The Analytic Conformal Bootstrap (and applications to large N theories)

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50 years of the Veneziano model

What will this talk be about?

We will study conformal field theories in $D > 2$

- Very relevant for Physics.
- Interesting interplay with Mathematics.
- Ubiquitous in dualities in string and gauge theory.

Unfortunately, studying CFT in $D > 2$ is not so easy...

- Symmetries are less powerful than in $D = 2$...
- In general they do not have a Lagrangian description...
- In a Lagrangian theory we can use Feynman diagrams:

$$A(g) = A^{(0)} + gA^{(1)} + \cdots$$

- But generic CFTs don't have a small coupling constant!

In spite of all this, progress can be made!

Conformal bootstrap

- Conformal bootstrap: resort to consistency conditions!
  - Conformal symmetry
  - Properties of the OPE
  - Unitarity
  - Crossing symmetry
- As we know the idea of the bootstrap is not new, and Veneziano was using it, 50 years ago!
- Successfully applied to 2d CFT in the eighties! [Ferrara, Gatto, Grillo; Belavin, Polyakov, Zamolodchikov]
- 25 years later it was finally implemented in $D > 2$! [Rattazzi, Rychkov, Tonni, Vichi '08] The original approach was numeric.

Today: Analytic results for CFTs in the spirit of the bootstrap!

Analytic conformal bootstrap

Which kind of analytic results will you get today?

Analytic bootstrap

- Results for operators with spin in a generic CFT!

$$\mathcal{O}_{DT} \sim \varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi, \qquad \mathcal{O}_{ST} \sim Tr\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi$$

- Study their scaling dimension Δ for large values of the spin ℓ:

$$\Delta(\ell) = \ell + 2\Delta_\varphi + \frac{c_2}{\ell^2} + \cdots, \quad \text{double-trace}$$

$$\Delta(\ell) = d_0 \log \ell + \ell + d_1 + \frac{c_1}{\ell} + \cdots, \quad \text{single-trace}$$

- We will obtain analytic results to all orders in $1/\ell$ resorting only to consistency conditions.
- Even valid for finite values of the spin and vast families of CFTs!

CFT - Symmetries

Conformal algebra:

- Scale transformations → dilatation D
- Poincare Algebra: P_μ and $M_{\mu\nu}$
- Special conformal transformations: K_μ

Specific CFTs may have extra symmetries but we will keep the discussion very general.

CFT - Ingredients

Main ingredient:

- Conformal Primary local operators: $\mathcal{O}_{\Delta,\ell}(x)$, $[K_\mu, \mathcal{O}(0)] = 0$

Dimension

Lorentz spin

In addition we have descendants $P_{\mu_k}...P_{\mu_1}\mathcal{O}_{\Delta,\ell} = \partial_{\mu_k}...\partial_{\mu_1}\mathcal{O}_{\Delta,\ell}$.

Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k\in\text{prim.}} C_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\left(\mathcal{O}_k(0)\underbrace{+x^\mu\partial_\mu\mathcal{O}_k(0)+\cdots}_{\text{all fixed}}\right)$$

- CFT data: The set Δ_i and C_{ijk} characterizes the CFT.

CFT - Basics

Main observable:

Correlation functions of primary operators

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(1)\mathcal{O}_j(2)\mathcal{O}_k(3)\rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Four-point function of identical operators:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}$$

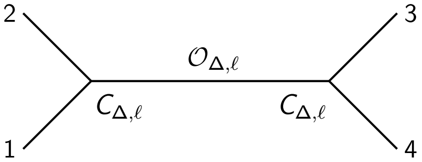
where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

Four-point function - properties

Conformal partial wave decomposition

- OPE: $\mathcal{O} \times \mathcal{O} = \sum_i \mathcal{O}_i + \textit{descendants}$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Delta,\ell}$$



$$\mathcal{G}(u,v) = 1 + \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

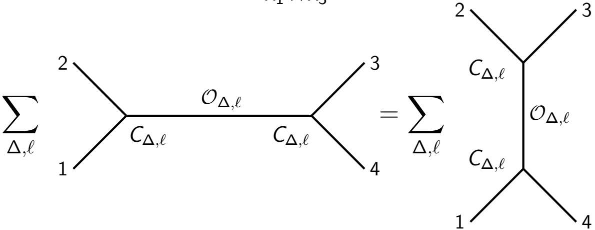
Identity operator Conformal primaries Conformal blocks

- Conformal blocks: For a given primary, take into account the contribution from all its descendants. Fully fixed function!

Conformal bootstrap

Crossing symmetry

$$v^{\Delta_{\mathcal{O}}}\mathcal{G}(u,v) \underbrace{=}_{x_1\leftrightarrow x_3} u^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u)$$

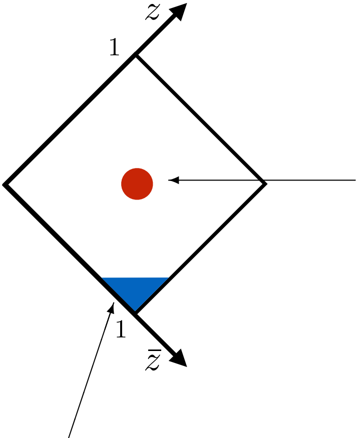


A remarkable...but hard equation!

$$\underbrace{v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell}C^2_{\Delta,\ell}G_{\Delta,\ell}(u,v)\right)}_{\text{Easy to expand around } u=0,\, v=1} = \underbrace{u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell}C^2_{\Delta,\ell}G_{\Delta,\ell}(v,u)\right)}_{\text{Easy to expand around } u=1,\, v=0}$$

Numerical vs Analytic bootstrap

Study this equation in different regions, $u = z\bar{z}, v = (1-z)(1-\bar{z})$



- In the Euclidean regime $\bar{z} = z^*$.
- We can study crossing around $u = v = \frac{1}{4}$
- Starting point of the numerical bootstrap.

- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$.
- Starting point of the analytic (light-cone) bootstrap!

Analytic bootstrap

Analytic bootstrap

- Why is this a good idea?

$$v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)\right) \quad = \quad u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)\right)$$

$$\text{Direct channel} \quad \Leftrightarrow \quad \text{Crossed channel}$$

- Very complicated interplay between l.h.s. and r.h.s. ... but:

Operators with large spin

DT operators with large spin	⇔	Identity operator
ST operators with large spin	⇔	ST operators with large spin

Conformal blocks - technicalities

- Eigenfunctions of a Casimir operator

$$\mathcal{C}\, G_{\Delta,\ell}(u,v) = J^2 G_{\Delta,\ell}(u,v)$$

  where $J^2 = (\ell+\Delta)(\ell+\Delta-1) \sim \ell^2$

- Small u limit:

$$G_{\Delta,\ell}(u,v) \sim u^{\tau/2} f^{coll}_{\tau,\ell}(v), \quad \tau = \Delta - \ell$$

We will introduce the notation

$$G_{\Delta,\ell}(u,v) \equiv u^{\tau/2} f_{\tau,\ell}(u,v)$$

- Small v limit:

$$f_{\tau,\ell}(u,v) \sim \log v$$

Necessity of a large spin sector

- Consider the $v \ll 1$ limit of the crossing equation: $C^2_{\Delta,\ell} \to a_{\tau,\ell}$

$$v^{\Delta_{\mathcal{O}}} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \right) = u^{\Delta_{\mathcal{O}}} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v,u) \right)$$

$$\Downarrow$$

$$1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \underbrace{\text{subleading terms}}_{\text{rest of operators sorted by twist}}$$

- The r.h.s. is divergent as $v \to 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u,v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_{\mathcal{O}}$ (actually $\tau_n = 2\Delta_{\mathcal{O}} + 2n$)

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Example: Generalised free fields

- Simplest solution: Large N CFTs - Generalised free fields

$$\boxed{\mathcal{G}^{(0)}(u,v) = 1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}}}$$

- Intermediate ops: Double twist operators: $\mathcal{O}\Box^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}$

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n \\ a_{n,\ell} &= a^{(0)}_{n,\ell} \end{aligned}$$

- Their OPE coefficients are such that the divergence of a single conformal block ($\sim \log v$) , as $v \to 0$, is enhanced!

$$1 + \sum_{\tau,\ell} a^{(0)}_{\tau_n,\ell} u^{\tau_n/2} f_{\tau_n,\ell}(u,v) = 1 + \underbrace{\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}}_{\uparrow} + u^{\Delta_{\mathcal{O}}}$$

But this divergence is quite universal!

Large spin sector

Additivity property [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov; F.A. ,Maldacena]

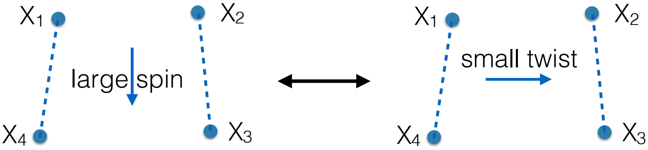
In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double twist operators with arbitrarily large spin and

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \mathcal{O}\left(\frac{1}{\ell}\right) \\ a_{n,\ell} &= a^{(0)}_{n,\ell}\left(1 + \mathcal{O}\left(\frac{1}{\ell}\right)\right) \end{aligned}$$

- All CFTs have a large spin sector, for which the operators become "free"!
- Can we do perturbations around large spin? YES!

What's going on?

- In Minkowski space we can have $x_{23}^2 \to 0, x_{23} \neq 0$.
- When some operators become null-separated the correlator develops singularities.

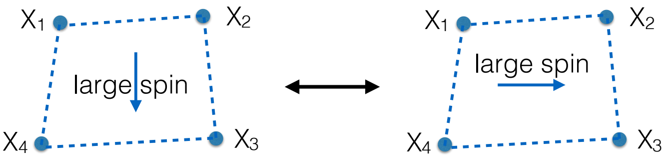


Dominated by high spin operators $\Leftrightarrow$ Dominated by low twist operators

(Double-trace) $(1, T_{\mu,\nu}, \cdots)$

Single trace operators

- We can also consider a sequential null limit $x_{i,i+12}^2 \to 0$.



ST with large spin $\Leftrightarrow$ ST with large spin

Large spin perturbation theory

- We would like to exploit the following idea

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \quad = \quad \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \cdots$$

Behaviour at large spin ⇔ Enhanced divergences as $v \to 0$

- The identity on the r.h.s. led to a remarkable result!
- Let's take this to the next level! Solve the following problem:

Given $Sing(u,v)$, find $a_{\tau,\ell}$ such that

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = Sing(u,v)$$

$$\uparrow$$

$$v^{-\Delta}, v^n \log^2 v, v^{1/2}, \cdots$$

Large spin perturbation theory

- Construct a basis of functions with specific enhanced singularities

$$
\begin{aligned}
H_\tau^{(0)}(u,v) &= \sum_\ell a_{\tau,\ell}^{(0)} u^{\tau/2} f_{\tau,\ell}(u,v) \sim \frac{u^{\Delta_\mathcal{O}}}{v^{\Delta_\mathcal{O}}} \\
H_\tau^{(1)}(u,v) &= \sum_\ell \frac{a_{\tau,\ell}^{(0)}}{J^2} u^{\tau/2} f_{\tau,\ell}(u,v) \\
H_\tau^{(2)}(u,v) &= \sum_\ell \frac{a_{\tau,\ell}^{(0)}}{J^4} u^{\tau/2} f_{\tau,\ell}(u,v) \\
&\vdots
\end{aligned}
$$

$$\mathcal{C} H_\tau^{(m+1)}(u,v) = H_\tau^{(m)}(u,v) \rightarrow \text{We can compute them!}$$

- Let's go back to our problem!

Large spin perturbation theory

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = Sing(u,v)$$

- Write $Sing(u,v)$ in our basis

$$\alpha_0 H_\tau^{(0)}(u,v) + \alpha_1 H_\tau^{(1)}(u,v) + \cdots = Sing(u,v)$$

- Then we have solved our problem!

$$a_{\tau,\ell} = a_{\tau,\ell}^{(0)}\left(\alpha_0 + \frac{\alpha_1}{J^2} + \cdots\right)$$

- Fixes the CFT data to all orders. Actually the series can be resummed and extrapolated to small spin!
- Allows to reconstruct the whole correlator!
- Note: we are assuming the CFT-data is analytic in the spin.

Inversion formula

- The series can be repackaged in a beautiful inversion formula [Caron-Huot]

$$a_{\tau,\ell} \sim \int dudv\, K(u,v,\tau,\ell)\, Sing(u,v)$$

- Which is explicitly analytic in the spin!
- It can be extrapolated down to $spin = 2$ [As a consequence of the Regge behaviour!]

Wider perspective on CFT

- Large spin perturbation theory allows to reconstruct the CFT-data from the enhanced singularities, but... the structure of singularities can be extremely complicated!

If two operators $\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}$ of twists τ_1 and τ_2 are part of the spectrum then there is a tower of operators $[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell}$ of twist

$$\tau_{[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell}} = \tau_1 + \tau_2 + 2n + \mathcal{O}\left(\frac{1}{\ell}\right)$$

- This should make you happy and sad at the same time!

The spectrum of generic CFTs is hard!

- ▷ $\mathcal{O}$ is part of the spectrum.
- ▷ $[\mathcal{O}, \mathcal{O}]_{n,\ell}$ is also part of the spectrum.
- ▷ And $[[\mathcal{O}, \mathcal{O}]_{n_1,\ell_1}, [\mathcal{O}, \mathcal{O}]_{n_2,\ell_2}]_{n_3,\ell_3}$ too, and so on!

Large spin expansions for non-perturbative CFTs

In non-perturbative CFTs the spectrum is very rich. Hard (but not impossible!) to apply our idea

$$\begin{aligned}\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \quad &= \quad \text{Rich spectrum in the crossed channel} \\ \text{Behaviour at large spin} \quad &\Leftrightarrow \quad \text{complicated divergences as } v \to 0\end{aligned}$$

- If the CFT has a small parameter we are better of, as this parameter further organises the problem.

LSPT: Strategy

Strategy

1. Use crossing symmetry to determine the enhanced singularities

$$\mathcal{G}(u,v) \leftarrow \mathcal{G}(u,v)|_{en.sing.} = \left(\frac{u}{v}\right)^{\Delta_0} \mathcal{G}(v,u)\Big|_{en.sing.}$$

   In theories with small parameters the latter follows from CFT-data at lower orders! (maybe including other correlators)

2. Then use LSPT to reconstruct the CFT-data from the enhanced singularities.
3. Go to next order and repeat.

This can be turned into an efficient machinery!

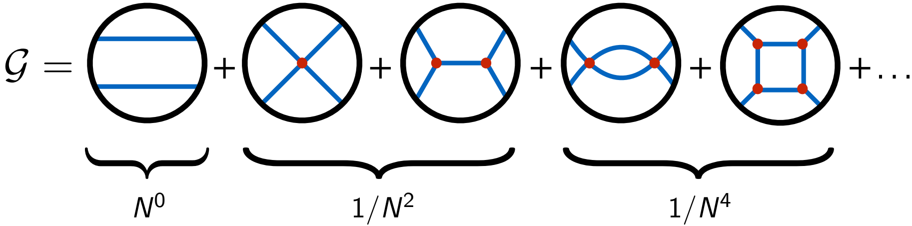
- ▷ Let's apply it to find $1/N$ corrections to GFF!

Large N CFTs

AdS/CFT

Large N CFT in D-dimensions (GFF + corrections) $\Leftrightarrow$ Gravitational theory in AdS_{D+1}

$\frac{1}{N^2}$ expansion in CFT $\leftrightarrow$ loops in AdS/powers of G_N.



- Diagrams in AdS are hard to compute...Use crossing for the CFT!

Large N holographic CFTs

- Let us compute $1/N$ corrections to large N CFTs/GFF!

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\mathcal{G}^{(1)}(u,v) + \cdots$$

Two Sources of corrections

1. Double twist operators will acquire corrections:

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma_{n,\ell} + \cdots \\ a_{n,\ell} &= a^{(0)}_{n,\ell} + \frac{1}{N^2}a^{(1)}_{n,\ell} + \cdots \end{aligned}$$

2. We can also have new intermediate operators at order $1/N^2$.

Which corrections are consistent with crossing symmetry?

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u)$$

Large N holographic CFTs

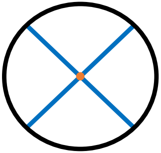
Case 1: No new operators at order $1/N^2$

- Double-trace operators don't produce enhanced divergences!

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \underbrace{\mathcal{G}^{(1)}(v,u)}_{f_{DT}(v,u)\sim v^{\Delta\mathcal{O}+n}}$$

- $\gamma_{n,\ell}, a^{(1)}_{n,\ell}$ vanish to all orders in $1/\ell$!
- On the other hand, we can have truncated solutions in the spin.
- Truncated solutions ↔ local interactions in the bulk.

$$\mathcal{G}^{(1)}_{trunc}(u,v) \sim$$



Large N holographic CFTs

Case 2: New single-trace operators at order $1/N^2$, *e.g.* $\mathcal{O}$ itself:

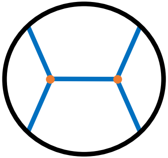
$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O}, \mathcal{O}]_{n,\ell} + \frac{1}{N^2}\mathcal{O}$$

- Now the situation is more interesting:

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u) \supset \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} v^{\Delta_{\mathcal{O}}/2} f_{\mathcal{O}}(v,u)$$

- The enhanced divergences fixes $\gamma_{n,\ell}, a^{(1)}_{n,\ell}$ to all orders in $1/\ell$!
- Non-truncated solutions correspond to *AdS* exchanges

$$\mathcal{G}^{(1)}_{non-tr}(u,v) \;\; \sim$$



Going to higher orders...

- Loops in *AdS* are a largely unexplored subject.
- We can approach this by studying $1/N^4$ corrections to GFF!

$$\tau_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma_{n,\ell}^{(1)} + \frac{1}{N^4}\gamma_{n,\ell}^{(2)} + \cdots$$

- New double-trace operators also produce enhanced-singularities

$$\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}(v,u) \sim \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} a_\ell v^{\Delta_{\mathcal{O}}+\frac{1}{N^2}\frac{\gamma^{(1)}}{2}+\cdots} f_\ell(v,u) \sim \frac{(\gamma^{(1)})^2}{N^4}\log^2 v$$

- This can be computed from the OPE data to previous order, and then we can find $\gamma_{n,\ell}^{(2)}$!

Twist four, spin two operator for $\mathcal{N}=4$ SYM

$$\Delta_{0,2} = 6 - \frac{4}{N^2} - \frac{45}{N^4} + \cdots$$

Large N Chern-Simons Vector Models [with Aharony, Bissi, Yacoby]

Models with HS symmetry at large N!

Spectrum at large N

- Scalar operator $J^{(0)}$ of dimension 1 or 2.
- Tower of HS conserved currents $J^{(s)}$, $s = 1, 2, \cdots$ of twist 1.
- Multitrace operators $[J^{(s)}, J^{(s')}]_{n,\ell}$.

We study crossing constraints on the correlator

$$\langle J^{(0)} J^{(0)} J^{(0)} J^{(0)} \rangle$$

- To zero order order only double trace operators $[\sigma, \sigma]_{n,\ell}$
- To order $1/N$ all the currents $J^{(s)}$ appear! their OPE is fixed by crossing.
- $\langle J^{(0)} J^{(0)} J^{(s)} \rangle$ agrees with Maldacena and Zhiboedov!
- Many new results to order $1/N^2$!

Conclusions

- Generic CFTs have a large spin sector which becomes essentially free and we have shown how to perform a perturbation around that sector.
- The method applies to vast families of CFTs and is based on symmetries and consistency conditions.
- Similar ideas guided Veneziano 50 years ago!
- Thrilling to see what happens to all this in the next 50 years!