# Towards mu-e Scattering @NNLO in QED: intro

Pierpaolo Mastrolia

*Muon-electron Scattering: Theory kickoff workshop*

*Padova, Sept. 4-5, 2017*

*In collaboration with:*

**- Passera, Peraro, Primo, Ossola, Schubert, Torres-Bobadilla**

Physics and Astronomy Department
*Galileo Galilei*
University of Padova - Italy

# Outline

- Motivations
- What we need and what we can provide
- Feynman Integrals in Dimensional Regularization
  - Integration-by-parts identities, Master Integrals & Differential Equations
  - Magnus Exponential Matrix and Canonical Forms
- Adaptive Integrand Decomposition
  - Improved reduction @ 1- and 2-loop
  - Automated two-loop corrections for generic processes
- Extra: on the scattering amplitudes involving massive particles
  - Algebra of Elliptic integrals from IBPs on the cuts
- Results
- Conclusions/Outlook

# Motivations

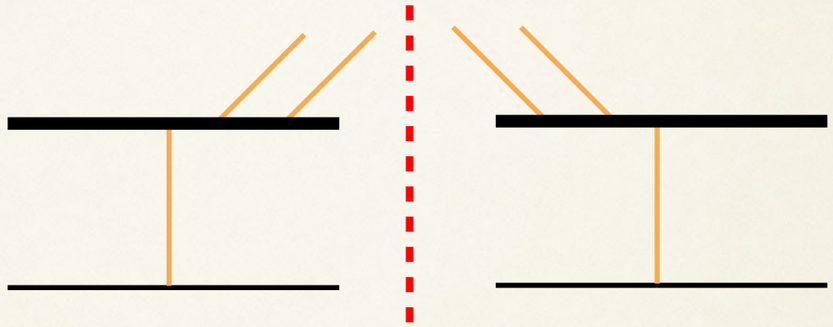
>> see **Venanzoni**'s talk

>> see **Marconi**'s talk

# What we need :: Anatomy of NNLO

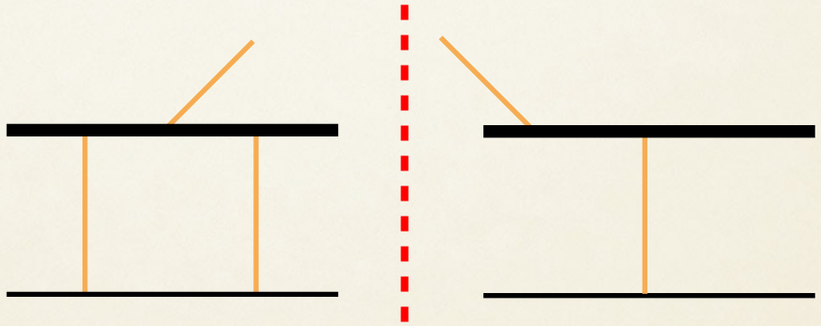
- **Double-real Radiation tree-level matrix elements**

  $\mathrm{d}\hat{\sigma}^{RR}_{NNLO}$



- **Single-real Radiation 1-loop matrix elements**

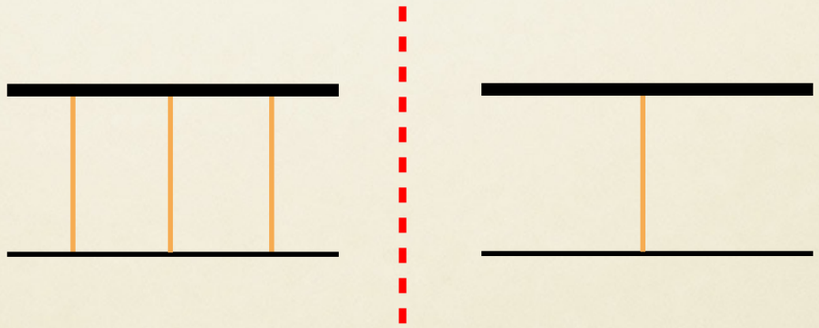
  $\mathrm{d}\hat{\sigma}^{RV}_{NNLO}$



>> see **Carloni-Calame, Fael** and **Ossola**'s talks

- **Virtual 2-loop matrix element**

  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$



>> see **this**, **Primo** and **Schubert**'s talks

# What we need :: Anatomy of NNLO

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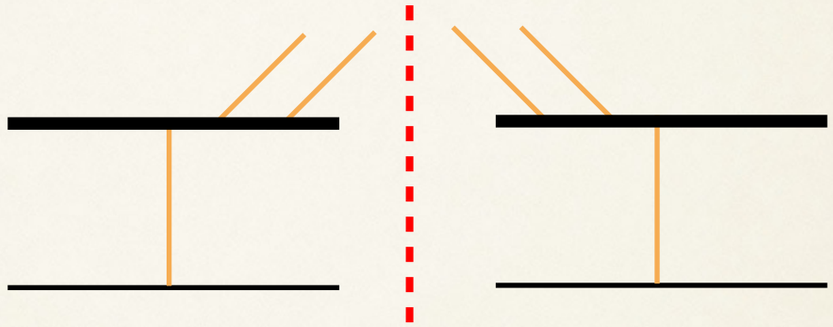

- **Single-real Radiation 1-loop matrix elements**

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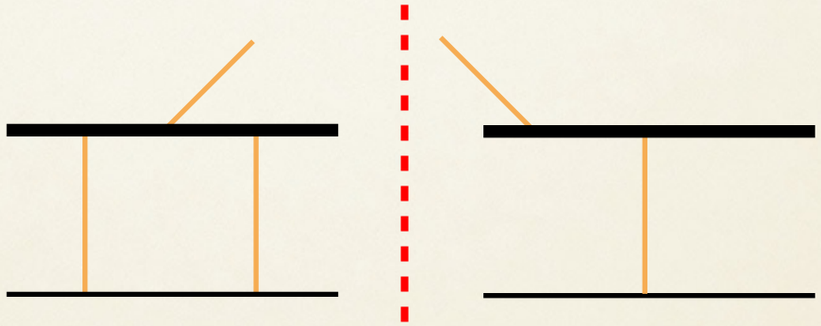
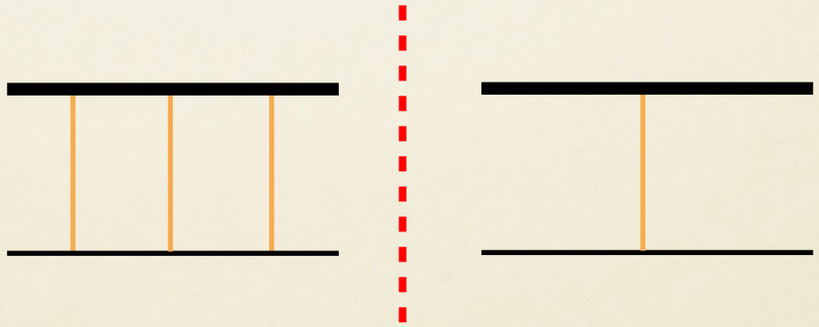


- **Virtual 2-loop matrix element**

  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$



>> see **Carloni-Calame, Fael** and **Ossola**'s talks

>> see **this, Primo** and **Schubert**'s talks

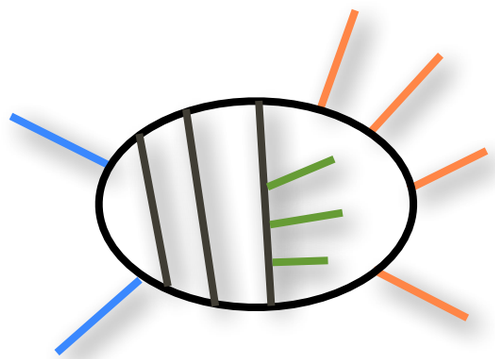
$$\mathrm{d}\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

- **Subtractions and MC-integration?**

>> left as exercise to Pavia's people :)

# Dimensionally Regulated Integrals

# Graph Topology & Integrals



$e =$ # legs :: $p_i, \quad (i = 1, \ldots, e);$

$\ell =$ # loops :: $q_i \quad (i = 1, \ldots, \ell);$

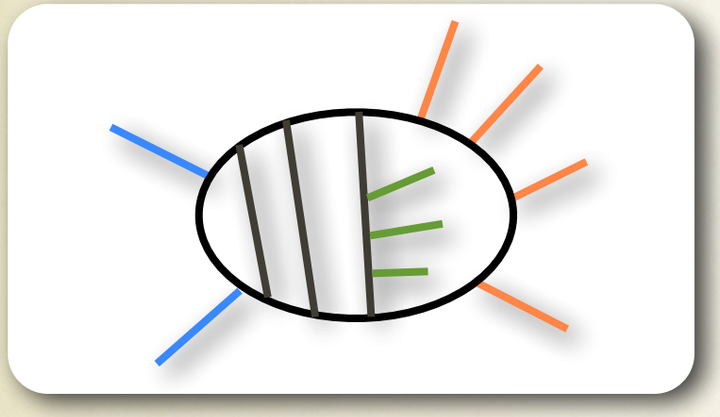
$n =$ # denominators :: $D_i \quad (i = 1, \ldots, n);$

$N =$ # scalar products (of types $q_i \cdot p_j$ and $q_i \cdot q_j$ ) $\qquad N = \ell(e-1) + \dfrac{\ell(\ell+1)}{2}$

$n =$ # **reducible** scalar products (expressed in terms of denominators);

$m =$ # **irreducible** scalar products $= N - n$ :: $S_i \quad (i = 1, \ldots, m)$

# Graph Topology & Integrals



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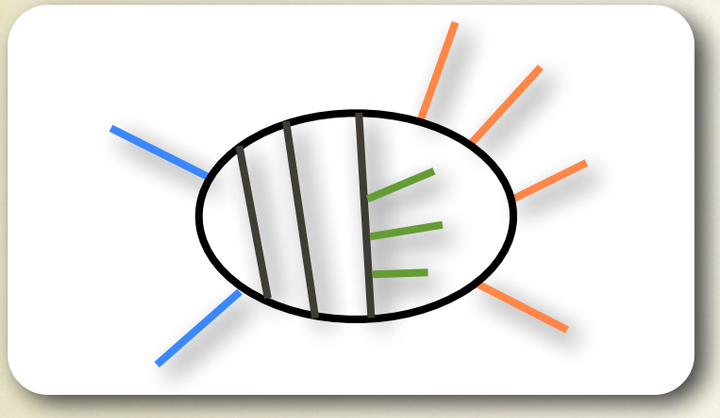

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**Associated Integrals** ::

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \ , \qquad \int_{q_1 \ldots q_\ell} \equiv \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d q_\ell}{(2\pi)^d}$$

$$f_{n,m}(\mathbf{x}, \mathbf{y}) = \frac{S_1^{y_1} \cdots S_m^{y_m}}{D_1^{x_1} \cdots D_n^{x_n}}$$

# Integration-by-parts Identities (IBPs)

Tkachov; Chetyrkin Tkachov; Laporta;

$$\int_{q_1 \ldots q_\ell} \frac{\partial}{\partial q_i^\mu} \Big( v^\mu \; f_{n,m}(\mathbf{x}, \mathbf{y}) \Big) = 0 \; , \qquad v = q_1, \ldots, q_\ell, \; p_1, \ldots, p_{e-1}.$$

$\forall (n,m)$, $N_{\rm IBP}$ = # of IBP relations $= \ell(\ell + e - 1)$

Relations between integrals associated to the same topology (or subtopologies)

$$c_0 \; F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) + \sum_{i,j} c_{i,j} \; F^{[d]}_{n,m}(\mathbf{x_i}, \mathbf{y_j}) = 0 \; ,$$

$$\mathbf{x_i} = \{x_1, \ldots, x_i \pm 1, \ldots, x_n\}$$

$$\mathbf{y_j} = \{y_1, \ldots, y_j \pm 1, \ldots y_n\}$$

public codes :: AIR; Reduze2; FIRE; LiteRed;
private codes :: ... many authors ... Laporta, Sturm ...

# Master Integrals (MIs)

Independent set of integrals $M_i^{[d]}$,

$$M_i^{[d]} \equiv \int_{q_1 \ldots q_\ell} m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \ ,$$

with a definite set of powers $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ such that

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \overset{\text{IBP}}{=} \sum_k c_k M_k^{[d]} \ , \qquad \forall (n,m)$$

They form a *basis* for the integrals of the corresponding topology.

## Two special cases

Two types of integrals generated from the **master integrands**

- Polynomial insertion: $$\int_{q_1 \ldots q_\ell} P(q_i \cdot p_j, q_i \cdot q_j) \ m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \alpha_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \overset{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$
- External-leg derivatives: $$p_i^\mu \frac{\partial}{\partial p_j^\mu} M_k^{[d]} = \int_{q_1 \ldots q_\ell} p_i^\mu \frac{\partial}{\partial p_j^\mu} \ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \beta_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \overset{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$

# Dimensional Recurrence Relations for MIs

Tarasov; Baikov; Lee

**Gram determinant**

$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \cdots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \cdots & p_{e-1}^2 \end{vmatrix}$$

**Dimension-shifted integrals**

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y})$$

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$$F^{[d]}_{n,m}(\mathbf{x},\mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x},\mathbf{y}) \qquad \Rightarrow \int_{q_1 \ldots q_\ell} \mathbf{G}\ f_{n,m}(\mathbf{x},\mathbf{y}) = \Omega(d, p_i)\ F^{[d+2]}_{n,m}(\mathbf{x},\mathbf{y})$$

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**G-insertion** generates shifted dim. integrals: **d --> d+2**

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In the case of **Master integrals**

$$M_k^{[d+2]} = \Omega(d, p_i)^{-1} \int_{q_1 \ldots q_\ell} \mathbf{G}\ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \overset{\text{IBP}}{=} \sum_i c_{k,i}\ M_i^{[d]}$$

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which can be seen as a **Dimensional recurrence relation**

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**Dimension-shifted integrals**

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which can be seen as a **Dimensional recurrence relation**

In general, $n$ MIs obey a **system of Dimensional recurrence relations**

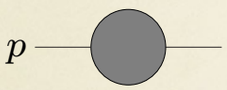
$$\mathbf{M}^{[d]} \equiv \begin{pmatrix} M_1^{[d]} \\ \vdots \\ M_n^{[d]} \end{pmatrix} \qquad \mathbf{M}^{[d+2]} = \mathbb{C}(d)\ \mathbf{M}^{[d]}$$

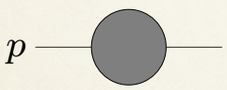
>> see **Laporta**'s talk

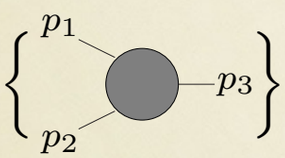
Recurrence on denominator powers

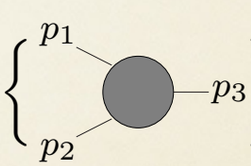
# Differential Equations for MIs

Kotikov; Remiddi;
Gehrmann Remiddi
Argeri Bonciani Ferroglia Remiddi **P.M.**
Weinzierl
……
Henn; Henn Smirnov
Lee; Papadopoulos;
Argeri diVita Mirabella Schlenk Schubert Tancredi **P.M.**
diVita Schubert Yundin **P.M.**
Remiddi Tancredi; Primo Tancredi
Papadopoulos Frellesvig
Zheng
……

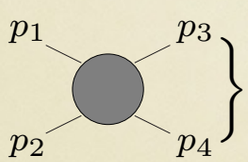
$p^2 \frac{\partial}{\partial p^2}$ 

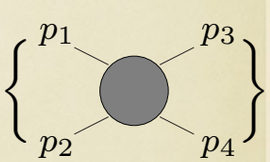
$= \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu}$ 


$P^2 \frac{\partial}{\partial P^2}$ 

$= \left[ A \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left( p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right]$ 


$$P = p_1 + p_2,$$

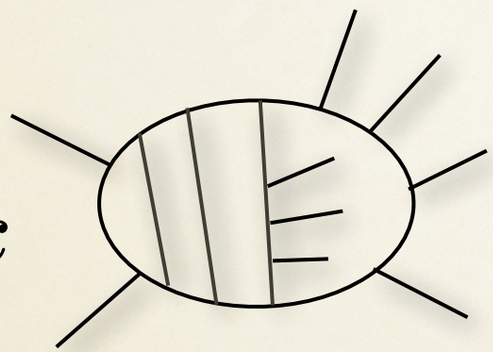
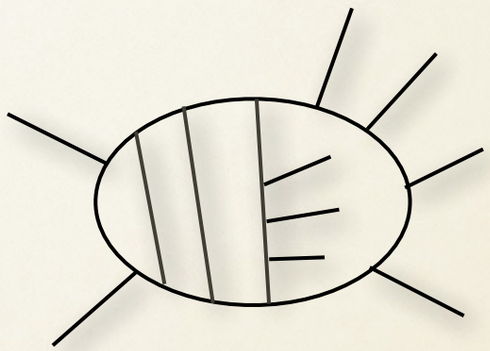
$P^2 \frac{\partial}{\partial P^2}$ 

$= \left[ C \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E(p_{1,\mu} + p_{3,\mu}) \left( \frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right]$ 


In general, $n$ MIs obey a **system of 1st ODE**

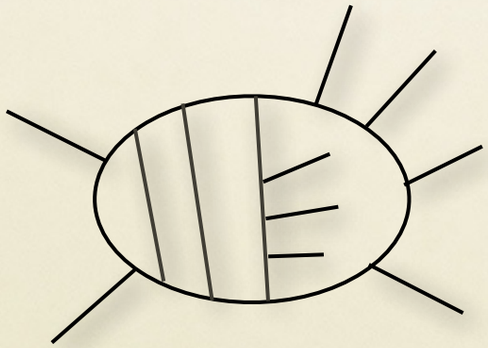
$$\partial_z \mathbf{M}^{[d]} = \mathbb{A}(d, z)\ \mathbf{M}^{[d]}$$

# Differential Equations for Master Integrals

$\partial_x$  $= A(d, x)$ 

kinematic variable
(s,t,u, masses)

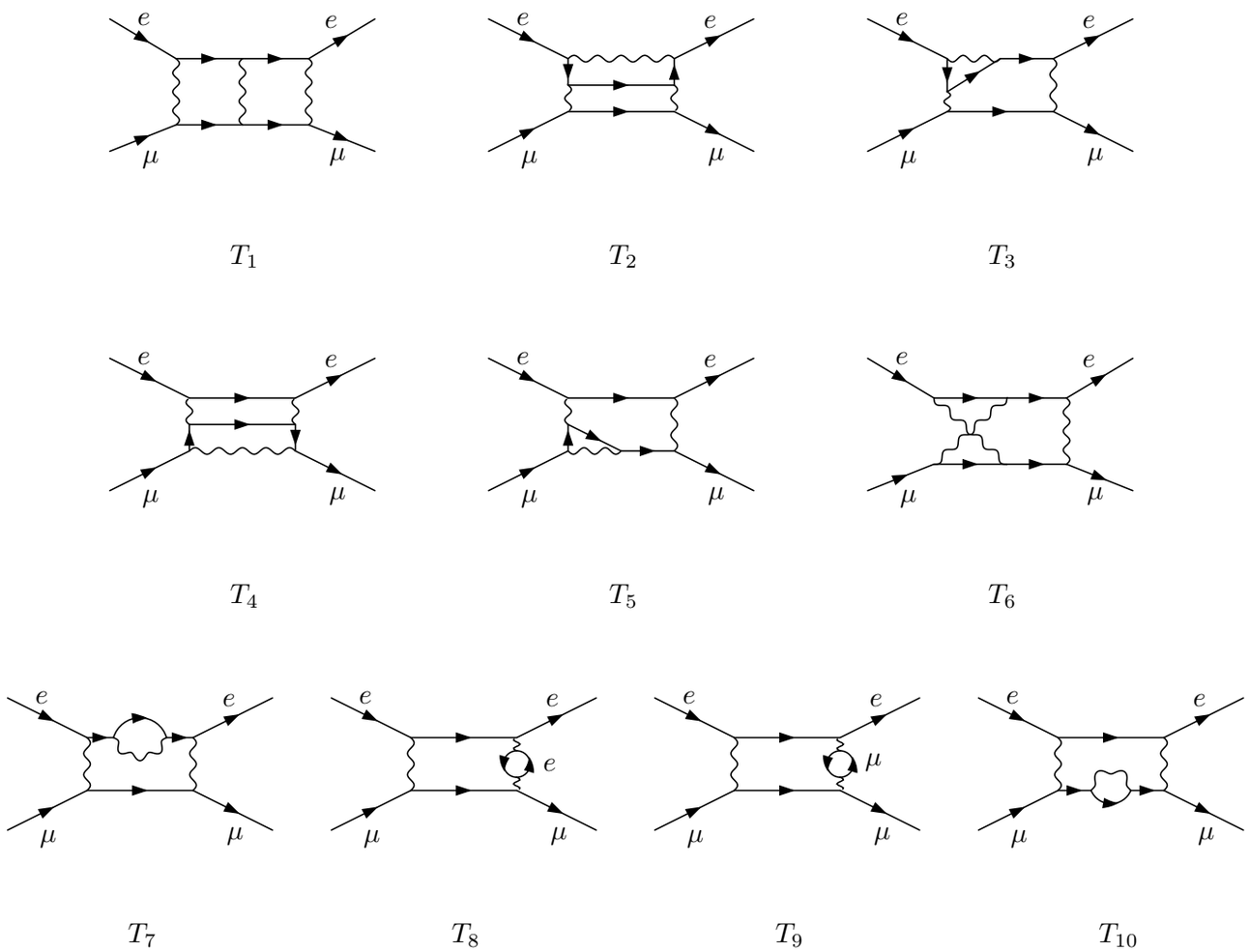
space-time
dimensions



= ?

# Two-Loop Integrals for Mu-E Scattering

# Feynman diagrams @ 2-loop



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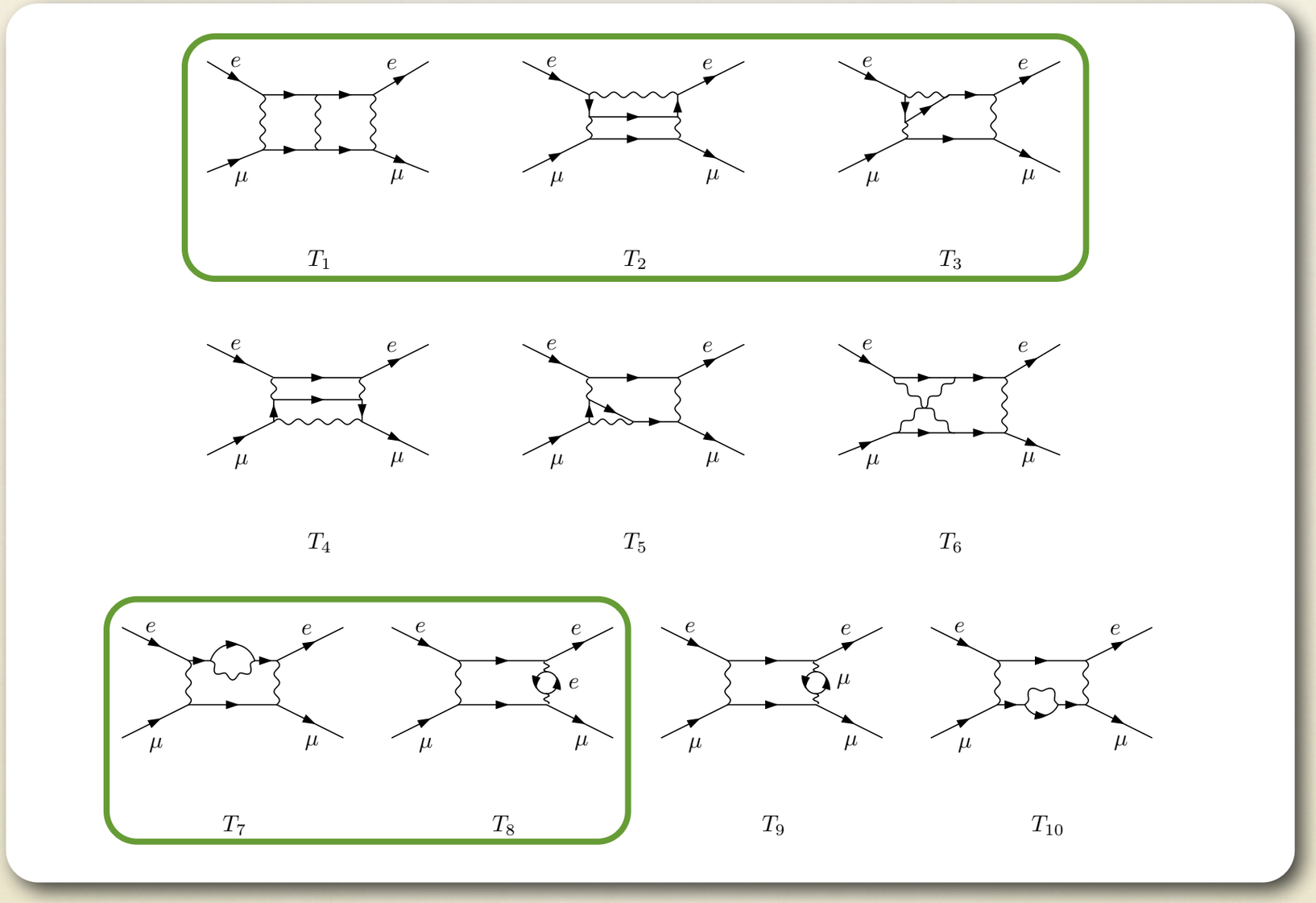
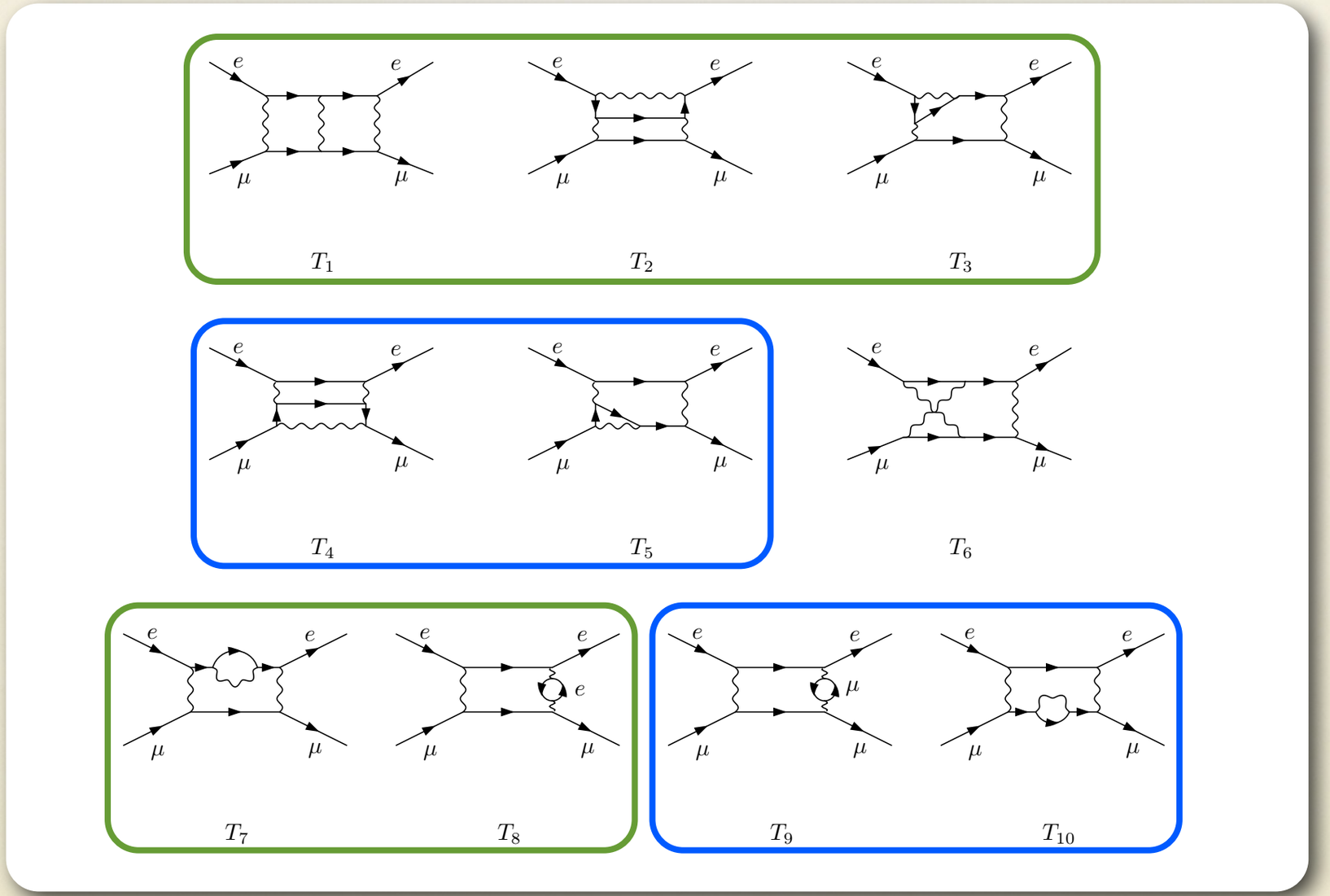


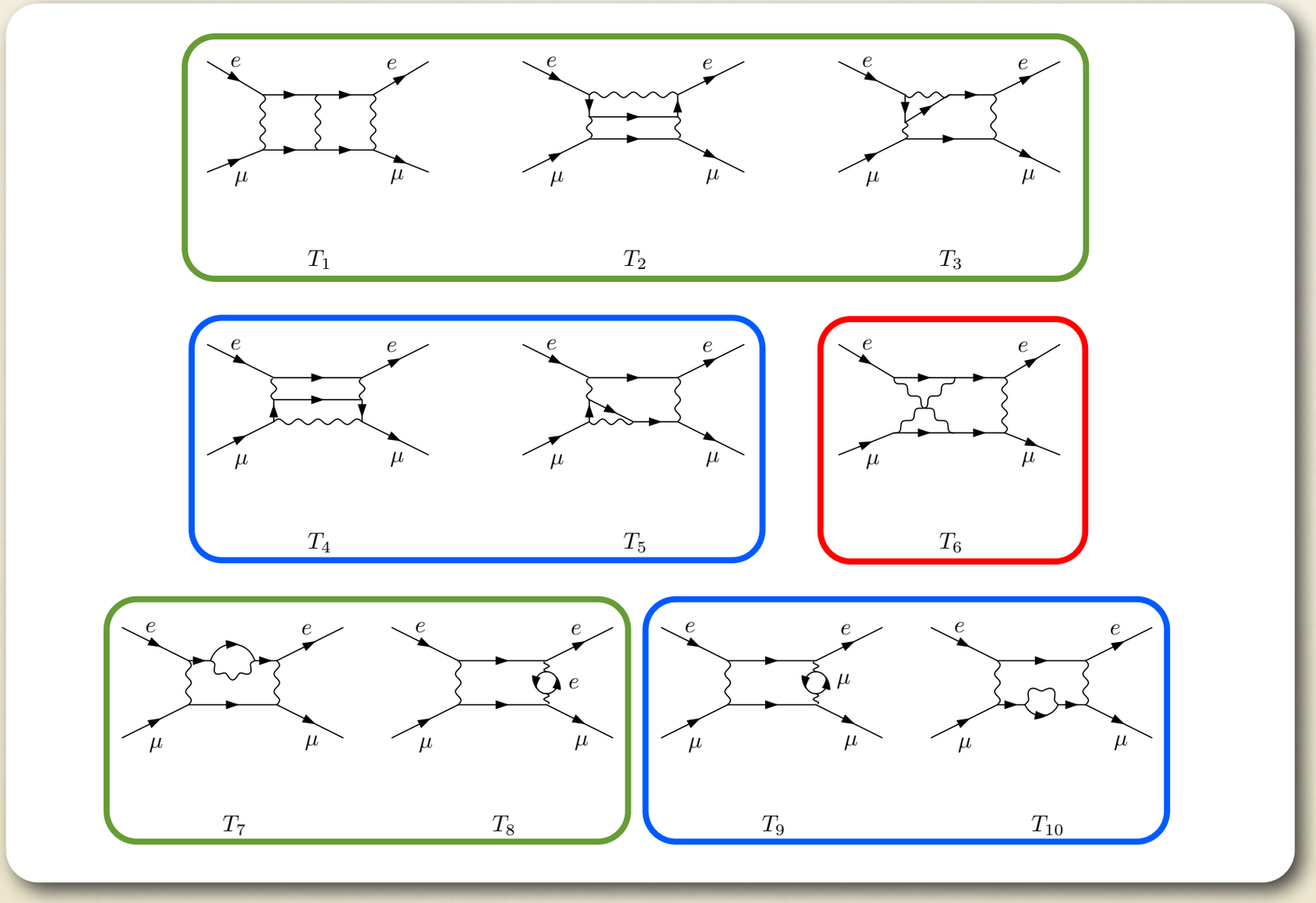

- Planar Integrals :: Family-1

# Feynman diagrams @ 2-loop




- Planar Integrals :: Family-1
- Planar Integrals :: Family-2

# Feynman diagrams @ 2-loop

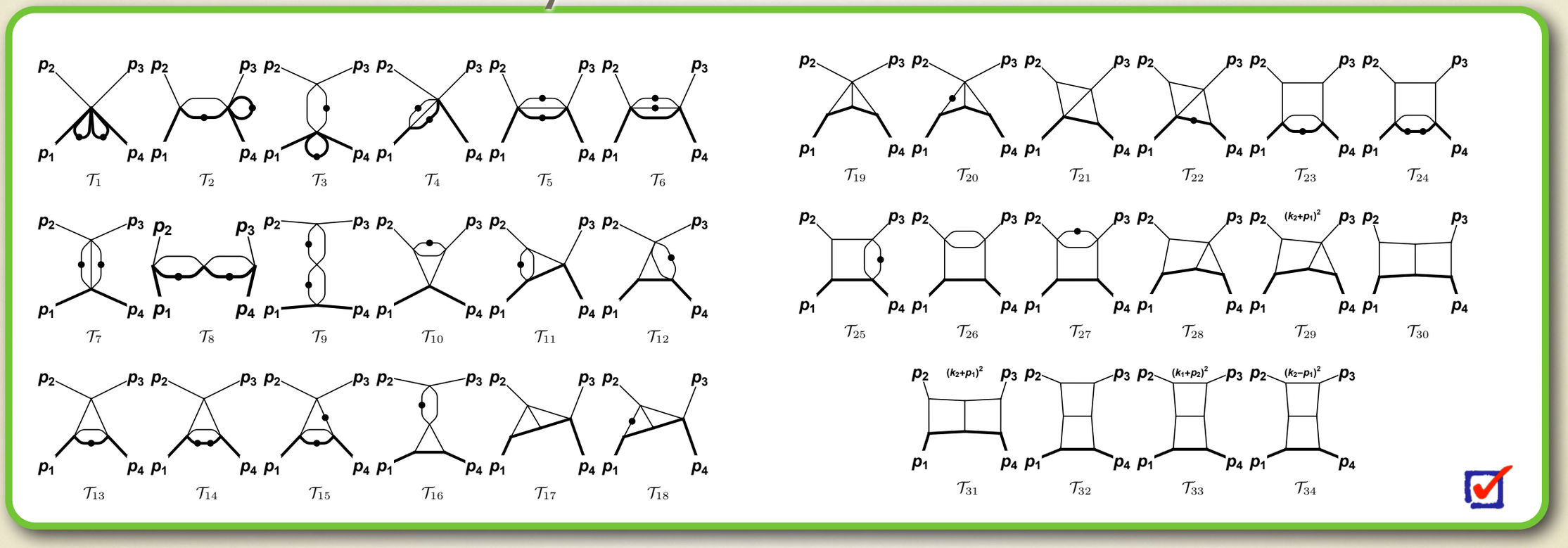



- Planar Integrals :: Family-1
- Planar Integrals :: Family-2
- Non-Planar Integrals

- massless **electron**
- **massive muon**

# 34 MIs for Family-1

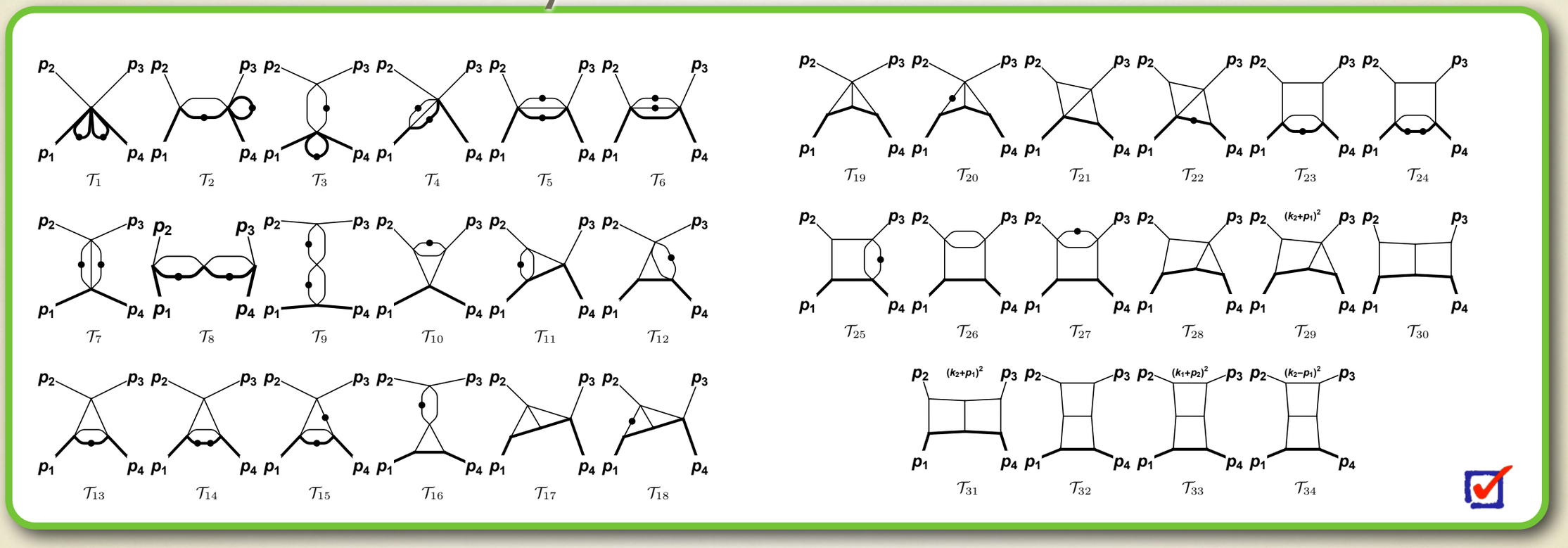
Passera Primo Schubert & **P.M.** (coming soon)




- **A different set of MIs already known** Bonciani Ferroglia Gehrmann vonManteuffel
- **We recomputed them with alternative techniques**

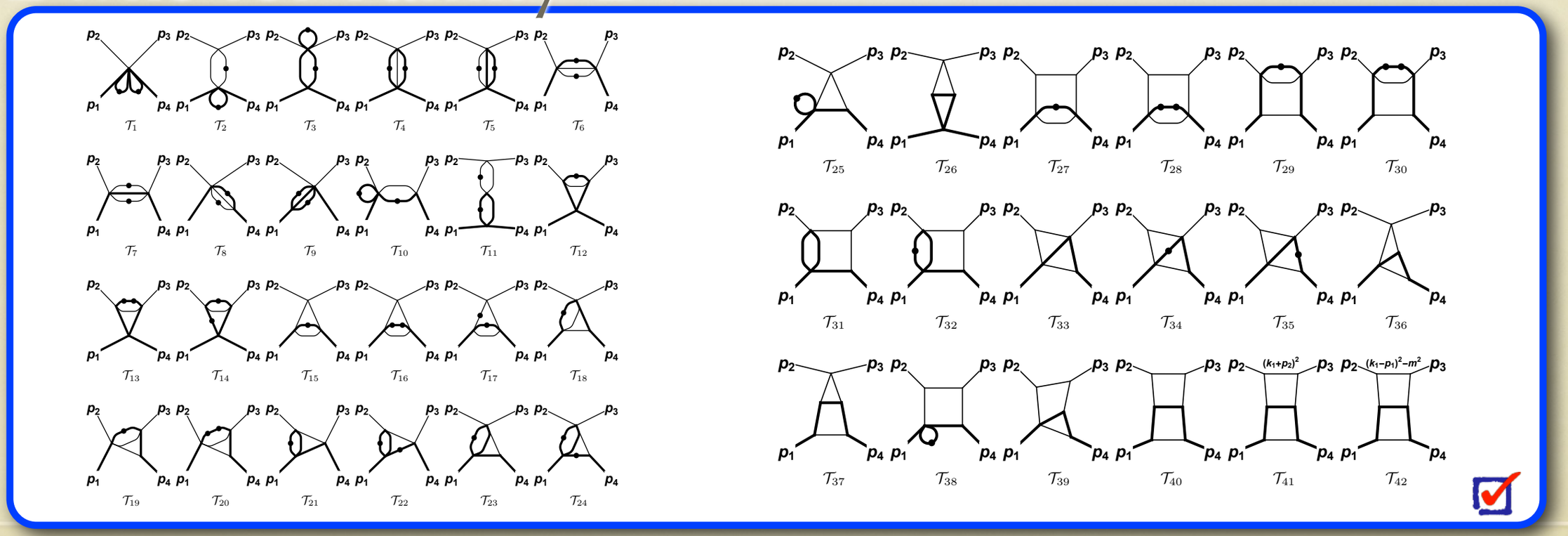
# 34 MIs for Family-1

Passera Primo Schubert & **P.M.** (coming soon)



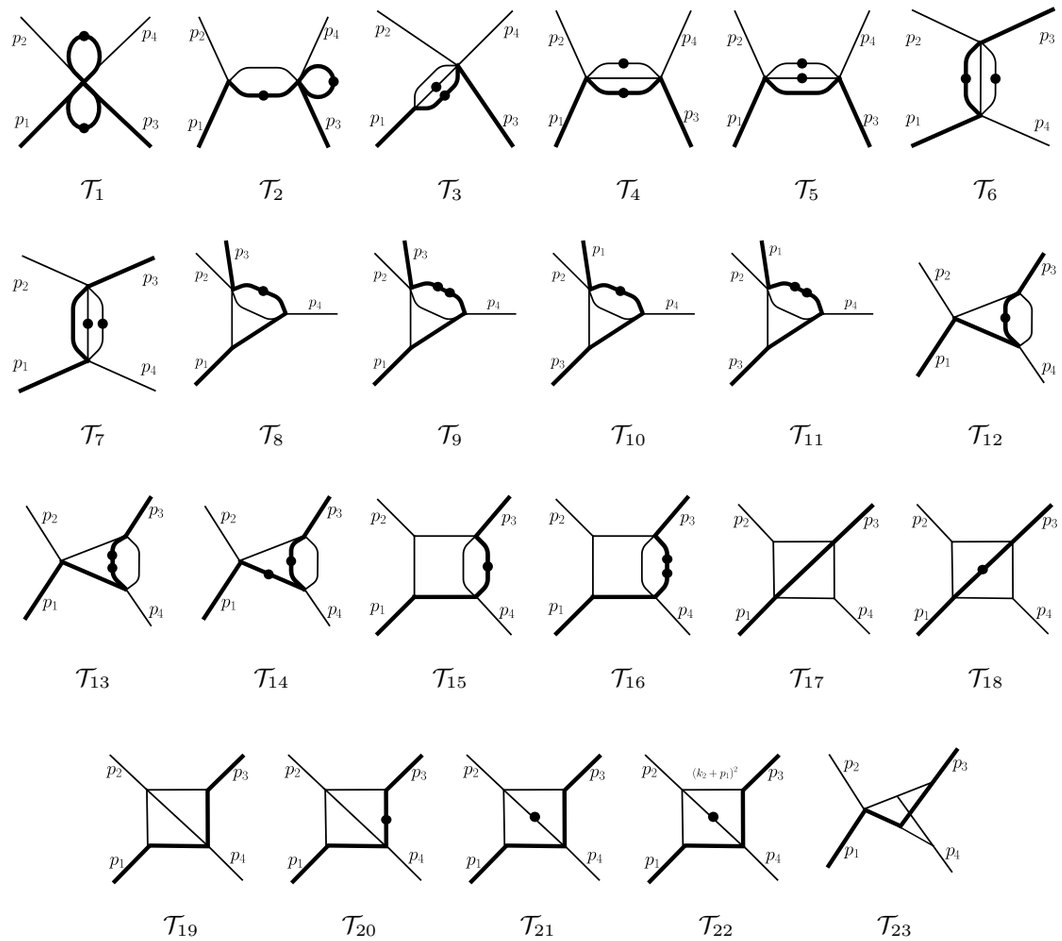

# 42 MIs for Family-2

Passera Primo Schubert & **P.M.** (coming soon)

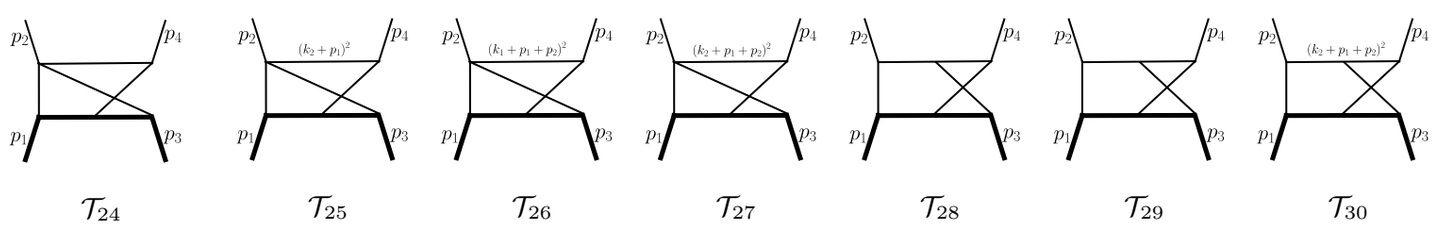


# 30 MIs for non-planar diagrams

Primo Schubert & **P.M.** (w.i.p.)




## The last missing pieces




>> see **Schubert**'s talk

# Quantum Mechanics

**Schroedinger Eq'n** (*eps-linear Hamiltonian*)

$$i\hbar\ \partial_t|\Psi(t)\rangle = H(\epsilon,t)|\Psi(t)\rangle\ , \qquad H(\epsilon,t) = H_0(t) + \epsilon H_1(t)$$

**Interaction Picture**

$$H_{i,I}(t) = B^\dagger(t)\ H_i(t)\ B(t)$$

**t-Evolution**

$$i\hbar\ \partial_t U_I(t) = \epsilon\, H_{1,I}(t) U_I(t) + \Big(H_{0,I}(t) - i\hbar\, B^\dagger(t)\ \partial_t B(t)\Big)\, U_I(t) \stackrel{!}{=} \epsilon\, H_{1,I}(t) U_I(t),$$

**Matrix Transform**

$$i\hbar\, \partial_t B(t) = H_0(t) B(t) \qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

**Schroedinger Eq'n** (*canonical form*)

$$i\hbar\ \partial_t|\Psi_I(t)\rangle = \epsilon\, H_{1,I}(t)|\Psi_I(t)\rangle\ ,$$

# Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

## System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

*A(x)* non-commutative

## solution: *Matrix Exponential*

$$Y(x) = e^{\Omega(x,x_0)} \ Y(x_0) \equiv e^{\Omega(x)} \ Y_0 \, , \qquad \Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

BCH-formula

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$

$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \ [A(\tau_1), A(\tau_2)] \ ,$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .$$

.....

## Iterated Integrals

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \equiv \int_\gamma d\log \eta_{i_1} \ldots d\log \eta_{i_k} \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k) \ldots g^\gamma_{i_1}(t_1) \, dt_1 \ldots \, dt_k \qquad g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$$

$$\mathcal{C}^{[\gamma]}_{\vec{m}} \mathcal{C}^{[\gamma]}_{\vec{n}} = \mathcal{C}^{[\gamma]}_{\vec{m}} \sqcup\!\sqcup \mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\vec{p} = \vec{m} \sqcup\!\sqcup \vec{n}} \mathcal{C}^{[\gamma]}_{\vec{p}} \qquad \mathcal{C}^{[\alpha\beta]}_{i_k,\ldots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\alpha]}_{i_k,\ldots,i_{p+1}} \mathcal{C}^{[\beta]}_{i_p,\ldots,i_1}$$

Chen
Goncharov
Remiddi Vermaseren
Gehrmann Remiddi
Bonciani Remiddi **P.M.**
Vollinga Weinzierl
Brown
Duhr Gangl Rhodes
.....

# Magnus & Dyson Series

- **Magnus**

$$Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0\,,$$

- **Dyson**

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)\ , \qquad Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n\ A(\tau_1)A(\tau_2)\cdots A(\tau_n)$$

$$\sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x)\right)$$

$$\begin{aligned}
Y_1 &= \Omega_1\ , \\
Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2\ , \\
Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3\ , \\
&\vdots \qquad\qquad \vdots \\
Y_n &= \Omega_n + \sum_{j=2}^{n} \frac{1}{j} Q_n^{(j)}\ .
\end{aligned}$$

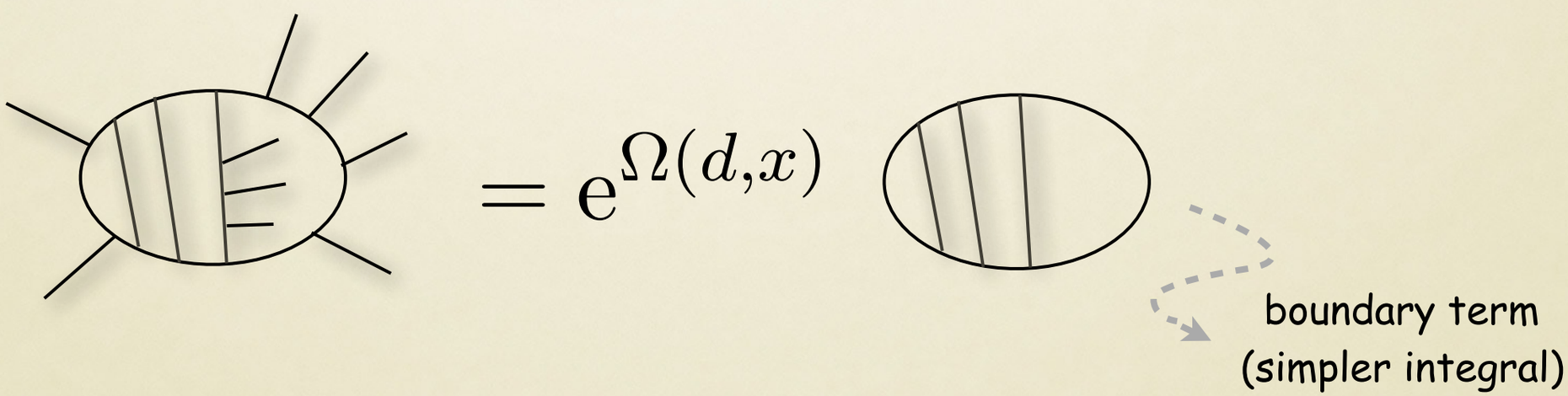
Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

## Quantum Mechanics

- Time-evolution in Perturbation Theory
- perturbation parameter: ε
- **Linear** Hamiltonian in ε
- **Unitary transform**
- Schroedinger Equation in the interaction picture (ε-factorization)
- solution: Dyson series

## Feynman Integrals

- Kinematic-evolution in Dimensional Regularization
- space-time dimensional parameter: ε = (4-d)/2
- **Linear** system in ε
- non-Unitary **Magnus transform**
- System of Differential Equations in canonical form (ε-factorization) Henn (2013)
- solution: Dyson/Magnus series




- Feynman integrals can be determined from differential equations that looks like **gauge transformations**

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

- Linear-eps Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$

- change of basis :: Magnus #1

$$f(\epsilon, x) = B_0(x)\ g(\epsilon, x)\ , \qquad B_0(x) \equiv e^{\Omega[A_0](x,x_0)}\ . \qquad \partial_x B_0(x) = A_0(x) B_0(x)\ ,$$

- *Canonical form* Henn (2013)

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)\ .$$

Uniform Transcendentality(*)

- Solution :: Magnus #2 (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x,x_0)}$$

Feynman integrals can be determined from differential equations that looks like gauge transformations

(*)dependng on boundary cond's

- eps-linear basis

$$\partial_x \; f(x,y,\epsilon) = \Big(A_{10}(x,y) + \epsilon A_{11}(x,y)\Big) \; f(x,y,\epsilon)$$
$$\partial_y \; f(x,y,\epsilon) = \Big(A_{20}(x,y) + \epsilon A_{21}(x,y)\Big) \; f(x,y,\epsilon)$$

- canonical form: Magnus #1

$$\partial_x \; g(x,y,\epsilon) = \epsilon \hat{A}_1(x,y) \; g(x,y,\epsilon)$$
$$\partial_y \; g(x,y,\epsilon) = \epsilon \hat{A}_2(x,y) \; g(x,y,\epsilon)$$

- Total Differential dLog-form

$$dg(x,y,\epsilon) = \epsilon \; d\hat{\mathcal{A}}(x,y) \; g(x,y,\epsilon) \; , \qquad d\hat{\mathcal{A}} \equiv \hat{A}_1 dx + \hat{A}_2 dy$$

$$dA = \sum_{i=1}^{n} M_i \; d\log \eta_i$$

**Alphabet**

**Constant matrices**

# On the Canonical System of DEQ

1. **Total-differential system ⇔ Path parametrization**
   - *a posteriori* (standard) ::
     parametrizing the kinematic variables *after* deriving the corresponding diff. eqs. (as shown before)
   - *a priori* (novel) ::
     introducing a parameter-dependent external kinematics, say $p_i = p_i(\tau)$ (for a given $i$) and differentiating w.r.t. to $\tau$.

- *pre-Canonical form* ::
  Linear-**eps** Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$

- *Canonical form*

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)$$

- change of basis ::
  Adjoint system of Diff.Eqs.

$$\partial_x B_0(x) = A_0(x) B_0(x)\ , \qquad f(\epsilon, x) = B_0(x)\ g(\epsilon, x)$$

# On the Canonical System of DEQ

1. **Total-differential system ⇔ Path parametrization**
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  Linear-**eps** Matrix
  $$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$
- *Canonical form*
  $$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)$$
- change of basis ::
  Adjoint system of Diff.Eqs.
  $$\partial_x B_0(x) = A_0(x) B_0(x)\ , \qquad f(\epsilon, x) = B_0(x)\ g(\epsilon, x)$$

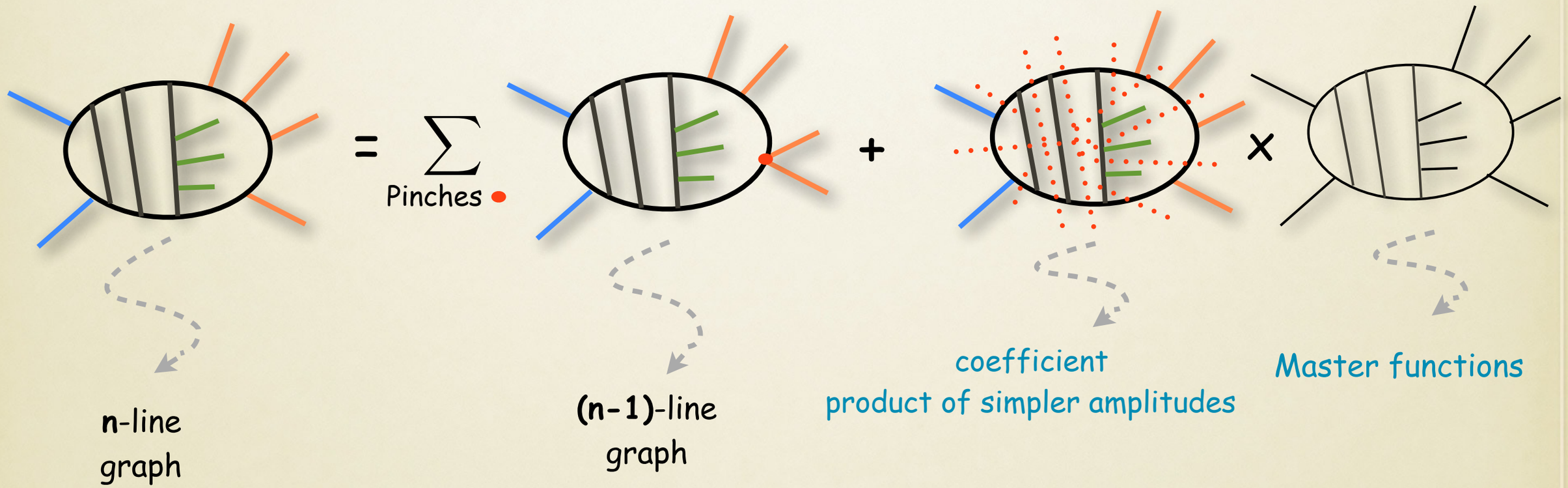
2. The Wronski matrix $W$ of the **homogeneous solutions** obeyes the adjoint equation
   $\Longleftrightarrow B_0 = W$ Remiddi Tancredi
3. The **homogeneous solutions** ⇔ **maximal cuts** of the integrals Primo Tancredi; Bosma Sogaard Zhang
4. The **maximal cuts** ⇐ **Baikov parametrization** Papadopoulos Frellesvig (Primo Schubert & **P.M.**)
5. The **homogeneous solutions** ⇔ **kernels** of iterated integrals Euler
6. **IBPs on the cuts** ⇔ **algebraic relations** for iterated integrals
   ⇒ Elliptic-integrals relations from IBPs on the cuts. Primo Schubert & **P.M.**

# Multi-loop Integrand Decomposition

# Multi-Loop Integrand Recurrence

Ossola & **P.M.** (2011);
Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

## $\ell$-Loop Recurrence Relation




- ☑ **all orders** (any number of loops and legs)
- ☑ **any topology** (planar and non-planar)
- ☑ **all kinematics** (massless and massive)
- ☑ **high-power of denominators**

# Longitudinal and Transverse Space

Peraro Primo & ***P.M.*** (2016)

- **Dimensional Regularization**

$$d = 4 - 2\epsilon$$

- **if n-legs < 5**

$$d = d_{//} + d_{\perp}$$

Collins; Kreimer

Transverse Space

Longitudinal space
**spanned by the (independent) legs**

☑ Denominators **do not depend on** "the angular variables" of the Transverse Space

☑ Numerators **depend on** "all" loop variables

# Adaptive Integrand Decomposition

Peraro Primo & ***P.M.***

- Integrand reduction beyond polynomial division

$$d = d_{//} + d_{\perp}$$

**idea n.1** **Integrating over Transverse Space**

**idea n.2** **Cutting in the Longitudinal Space**

# 1&2-loop Automation :: AIDA

Peraro Primo TorresBobadilla & ***P.M.***

- Application to Mu-e scattering

Ossola Peraro Primo TorresBobadilla & ***P.M.***

**>>** see **Primo**'s talk

# Summary ...

- ☑ After Amedeo & Uli's talks

# ...Outlook

- ☑ Progress in Mu-e Scattering @ 2-loop ==> Progress in pp —> t T @ 2-loop