Preliminary Considerations on Hadronic Contributions to μ-e Scattering at NLO

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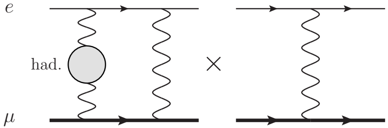
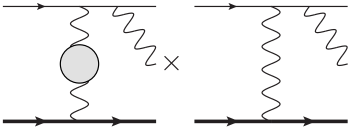
Padova, 4 - 5 September 2017

Muon-electron scattering: theory kickoff workshop

in collaboration with M. Passera

How to calculate these diagrams of $O(\alpha^4)$?





Dispersion Relation

Factorizable and non-factorizable terms

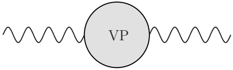
$$\Pi_{\rm had}(q^2) = \frac{q^2}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{{\rm Im}\Pi_{\rm had}(s)}{s(s - q^2 - i\varepsilon)}$$

$$\frac{d\sigma}{dt} = \frac{d\sigma^0}{dt} \left| \frac{\alpha(t)}{\alpha(0)} \right|^2$$

see also:

Kühn, Uccirati, NPB 806 (2009) 300.

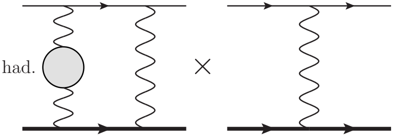
Jegerlehner, The Anomalous Magnetic Moment of the Muon, 2007 & 2017



$$i\,\Pi^{\mu\nu}(q^2) = i\,\Pi(q^2)\,(g^{\mu\nu}q^2 - q^\mu q^\nu)$$
$$= \int d^4x\, e^{iqx}\,\langle 0|\, T\{j^\mu(x)\,j^\nu(0)\}\,|0\rangle$$

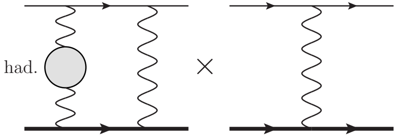
$$\Pi^{\mathrm{Ren}}_{\mathrm{had}}(q^2) = -\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\,\frac{\mathrm{Im}\Pi_{\mathrm{had}}(z)}{q^2 - z + i\varepsilon}$$

$$\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \to \frac{-ig^{\mu\alpha}}{q^2 + i\varepsilon} i\,\Pi_{\mathrm{had}}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta)\frac{-ig^{\beta\nu}}{q^2 + i\varepsilon}$$



$$\frac{-ig^{\mu\nu}}{q^2+i\varepsilon} \to \frac{-ig^{\mu\alpha}}{q^2+i\varepsilon} i\,\Pi_{\rm had}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta)\frac{-ig^{\beta\nu}}{q^2+i\varepsilon}$$

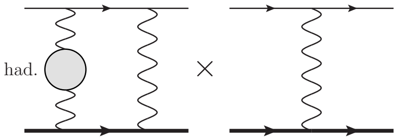
$$\to \frac{-ig^{\mu\nu}}{q^2+i\varepsilon}\Pi_{\rm had}(q^2)$$



$$\frac{-ig^{\mu\nu}}{q^2+i\varepsilon} \to \frac{-ig^{\mu\alpha}}{q^2+i\varepsilon} i\,\Pi_{\rm had}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta)\frac{-ig^{\beta\nu}}{q^2+i\varepsilon}$$

$$\to \frac{-ig^{\mu\nu}}{q^2+i\varepsilon}\Pi_{\rm had}(q^2)$$

$$\to -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{dz}{z}\, {\rm Im}\Pi_{\rm had}(z)\, \left[\frac{-ig^{\mu\nu}}{q^2 - z + i\varepsilon}\right]$$

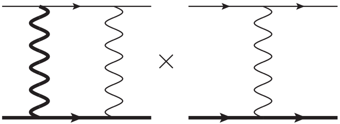


$$\frac{-ig^{\mu\nu}}{q^2+i\varepsilon} \to \frac{-ig^{\mu\alpha}}{q^2+i\varepsilon} i\,\Pi_{\rm had}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta)\frac{-ig^{\beta\nu}}{q^2+i\varepsilon}$$

$$\to \frac{-ig^{\mu\nu}}{q^2+i\varepsilon}\Pi_{\rm had}(q^2)$$

$$\to -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\,{\rm Im}\Pi_{\rm had}(z)\,\left[\frac{-ig^{\mu\nu}}{q^2-z+i\varepsilon}\right]$$

$$-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz}{z}\,{\rm Im}\Pi^{\rm had}(z)$$



$$-\mathrm{Im}\Pi_{\mathrm{had}}(s) = \frac{\alpha}{3} R_{\mathrm{had}}(s) = \frac{\alpha s}{4\pi|\alpha(s)|^2}\sigma_{\mathrm{tot}}(e^+e^- \to \gamma^\star \to \mathrm{anything})$$

$$R_{\mathrm{had}}(s) = \sigma_{\mathrm{tot}} / \frac{4\pi|\alpha(s)|^2}{3s}$$

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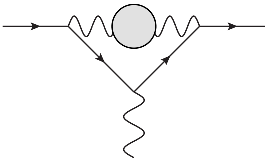
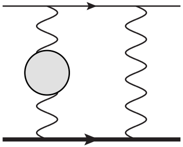
- Fortran package `alphaQED` by F. Jegerlehner
  www-com.physik.hu-berlin.de/fjeger/alphaQED16.tar.gz

  ```
  complex(8) function cggvap(q2,error)
  ```

- Hagiwara, Teubner et al., J. Phys. G38 (2011) 085003

Building blocks



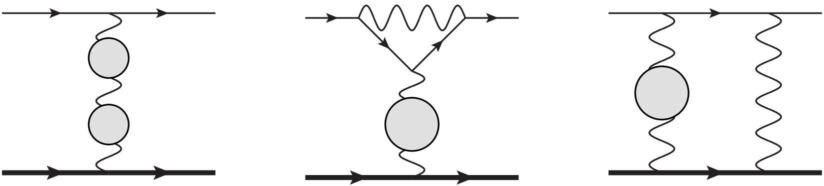


Kühn, Uccirati NPB 806 (2009) 300

Numerical issue

- Change of variable $z \to y = \frac{4m_\pi^2}{z}$, $0 < y < 1$.
- Limit $y \to 0$.
- Narrow resonances.
- Check with QED NNLO: $\Pi_{\rm had} \to \Pi_{\rm qed}^{1-\rm loop}$.

Factorizable and non-factorizable terms



$$\frac{d\sigma}{dt} \propto \alpha^2 \, \mathrm{Born}$$

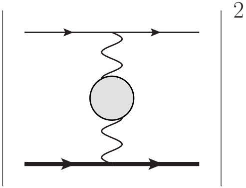
$$\frac{d\sigma}{dt} \propto \alpha^2(t)\,\mathrm{Born} \qquad\qquad \alpha(t) = \frac{\alpha}{1 - \Pi(t)}$$

$$\Pi(q^2) = \alpha\,\Pi_{\mathrm{had}} + \alpha\,\Pi_{\mathrm{qed1}} + \alpha^2\,\Pi_{\mathrm{qed2}} + \ldots$$

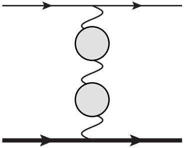
$$\frac{d\sigma}{dt} \propto \alpha^2(t)\,\mathrm{Born} \qquad \alpha(t) = \frac{\alpha}{1 - \Pi(t)}$$

$$\Pi(q^2) = \alpha\,\Pi_{\mathrm{had}} + \alpha\,\Pi_{\mathrm{qed1}} + \alpha^2\,\Pi_{\mathrm{qed2}} + \ldots$$

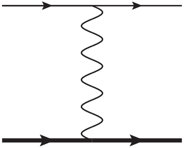
$$\left.\frac{d\sigma}{dt}\right|_{O(\alpha^4)} \propto \alpha^4 \left[3\Pi_{\mathrm{had}}^2 + 6\Pi_{\mathrm{had}}\Pi_{\mathrm{qed1}} + 3\Pi_{\mathrm{qed1}}^2 + 2\Pi_{\mathrm{qed2}}\right] \mathrm{Born}$$

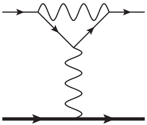
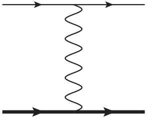


$+ 2\,\mathrm{Re}$

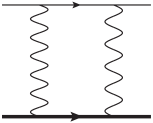


$\times$

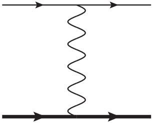


$2\,\mathrm{Re}$  $\times$  $= \alpha(t)^2 \alpha\,\mathrm{vert}$

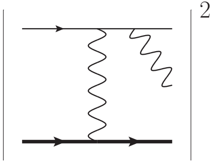
$2\,\mathrm{Re}$



$\times$



$$= \alpha\,\alpha(t)^2\,\mathrm{box}_{\mathrm{IR}} + \alpha^2\,\alpha(t)\,\mathrm{box}_{\mathrm{fin}}$$



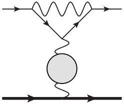
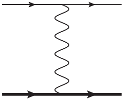
$$= \alpha\,\alpha(t)^2\,\mathrm{soft} + \alpha^3\,\mathrm{hard}$$

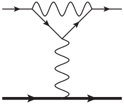
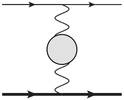
$$\frac{d\sigma}{dt} \propto \alpha^2 \left[\mathrm{Born} + \alpha(\mathrm{virt} + \mathrm{real})\right]$$

$$\frac{d\sigma}{dt} \propto \alpha^2(t) \left[\mathrm{Born} + \alpha(\mathrm{vert} + \mathrm{box}_{\mathrm{IR}} + \mathrm{soft})\right]$$
$$+\alpha^2\alpha(t)\,\mathrm{box}_{\mathrm{fin}} + \ldots$$

$$\left.\frac{d\sigma}{dt}\right|_{O(\alpha^4)} \propto \left[3\Pi^2_{\mathrm{had}} + 6\Pi_{\mathrm{had}}\Pi_{\mathrm{qed1}} + 3\Pi^2_{\mathrm{qed1}} + 2\Pi_{\mathrm{qed2}}\right] \mathrm{Born}$$
$$+ 2\,\Pi_{\mathrm{had}}\left[\mathrm{vert} + \mathrm{box}_{\mathrm{IR}} + \mathrm{soft}\right] + \Pi_{\mathrm{had}}\,\mathrm{box}_{\mathrm{fin}}$$
$$+ \text{non factorizable terms}$$

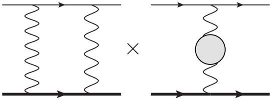
$2\,\Pi_{\mathrm{had}} \times \mathrm{vert}$

$2\,\mathrm{Re}$  $\times$ 

$2\,\mathrm{Re}$  $\times$ 

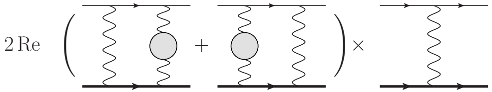
$$2\,\Pi_{\text{had}} \times \text{box}_{\text{IR}}$$

$2\,\text{Re}$



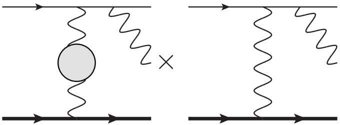
$$= \Pi_{\text{had}}(t)\Big[\text{box}_{\text{IR}} + \text{box}_{\text{fin}}\Big]$$

$$2\,\Pi_{\mathrm{had}} \times \mathrm{box}_{\mathrm{IR}}$$



$$= \Pi_{\mathrm{had}}(t)\,\mathrm{box}_{\mathrm{IR}} + \int \frac{dz}{z}\,\mathrm{Im}\Pi_{\mathrm{had}}(z)\,\mathrm{box}_{\mathrm{fin}}(z)$$

$2\,\Pi_{\rm had} \times \textit{soft}$

$2\,\mathrm{Re}$ 

$$= 2\,\Pi_{\rm had}(t)\,\mathrm{soft} + \int \frac{dz}{z}\,\mathrm{Im}\Pi_{\rm had}(z)\,\mathrm{hard}(z)$$

Conclusions

- Dispersion relation of vacuum polarization.
- Two building blocks: the vertex and the boxes.
- Numerical evaluation.
- Can we reabsorb factorizable terms into $\alpha(t)$?
- External data input – read $R(s)$ – only for non-factorizable terms?