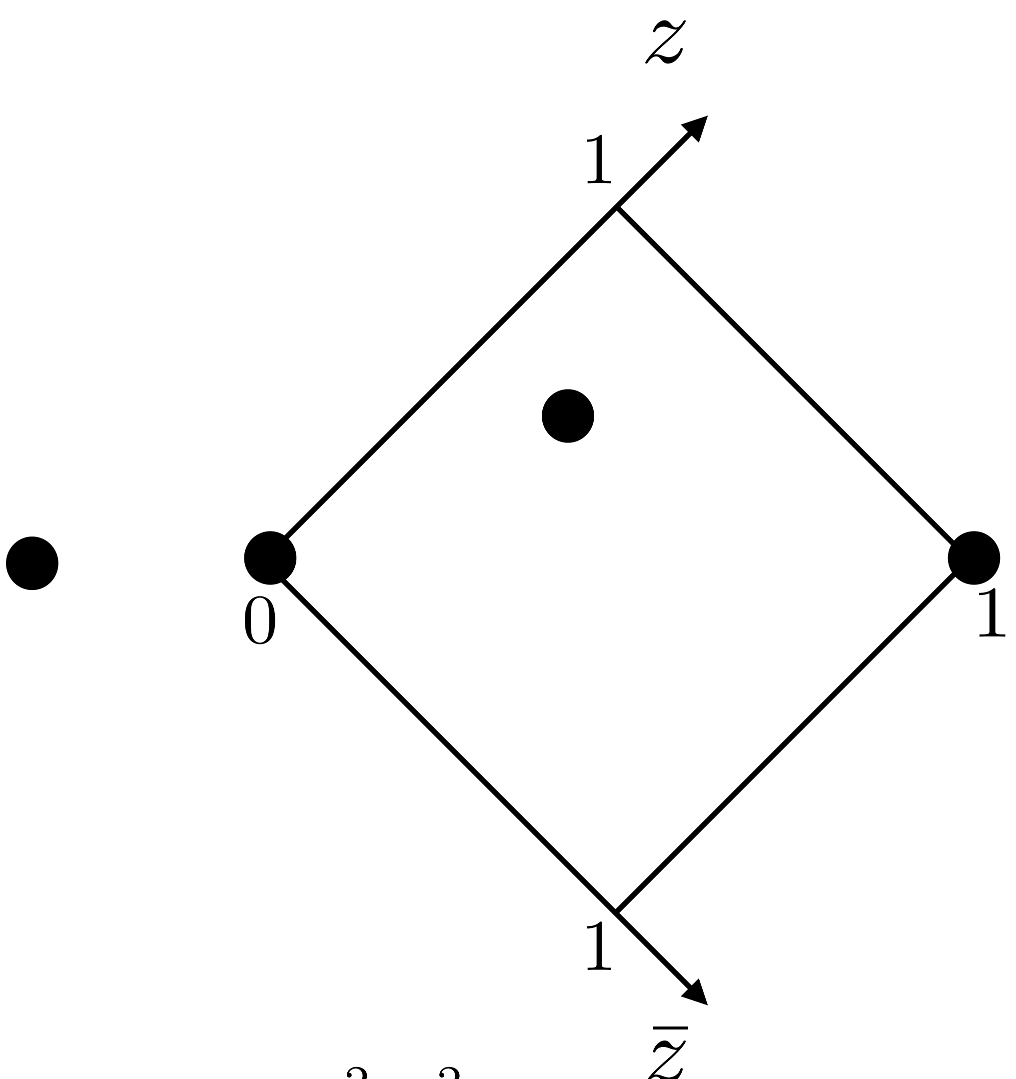
$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$



CFT data: $(\Delta, \mathbf{J})$ λ_{ijk}

Conformal Bootstrap

$$\langle \mathcal{O}(0)\mathcal{O}(z,\bar{z})\mathcal{O}(1)\mathcal{O}(\infty)\rangle$$



$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad ,$$

$$v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

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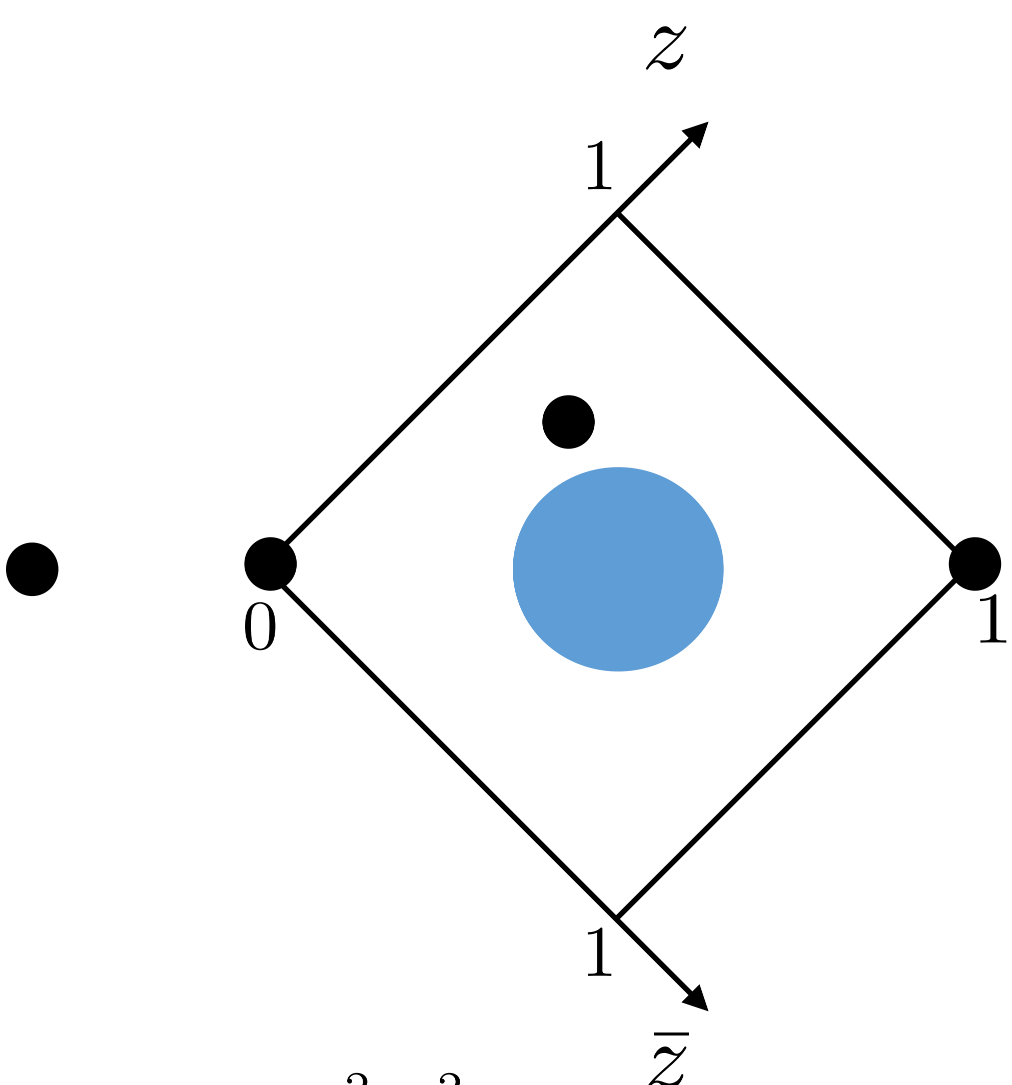

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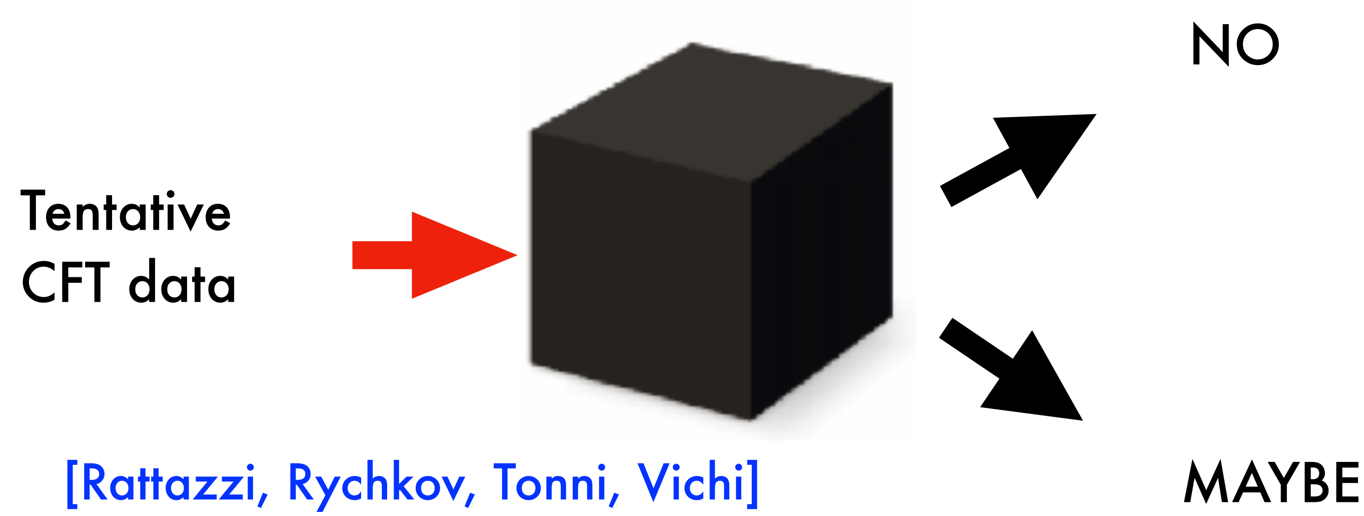

[Rattazzi, Rychkov, Tonni, Vichi]
[Rychkov talk]

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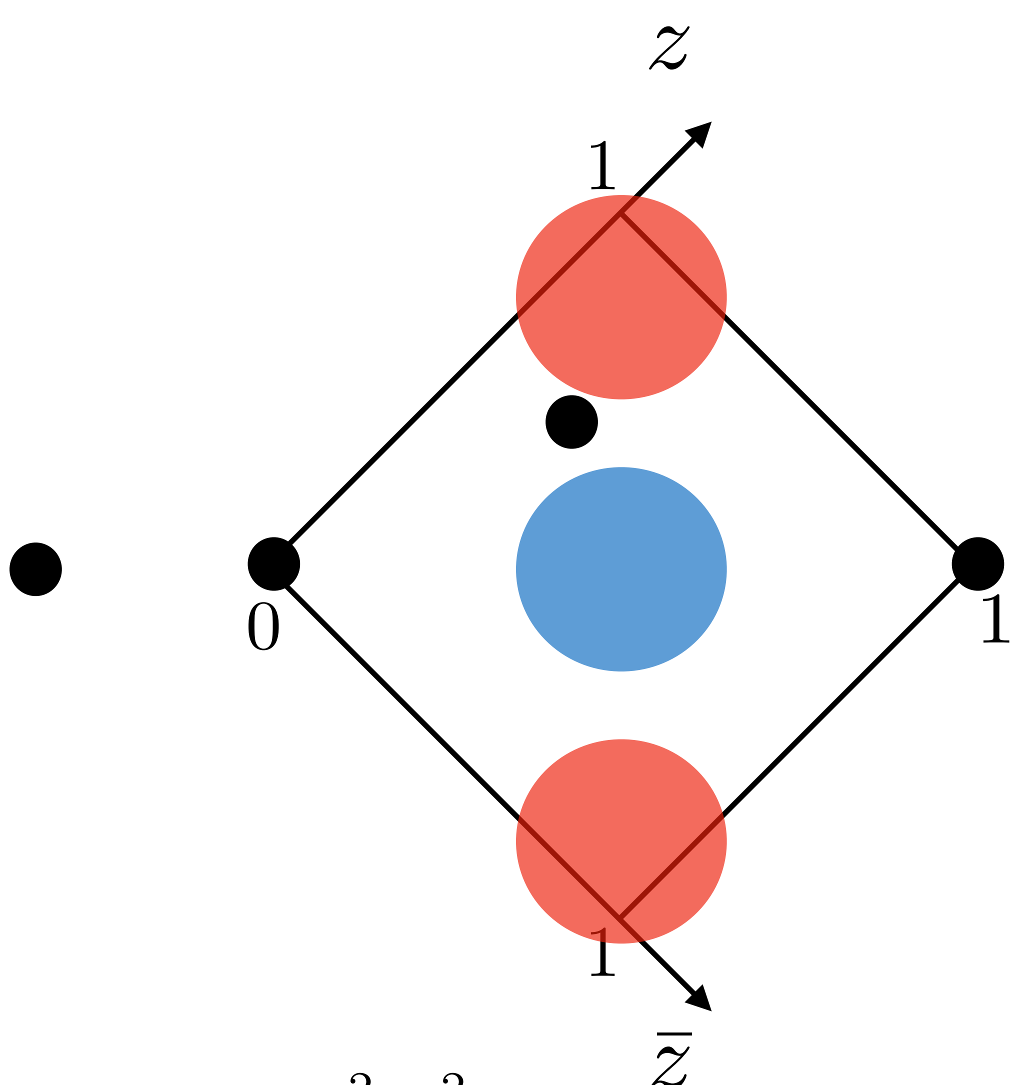

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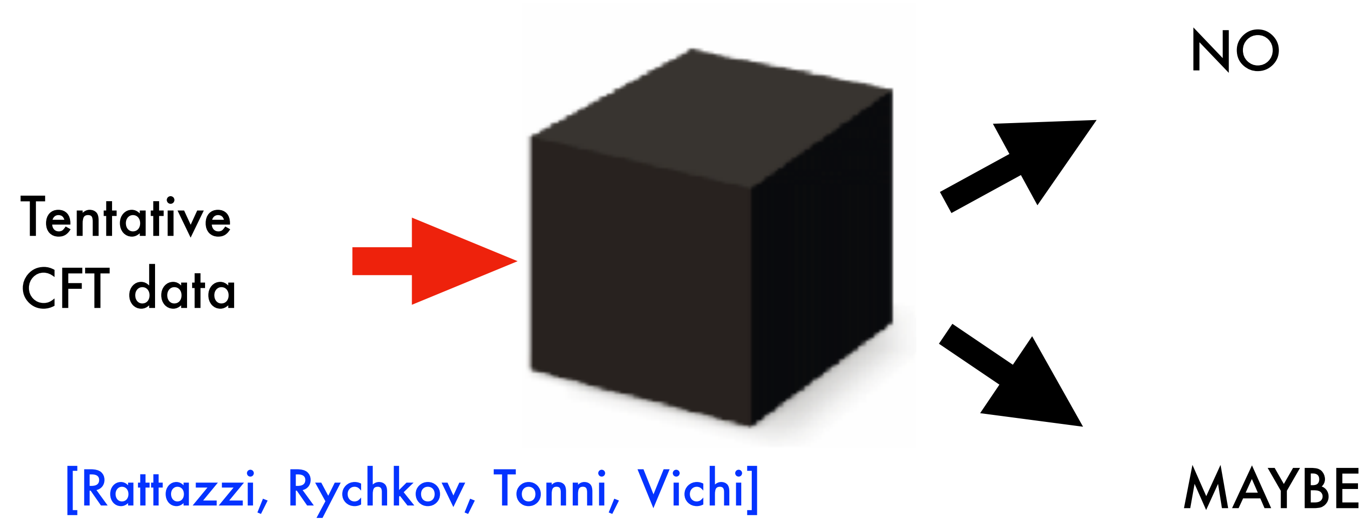

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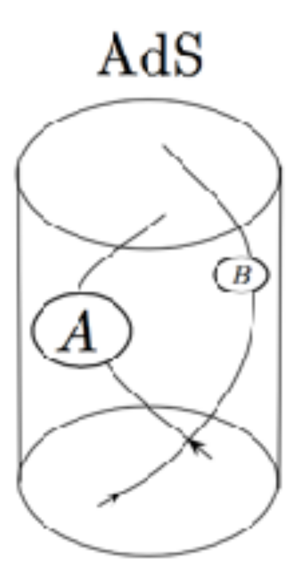
- analytic bootstrap (expansion in spin)



light-cone OPE

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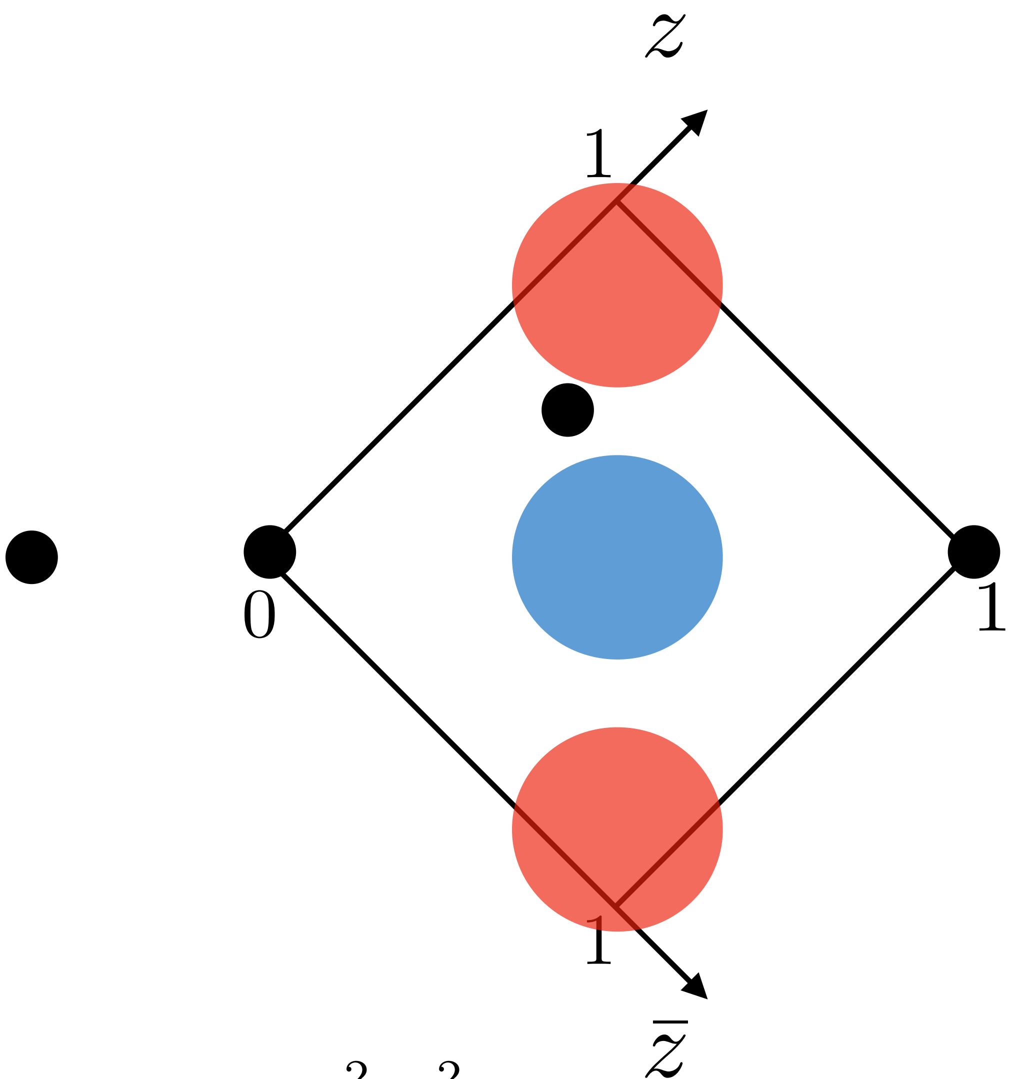

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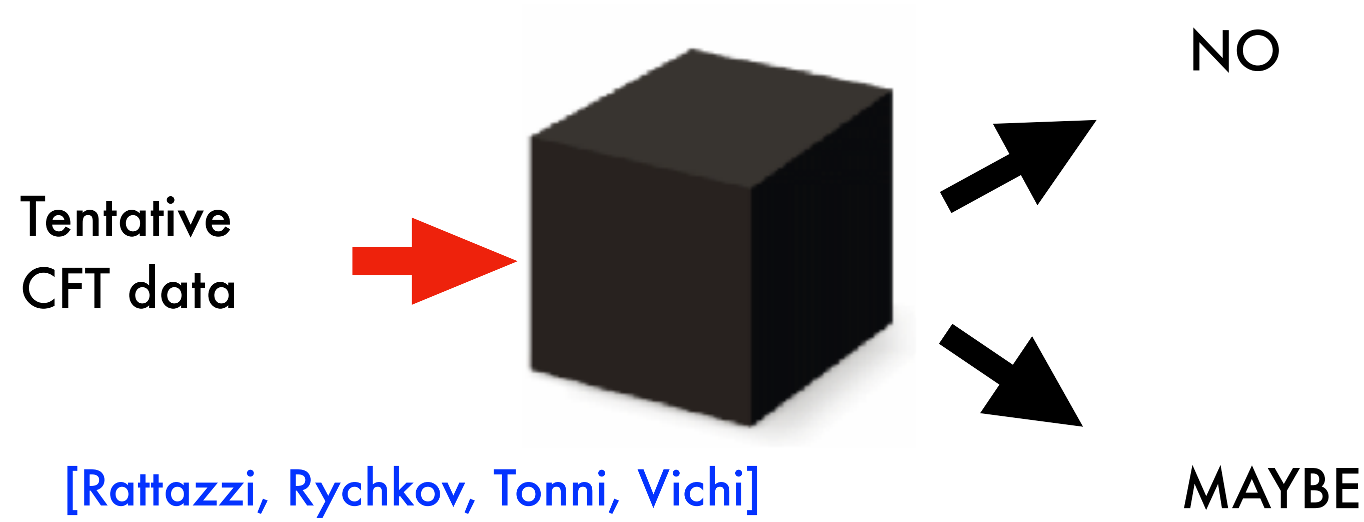

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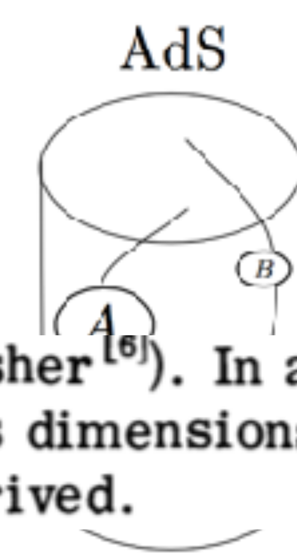

light-cone OPE

Fisher[6]). In addition, general properties of the anomalous dimensions of operators with high spins have been derived.

[Parisi 72'] [Callan, Gross 73'] [Polyakov 74']

Tauberian Theorem and Euclidean Bootstrap

[Pappadopulo, Rychkov, Espin, Rattazzi 12']

[Rychkov, Yvernay 15']

[Qiao, Rychkov 17']

Consider the limit of cross ratios $z, \bar{z} \to 1$

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$$\int_0^\infty dE\ f(E) e^{-E\beta} \sim \frac{1}{\beta^{2\Delta_{\mathcal{O}}}} \left(1 + O(\beta^{\Delta^*})\right) \qquad (\beta \to 0)$$

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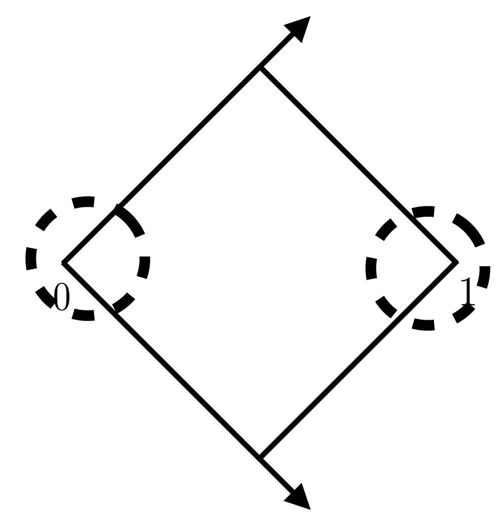

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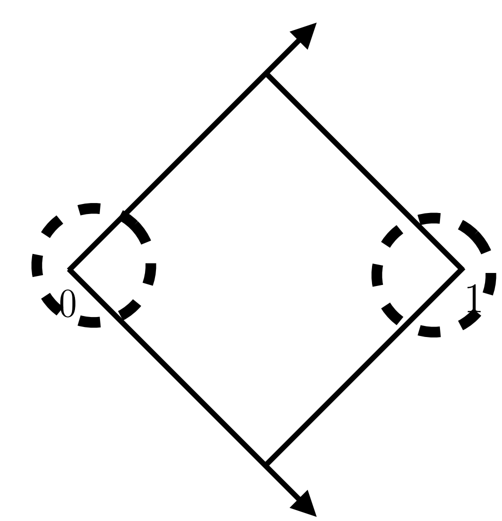

Hardy-Littlewood Tauberian Theorem:

$$\Rightarrow \int_0^E dE'\ f(E') \sim \frac{E^{2\Delta_{\mathcal{O}}}}{\Gamma(2\Delta_{\mathcal{O}} + 1)} \left[1 + O(\frac{1}{\log E})\right]$$

Example

Asymptotic of the integrated density does not fix asymptotic of the density itself

$$f(E) = \frac{E^{2\Delta-1}}{\Gamma(2\Delta)} (1 + \alpha \sin E\tau)$$

Asymptotic of the density depends on alpha.

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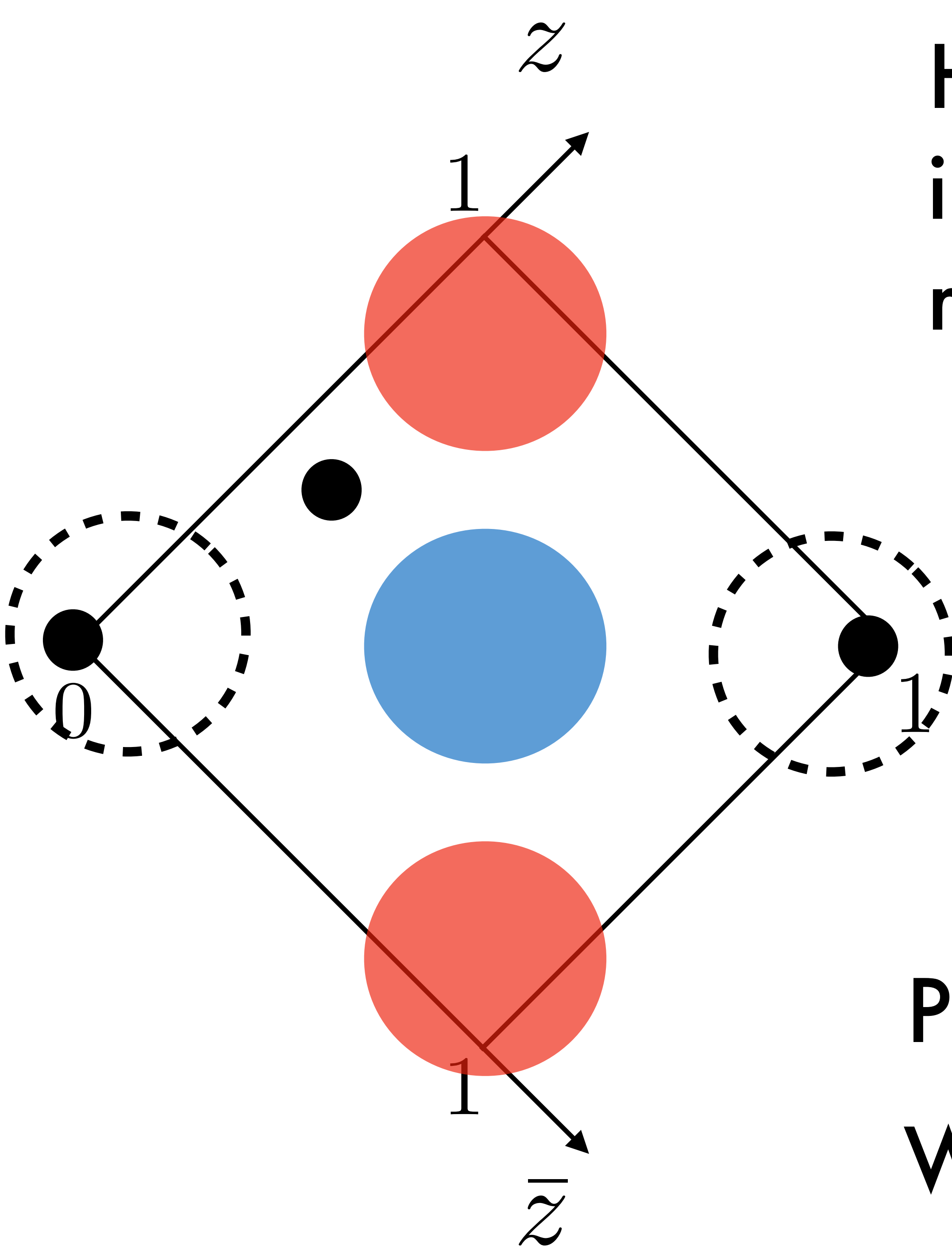
Asymptotic of the density depends on alpha.

Asymptotic of the integrated density is universal.

$$\int_0^E dE' f(E') = \frac{E^{2\Delta}}{\Gamma(2\Delta+1)}\left(1-\alpha\frac{2\Delta\cos E\tau}{E\tau}+...\right)$$

Analytic Euclidean Bootstrap



How do we solve crossing in the Euclidean OPE region more efficiently?

Primaries of different spin mix

What happens to corrections?

Conformal Partial Wave Expansion

There is a more primitive expansion of the correlator

$$\mathcal{G}(z,\bar{z}) = \sum_{J=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i}\; c(J,\Delta) F_{\Delta,J}(z,\bar{z}) + (\mathrm{non-norm})$$

Conformal partial waves are given by

$$F_{\Delta,J} = \frac{1}{2}\left(K_{J,\Delta} G_{\Delta,J} + K_{J,d-\Delta} G_{d-\Delta,J}\right)$$

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By closing the contour we recover the OPE expansion.

This is achieved by $c(J,\Delta)$ being **meromorphic** with its **residues** fixed in terms of the three-point functions.

Properties of Partial Waves

* Shadow symmetry

$$c(J, \Delta) = c(J, d - \Delta)$$

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universal singularities
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$$\frac{1}{K_{J,\Delta}} \sim 4^{\Delta}, \quad \Delta \gg 1$$

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* Polynomial boundedness (Caron-Huot's inversion formula)

Lorentzian Inversion Formula

[Caron-Huot 17']

$$c(J,\Delta) = c^t(J,\Delta) + (-1)^J c^u(J,\Delta)$$

$$c^t(J,\Delta) = \int_0^1 dz d\bar{z} \mu(z,\bar{z}) G_{J+d-1,\Delta+1-d}(z,\bar{z}) \mathrm{dDisc}[\mathcal{G}(z,\bar{z})]$$

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* Valid for J>1 (in the planar limit J>2)
* Equal to the square of a commutator

$$\mathrm{dDisc}[G(z,\bar{z})] = -\frac{1}{2}\langle [\mathcal{O}_2(-1),\mathcal{O}_3(-\rho)][\mathcal{O}_1(1),\mathcal{O}_4(\rho)]\rangle \geq 0$$

$$\mathrm{dDisc}[G(z,\bar{z})] \sim \sum_{\mathcal{O}',J'} \sin^2\left(\frac{\pi(\Delta'-2\Delta-J')}{2}\right)\lambda^2_{\mathcal{O}',J'}\left|\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right|^{\Delta'+J'}\left|\frac{1-\sqrt{\bar{\rho}}}{1+\sqrt{\bar{\rho}}}\right|^{\Delta'-J'}$$

$$z = \frac{4\rho}{(1+\rho)^2}$$

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s-channel data

- Valid for J>1 (in the planar limit J>2)
- Equal to the square of a commutator

$$\mathrm{dDisc}[G(z,\bar{z})] = -\frac{1}{2}\langle[\mathcal{O}_2(-1),\mathcal{O}_3(-\rho)][\mathcal{O}_1(1),\mathcal{O}_4(\rho)]\rangle \geq 0$$

$$\mathrm{dDisc}[G(z,\bar{z})] \sim \sum_{\mathcal{O}',J'} \sin^2\left(\frac{\pi(\Delta'-2\Delta-J')}{2}\right) \lambda^2_{\mathcal{O}',J'} \left|\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right|^{\Delta'+J'} \left|\frac{1-\sqrt{\bar{\rho}}}{1+\sqrt{\bar{\rho}}}\right|^{\Delta'-J'}$$

$$z = \frac{4\rho}{(1+\rho)^2}$$

Lorentzian Inversion Formula

[Caron-Huot 17']

$$c(J,\Delta) = c^t(J,\Delta) + (-1)^J c^u(J,\Delta)$$

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Lorentzian Inversion Formula and OPE

Let us plug **t-channel OPE** in the inversion formula:

$$c(J,\Delta) = \sum_{i,\tilde{J}} f^{t}_{\Delta_i,\tilde{J}}(J,\Delta)$$

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Euclidean Analytic Bootstrap: expansion of $c(J,\Delta)$ at **large scaling dimension** Δ and **fixed spin** J is controlled by the short-distance OPE

(1/dimension expansion)

Unit Operator

Let us invert the unit operator in the t-channel

$$\mathcal{G}(z,\bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_{\mathcal{O}}} + \ldots \quad \Rightarrow \quad \mathrm{dDisc}\mathcal{G}(z,\bar{z}) = 2\sin^2(\pi\Delta_{\mathcal{O}})\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_{\mathcal{O}}} + .$$

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extra poles physical poles

$$c(J,\Delta) \sim f(\Delta) f(d-\Delta) \qquad f(\Delta) = \frac{\Gamma(\frac{\Delta+J+1}{2})\Gamma\left(\frac{-\Delta+2\Delta_{\mathcal{O}}+J}{2}\right)}{\Gamma\left(\frac{\Delta+J}{2}\right)\Gamma\left(\frac{\Delta+J}{2}-\Delta_{\mathcal{O}}+\frac{d}{2}\right)}$$

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This also represents the full answer in the theory of **generalized free fields**

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle = \langle \mathcal{O}\mathcal{O}\rangle\langle \mathcal{O}\mathcal{O}\rangle + \text{perm}$$

Dispersion Relations

Having identified a meromorphic polynomially bounded function it is natural to consider dispersion relations

$$\frac{1}{n!}\partial_{\Delta}^{n} c(J,\Delta) = \oint \frac{d\Delta'}{2\pi i} \; \frac{c(J,\Delta')}{(\Delta'-\Delta)^{n+1}}$$

We can then drop the contribution from infinity.

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exponential enhancement $4^\Delta \lambda^2_{\Delta,J}$

Dispersion Relations

Consider for simplicity the case without subtractions

$$0 < \Delta_{\mathcal{O}} - \frac{d-2}{2} \leq \frac{3}{4}$$

$$\int_0^\infty d\nu' \rho_J^{OPE}(d/2 + \nu') \frac{2\nu'}{\nu'^2 + \nu^2} = c(J, d/2 + i\nu) + \text{extra}$$

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$$\text{extra} = \sum_{n=0}^{\infty} \frac{2(J+n)+d}{\left(J+\frac{d}{2}+n\right)^2+\nu^2} \alpha_n c(J+n+1, J+d-1)$$

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controlled by the Euclidean OPE at large n

Dispersion Relations

We therefore arrive at the following picture

$$\int_0^\infty d\nu' \rho_J^{OPE}(d/2+\nu') \frac{2\nu'}{\nu'^2+\nu^2} = \sum_i \frac{c_i}{\nu^{\gamma_i}} + \sum_{n=1}^{\infty} \frac{d_n}{\nu^{2n}}$$

OPE-computable ``tails'': come from an infinite family operators in the s-channel

Non-OPE terms: come from individual operators in the s-channel

We can now apply tauberian theorem to put it into a more familiar form.

A(symptotic)ESR in CFTs

$$\Delta_{\mathcal{O}} = \frac{d-2}{2} + \gamma$$

The result is that in any CFT the following relations are true:

$$\int_0^{\Delta} d\tilde{\Delta}\ \tilde{\Delta}^2 \rho_J^{OPE}(\tilde{\Delta}) \sim \beta_J \frac{\Delta^{4\gamma}}{4\gamma} \qquad 0 < \gamma \leq \frac{1}{4}$$

$$\int_0^{\Delta} d\tilde{\Delta}\ \tilde{\Delta} \rho_J^{OPE}(\tilde{\Delta}) \sim \beta_J \frac{\Delta^{4\gamma-1}}{4\gamma-1} \qquad \frac{1}{4} < \gamma \leq \frac{1}{2} \qquad (\Delta \to \infty)$$

$$\int_0^{\Delta} d\tilde{\Delta} \rho_J^{OPE}(\tilde{\Delta}) \sim \beta_J \frac{\Delta^{4\gamma-2}}{4\gamma-2} \qquad \gamma > \frac{1}{2}$$

$$\beta_J = 2^{3+2J+d-4\gamma} \pi^2 \frac{\Gamma\left(J + \frac{d}{2}\right)}{\Gamma(J+1)\Gamma(\gamma)^2 \Gamma(\Delta_{\mathcal{O}})^2}$$

``All CFTs are GFF at large dimension''

Finiteness and Corrections

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What happens at finite scaling dimension? (``operator deserts”)

2d Ising

$$z = \frac{4\rho}{(1+\rho)^2}$$

Consider the four-point function of spin fields

$$\mathcal{G}^{2d\ \text{Ising}}(z,\bar{z}) = \frac{1+\sqrt{\rho}\sqrt{\bar{\rho}}}{(1-\rho^2)^{1/4}(1-\bar{\rho}^2)^{1/4}}$$

$$\mathcal{G}^{GFF}(z,\bar{z}) = 1 + \left(\frac{16\rho\bar{\rho}}{(1-\rho)^2(1-\bar{\rho})^2}\right)^{1/8} + \left(\frac{16\rho\bar{\rho}}{(1+\rho)^2(1+\bar{\rho})^2}\right)^{1/8}$$

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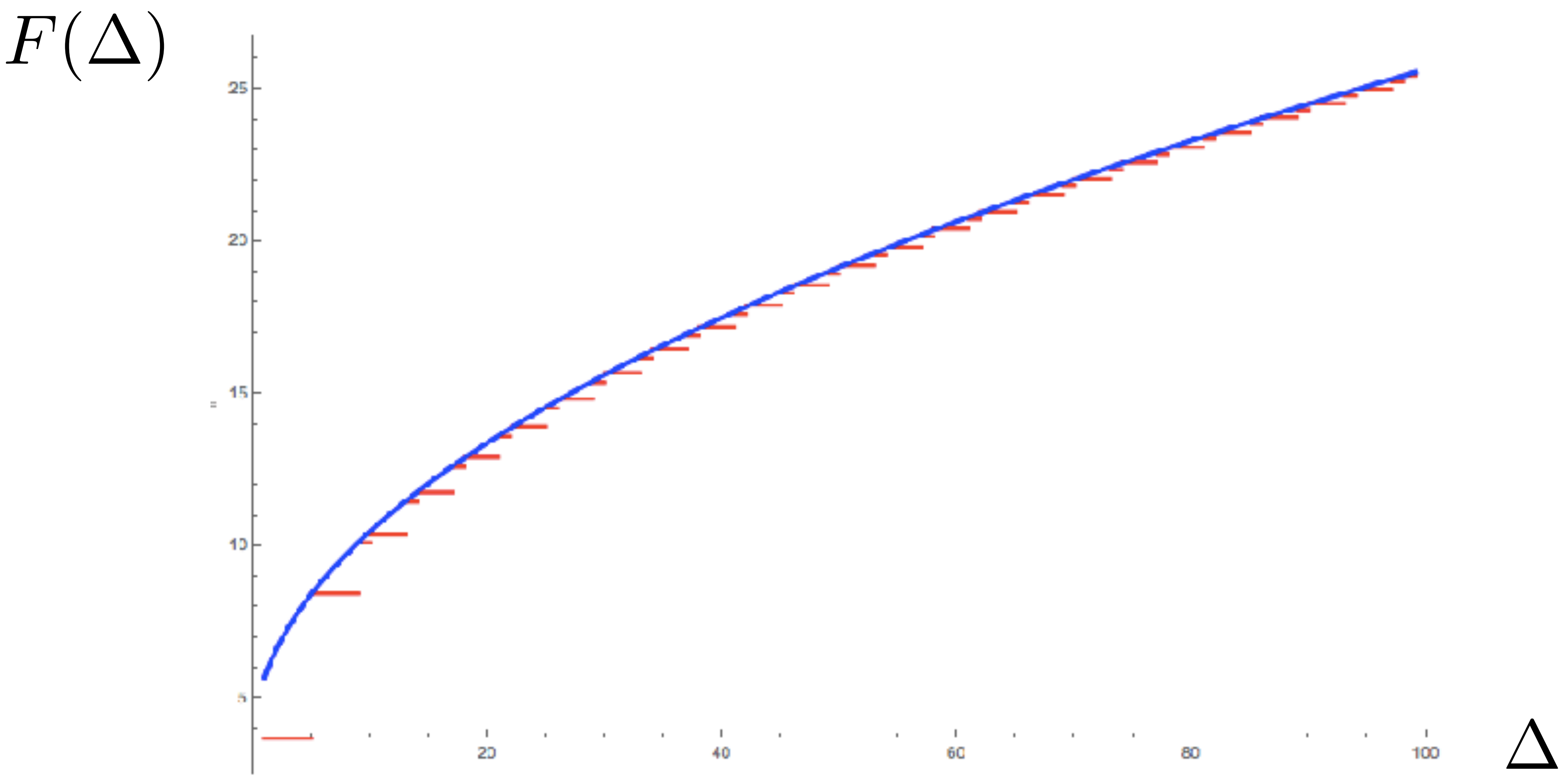
Let us plot the renormalized spectral density

$$\rho_J^{OPE}(\Delta) = \sum_i \frac{\lambda^2_{\Delta_i,J}}{K_{J,\Delta_i}}\delta(\Delta-\Delta_i)$$

[Belavin, Polyakov, Zamolodchikov 84']

2d Ising

Spin 2 integrated spectral density (similar for other spins)

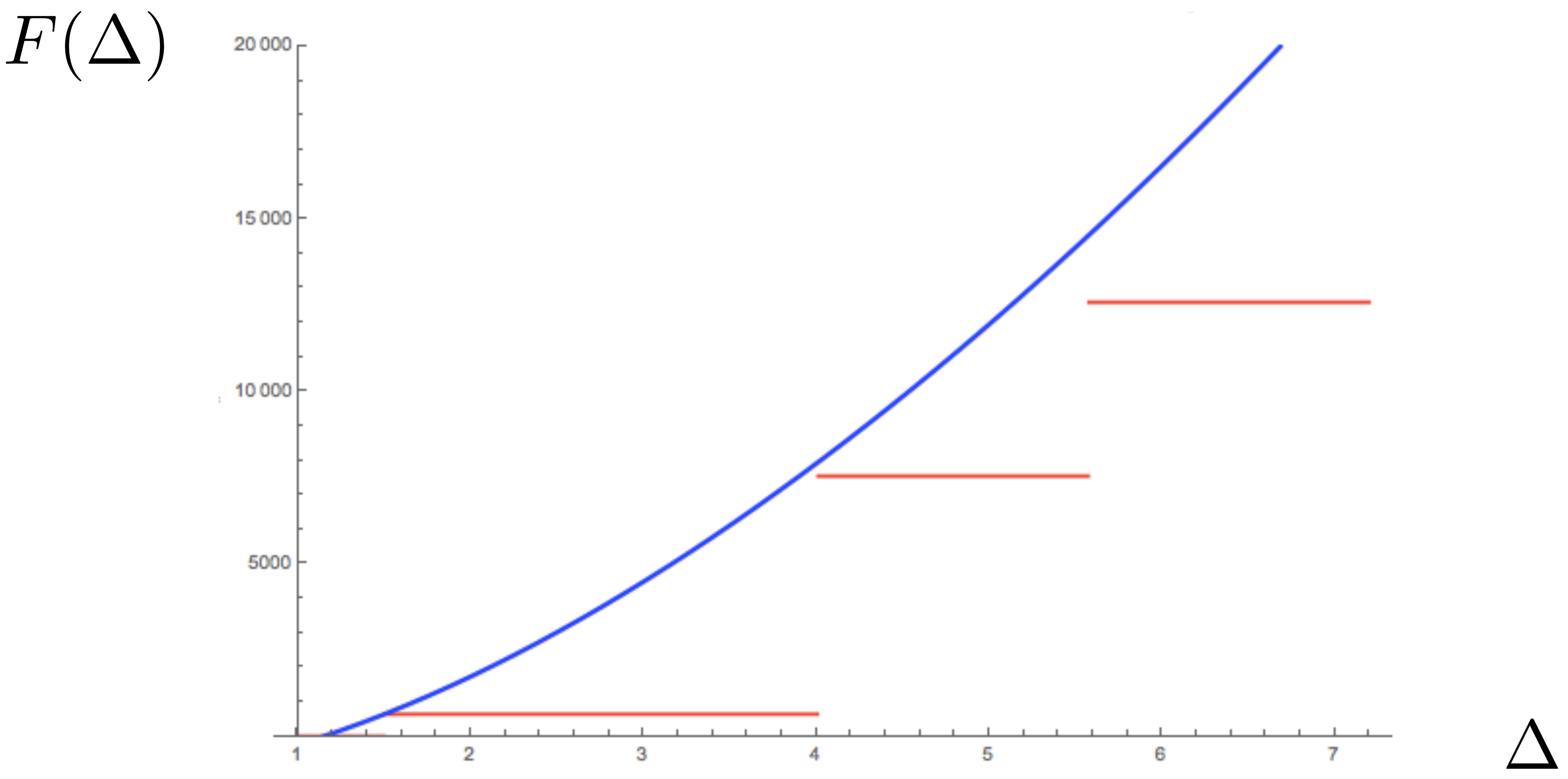


$$F(\Delta) = \int_0^{\Delta} d\tilde{\Delta}\ \tilde{\Delta}^2 \rho_J^{OPE}(\tilde{\Delta}) = c_{GFF}\nu^{1/2} + d$$

3d Ising

[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi 12′]
[Simmons-Duffin 16′]

Spin 2 integrated spectral density $\Delta_\epsilon \simeq 1.41$



$$F(\Delta) = \int_0^\Delta d\tilde{\Delta}\ \rho_J^{OPE}(\tilde{\Delta}) = c_{GFF}\nu^{1.65} + d$$

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* Something interesting happens at finite Δ
(Veneziano amplitude, 2d Ising, 3d Ising, large N QCD…)

Generalizations and Open Questions

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What are the general lessons to be learned from the Veneziano amplitude?

Thank You!