

Preliminary Considerations on Hadronic Contributions to μ - e Scattering at NLO

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Muon-electron scattering: theory kickoff workshop

in collaboration with M. Passera

How to calculate these diagrams of $O(\alpha^4)$?



Dispersion Relation

$$\Pi_{\text{had}}(q^2) = \frac{q^2}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\text{Im}\Pi_{\text{had}}(s)}{s(s - q^2 - i\varepsilon)}$$

Factorizable and non-factorizable terms

$$\frac{d\sigma}{dt} = \frac{d\sigma^0}{dt} \left| \frac{\alpha(t)}{\alpha(0)} \right|^2$$

see also:

Kühn, Uccirati, NPB 806 (2009) 300.

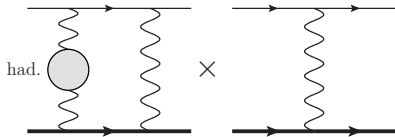
Jegerlehner, The Anomalous Magnetic Moment of the Muon, 2007 & 2017



$$\begin{aligned}
 i\Pi^{\mu\nu}(q^2) &= i\Pi(q^2)(g^{\mu\nu}q^2 - q^\mu q^\nu) \\
 &= \int d^4x e^{iqx} \langle 0| T\{j^\mu(x)j^\nu(0)\}|0\rangle
 \end{aligned}$$

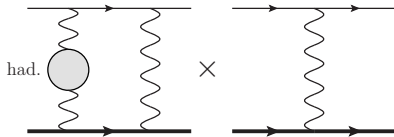
$$\Pi_{\text{had}}^{\text{Ren}}(q^2) = -\frac{q^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z} \frac{\text{Im}\Pi_{\text{had}}(z)}{q^2 - z + i\epsilon}$$

$$\frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \rightarrow \frac{-ig^{\mu\alpha}}{q^2 + i\epsilon} i\Pi_{\text{had}}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta) \frac{-ig^{\beta\nu}}{q^2 + i\epsilon}$$

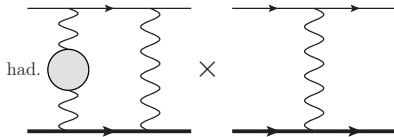


$$\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \rightarrow \frac{-ig^{\mu\alpha}}{q^2 + i\varepsilon} i\Pi_{\text{had}}(q^2)(q^2 g_{\alpha\beta} - q_\alpha q_\beta) \frac{-ig^{\beta\nu}}{q^2 + i\varepsilon}$$

$$\rightarrow \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \Pi_{\text{had}}(q^2)$$

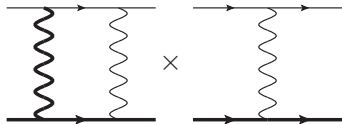


$$\begin{aligned}
\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} &\rightarrow \frac{-ig^{\mu\alpha}}{q^2 + i\varepsilon} i \Pi_{\text{had}}(q^2) (q^2 g_{\alpha\beta} - q_\alpha q_\beta) \frac{-ig^{\beta\nu}}{q^2 + i\varepsilon} \\
&\rightarrow \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \Pi_{\text{had}}(q^2) \\
&\rightarrow -\frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z} \text{Im} \Pi_{\text{had}}(z) \left[\frac{-ig^{\mu\nu}}{q^2 - z + i\varepsilon} \right]
\end{aligned}$$



$$\begin{aligned}
\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} &\rightarrow \frac{-ig^{\mu\alpha}}{q^2 + i\varepsilon} i \Pi_{\text{had}}(q^2) (q^2 g_{\alpha\beta} - q_\alpha q_\beta) \frac{-ig^{\beta\nu}}{q^2 + i\varepsilon} \\
&\rightarrow \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \Pi_{\text{had}}(q^2) \\
&\rightarrow -\frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z} \text{Im} \Pi_{\text{had}}(z) \left[\frac{-ig^{\mu\nu}}{q^2 - z + i\varepsilon} \right]
\end{aligned}$$

$$-\frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z} \text{Im} \Pi^{\text{had}}(z)$$



$$-\text{Im}\Pi_{\text{had}}(s) = \frac{\alpha}{3} R_{\text{had}}(s) = \frac{\alpha s}{4\pi|\alpha(s)|^2} \sigma_{\text{tot}}(e^+ e^- \rightarrow \gamma^* \rightarrow \text{anything})$$

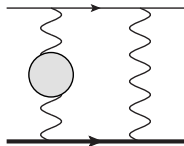
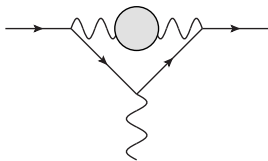
$$R_{\text{had}}(s) = \sigma_{\text{tot}} / \frac{4\pi|\alpha(s)|^2}{3s}$$

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$$R_{\text{had}}(s) = \sigma_{\text{tot}} / \frac{4\pi|\alpha(s)|^2}{3s}$$

- Fortran package alphaQED by F. Jegerlehner
www-com.physik.hu-berlin.de/fjeger/alphaQED16.tar.gz
`complex(8) function cggvap(q2,error)`
- Hagiwara, Teubner et al., J. Phys. G38 (2011) 085003

Building blocks

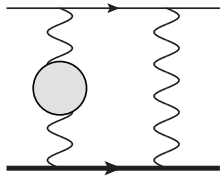
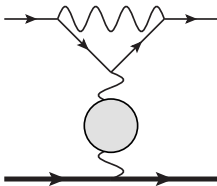
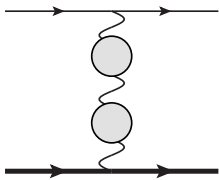


Kühn, Uccirati NPB 806 (2009) 300

Numerical issue

- Change of variable $z \rightarrow y = \frac{4m_\pi^2}{z}$, $0 < y < 1$.
- Limit $y \rightarrow 0$.
- Narrow resonances.
- Check with QED NNLO: $\Pi_{\text{had}} \rightarrow \Pi_{\text{qed}}^{1\text{-loop}}$.

Factorizable and non-factorizable terms



$$\frac{d\sigma}{dt} \propto \alpha^2 \text{ Born}$$

$$\frac{d\sigma}{dt} \propto \alpha^2(t) \text{ Born}$$

$$\alpha(t) = \frac{\alpha}{1 - \Pi(t)}$$

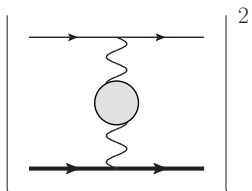
$$\Pi(q^2) = \alpha \Pi_{\text{had}} + \alpha \Pi_{\text{qed1}} + \alpha^2 \Pi_{\text{qed2}} + \dots$$

$$\frac{d\sigma}{dt} \propto \alpha^2(t) \text{ Born}$$

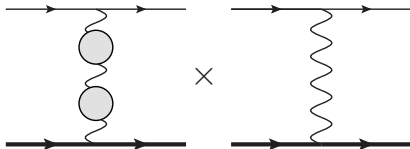
$$\alpha(t) = \frac{\alpha}{1 - \Pi(t)}$$

$$\Pi(q^2) = \alpha \Pi_{\text{had}} + \alpha \Pi_{\text{qed1}} + \alpha^2 \Pi_{\text{qed2}} + \dots$$

$$\left. \frac{d\sigma}{dt} \right|_{O(\alpha^4)} \propto \alpha^4 [3\Pi_{\text{had}}^2 + 6\Pi_{\text{had}}\Pi_{\text{qed1}} + 3\Pi_{\text{qed1}}^2 + 2\Pi_{\text{qed2}}] \text{ Born}$$

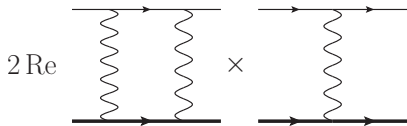


+ 2 Re

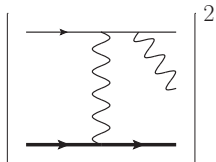


2Re
 $= \alpha(t)^2 \alpha \text{ vert}$

The diagram shows two Feynman diagrams separated by a multiplication sign (\times). The left diagram is a loop diagram with a thick horizontal line at the bottom and two thin horizontal lines at the top. A wavy line connects the bottom line to a vertex on the left thin line. From this vertex, a wavy line goes to a vertex on the right thin line. From the right thin line, a wavy line goes to a vertex on the left thin line. From this vertex, a wavy line goes to a vertex on the bottom line. The right diagram is a tree-level diagram with a thick horizontal line at the bottom and two thin horizontal lines at the top. A wavy line connects the bottom line to a vertex on the left thin line. From this vertex, a wavy line goes to a vertex on the right thin line.



$$= \alpha \alpha(t)^2 \text{box}_{\text{IR}} + \alpha^2 \alpha(t) \text{box}_{\text{fin}}$$



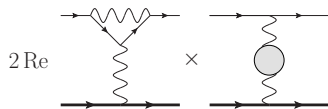
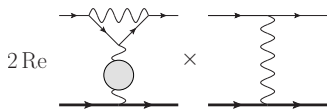
$$= \alpha \alpha(t)^2 \text{soft} + \alpha^3 \text{hard}$$

$$\frac{d\sigma}{dt} \propto \alpha^2 \left[\text{Born} + \alpha(\text{virt} + \text{real}) \right]$$

$$\frac{d\sigma}{dt} \propto \alpha^2(t) \left[\text{Born} + \alpha(\text{vert} + \text{box}_{\text{IR}} + \text{soft}) \right] \\ + \alpha^2 \alpha(t) \text{box}_{\text{fin}} + \dots$$

$$\left. \frac{d\sigma}{dt} \right|_{O(\alpha^4)} \propto [3\Pi_{\text{had}}^2 + 6\Pi_{\text{had}}\Pi_{\text{qed1}} + 3\Pi_{\text{qed1}}^2 + 2\Pi_{\text{qed2}}] \text{Born} \\ + 2\Pi_{\text{had}} [\text{vert} + \text{box}_{\text{IR}} + \text{soft}] + \Pi_{\text{had}} \text{box}_{\text{fin}} \\ + \text{non factorizable terms}$$

$2\Pi_{\text{had}} \times \text{vert}$



$$2 \Pi_{\text{had}} \times \text{box}_{\text{IR}}$$

$$2 \text{Re} \left[\text{Diagram 1} \times \text{Diagram 2} \right] = \Pi_{\text{had}}(t) \left[\text{box}_{\text{IR}} + \text{box}_{\text{fin}} \right]$$

The diagram shows two Feynman diagrams representing hadronic vacuum polarization. The first diagram is a photon loop with a fermion line. The second diagram is a photon loop with a fermion loop (a bubble). The diagrams are multiplied together, and the result is equated to the imaginary part of the hadronic vacuum polarization function, $\Pi_{\text{had}}(t)$, multiplied by the sum of the infrared and finite parts of the box diagram, $\text{box}_{\text{IR}} + \text{box}_{\text{fin}}$.

$$2\Pi_{\text{had}} \times \text{box}_{\text{IR}}$$

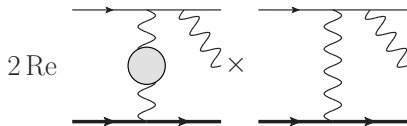
Diagrammatic representation of the equation: $2\text{Re} \left(\text{hadronic diagrams} \right) \times \text{box}_{\text{IR}}$

The first part shows the real part of the hadronic vacuum polarization function, Π_{had} , which is represented by two diagrams in parentheses. Each diagram shows a fermion loop (a grey circle) with a wavy line (representing a photon) attached to each vertex. The two diagrams correspond to the two possible ways of attaching the external photon lines to the fermion loop. The entire expression is multiplied by 2Re .

The second part is a box diagram, box_{IR} , consisting of two fermion lines forming a rectangle. A wavy line (representing a photon) connects the top and bottom vertices of the rectangle.

$$= \Pi_{\text{had}}(t) \text{box}_{\text{IR}} + \int \frac{dz}{z} \text{Im}\Pi_{\text{had}}(z) \text{box}_{\text{fin}}(z)$$

$$2 \Pi_{\text{had}} \times \text{soft}$$



$$= 2 \Pi_{\text{had}}(t) \text{soft} + \int \frac{dz}{z} \text{Im} \Pi_{\text{had}}(z) \text{hard}(z)$$

Conclusions

- Dispersion relation of vacuum polarization.
- Two building blocks: the vertex and the boxes.
- Numerical evaluation.
- Can we reabsorb factorizable terms into $\alpha(t)$?
- External data input – read $R(s)$ – only for non-factorizable terms?