Muon-electron scattering, Theory kickoff workshop
Padova, September 4, 2017

Towards muon-electron scattering @NNLO: Integrand decomposition

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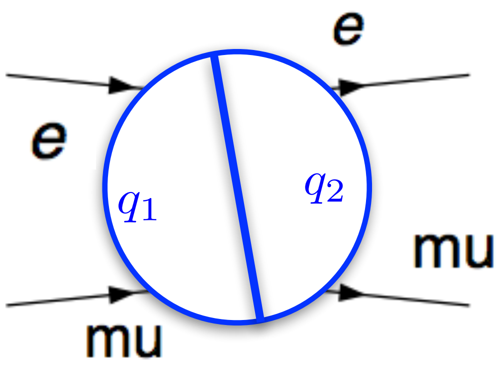
In collaboration with: P.Mastrolia, M.Passera, T.Peraro, U.Schubert and W.J.Torres Bobadilla

The Road Map to precision calculations

- To compute an amplitude, we start from (a lot of) Feynman diagrams



- We reduce Feynman diagrams to a sum of scalar integrals

$$\int \prod_{i=1}^{\ell} \frac{d^d\, q_i}{(2\pi)^d} \frac{(q_j \cdot p_r)^{\beta_1} \cdots (q_k \cdot p_s)^{\beta_m}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

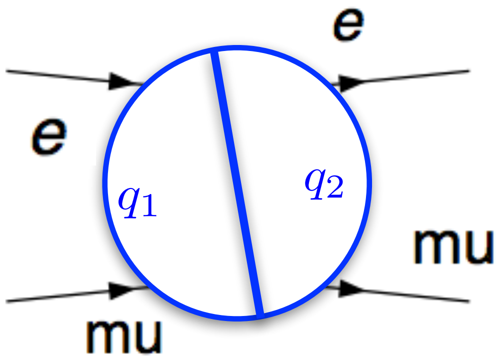
- We use identities between integrals (IBPs) to reduce them to a minimal basis of independent functions, the Master Integrals

$$\int \prod_{i=1}^{\ell} \frac{d^d\, q_i}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} \left[v^\mu \frac{(q_j \cdot p_r)^{\beta_1} \cdots (q_k \cdot p_s)^{\beta_m}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} \right] = 0 \quad v^\mu = q_i^\mu, p_j^\mu$$

- We compute the Master Integrals

The Road Map to precision calculations

- To compute an amplitude, we start from (a lot of) Feynman diagrams



- We reduce Feynman diagrams to a sum of scalar integrals (this talk)

$$\int \prod_{i=1}^{\ell} \frac{d^d\, q_i}{(2\pi)^d} \frac{(q_j \cdot p_r)^{\beta_1} \cdots (q_k \cdot p_s)^{\beta_m}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

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- We compute the Master Integrals (Ulrich’s talk)

Muon-electron scattering via Adaptive Integrand Decomposition

The Integrand Decomposition method

- Before integration, the amplitude is just a rational function

Ossola, Papadopoulos, Pittau(07)
Ellis, Giele, Kunszt, Melnikov (08)
Mastrolia,Ossola,
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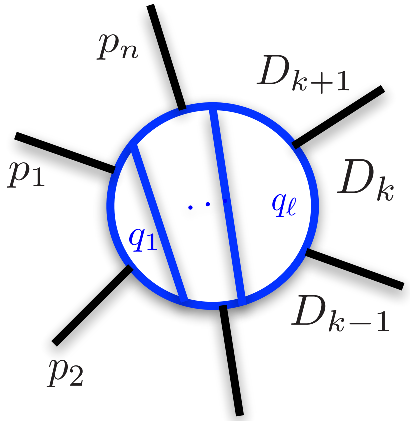

$$= \frac{\mathcal{N}_{i_1\cdots i_k \cdots i_m}(q_i)}{D_1(q_i)\cdots D_k(q_j)\cdots D_m(q_i)}$$

- Use Multivariate Polynomial Division to write

$$\mathcal{N}_{i_1\cdots i_k\cdots i_m}(q_i) = \sum_{k=1}^{m} \mathcal{N}_{1\cdots i_{k-1} i_{k+1}\cdots i_m}(q_i) D_k(q_i) + \Delta_{i_1\cdots i_m}(q_i)$$

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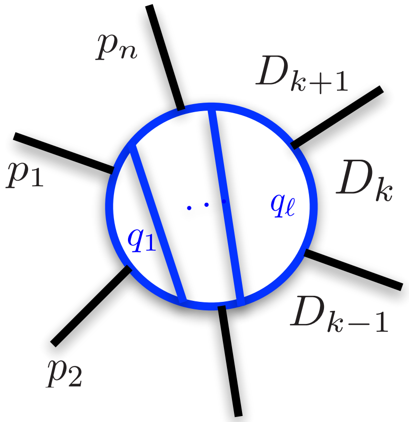

$$= \frac{\mathcal{N}_{i_1\cdots i_k\cdots i_m}(q_i)}{D_1(q_i)\cdots D_k(q_j)\cdots D_m(q_i)}$$

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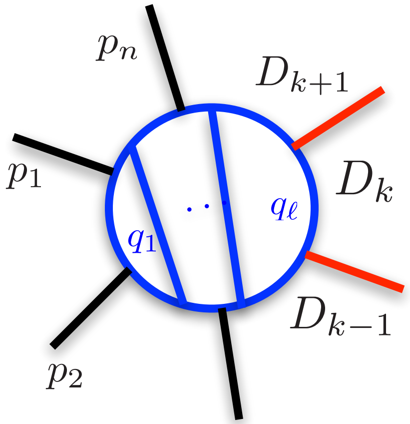

$$= \sum_{k=1}^{n} \frac{\mathcal{N}_{i\cdots i_{k-1}\, i_{k+1}\cdots i_n}(q_i)\cancel{D_k(q_i)}}{D_1(q_i)\cdots \cancel{D_k(q_i)}\cdots D_n(q_i)} + \frac{\Delta_{i_1\cdots i_n}(q_i)}{D_1(q_i)\cdots D_k(q_i)\cdots D_n(q_i)}$$

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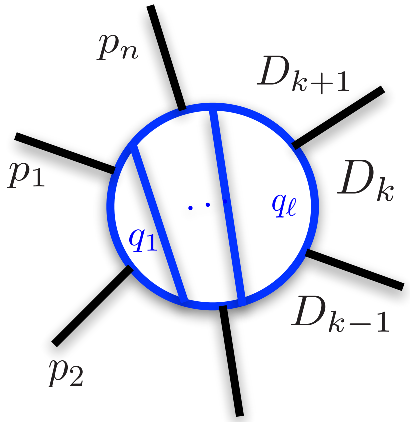

$$= \frac{\mathcal{N}_{i_1 \cdots i_k \cdots i_m}(q_i)}{D_1(q_i) \cdots D_k(q_j) \cdots D_m(q_i)}$$

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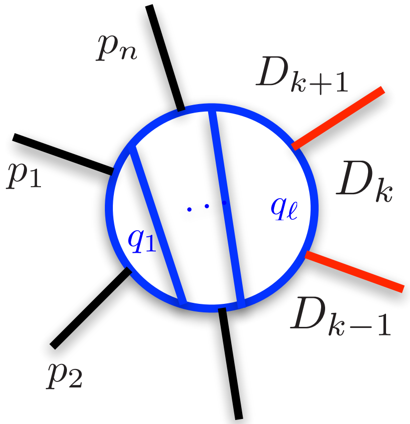
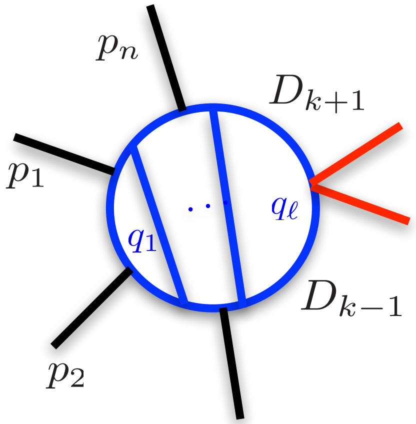
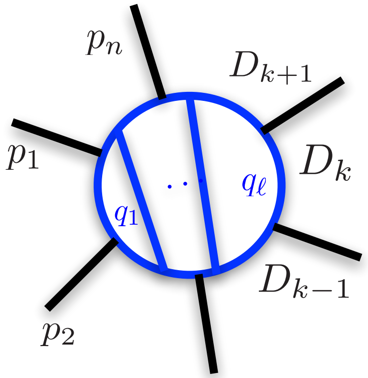


$$= \sum_{k=1}^{n}$$



$$+ \frac{\Delta_{i_1 \cdots i_m}(q_i)}{D_1(q_i) \cdots D_m(q_i)}$$

- Repeat recursively



$$= \sum_{k=0}^{n} \sum_{i_1 \cdots i_k} \frac{\Delta_{i_1 \cdots i_k}(q)}{D_{i_1}(q) \cdots D_{i_k}(q)}$$

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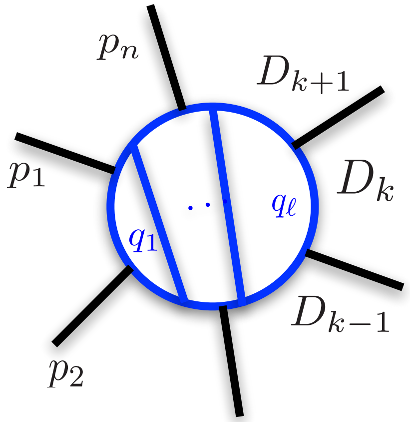

$$= \frac{\mathcal{N}_{i_1\cdots i_k\cdots i_m}(q_i)}{D_1(q_i)\cdots D_k(q_j)\cdots D_m(q_i)}$$

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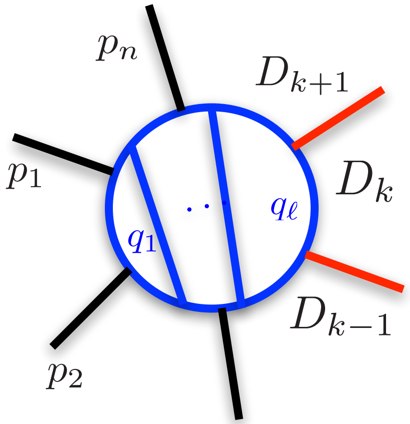
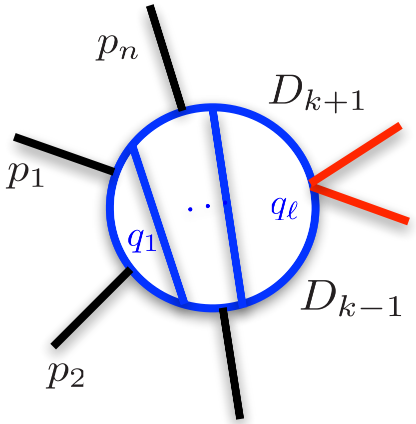


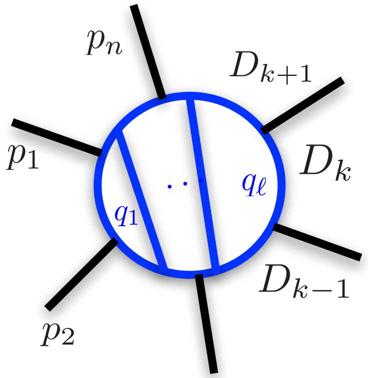
$$= \sum_{k=1}^{n}$$



$$+ \frac{\Delta_{i_1\cdots i_m}(q_i)}{D_1(q_i)\cdots D_m(q_i)}$$

- Repeat recursively, **integrate**

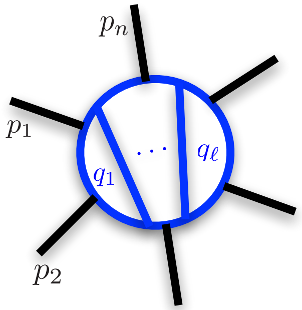
$$\int$$



$$= \sum_{k=0}^{n} \sum_{i_1\cdots i_k} \int \frac{\Delta_{i_1\cdots i_k}(q)}{D_{i_1}(q)\cdots D_{i_k}(q)}$$

Feynman Integrands

- **How many variables** do integrands in d depend on?



$$= \frac{\mathcal{N}(q_i, p_j)}{D_1(q_i, p_j) \ldots D_m(q_i, p_j)}$$

- If **external particles** are in 4-dim, use

$$\boxed{\begin{aligned} g_d^{\mu\nu} &= \begin{pmatrix} g_4^{\mu\nu} & \mathbf{0} \\ \mathbf{0} & g_{-2\epsilon}^{\mu\nu} \end{pmatrix} \\ q_i^\alpha &= q_{[4]\,i}^\alpha + q_{-2\epsilon\,i}^\alpha \end{aligned}}$$

$$q_i \cdot q_j = q_{[4]\,i} \cdot q_{[4]\,j} + \mu_{ij}$$

$$q_i \cdot p_j = q_{[4]\,i} \cdot p_j$$

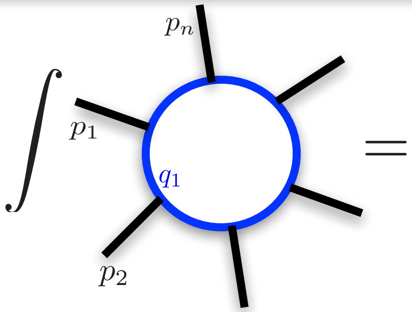
- Each $q_{i\,[4]}$ has four components

$$q_{i\,[4]}^\alpha = x_{1\,i} e_1^\alpha + x_{2\,i} e_2^\alpha + x_{3\,i} e_3^\alpha + x_{4\,i} e_4^\alpha$$

- There are $\ell(\ell+1)/2$ scalar products μ_{ij}
- The total number of variables is $\ell(\ell+9)/2$

$$\boxed{\mathbf{z} = \{x_{1\,i}, x_{2\,i}, x_{3\,i}, x_{4\,i}, \mu_{ij}\}}$$

One-loop Integrand decomposition



$$= \int d\mathbf{z} \frac{\mathcal{N}_{i\cdots i_n}(\mathbf{z})}{D_1(\mathbf{z})\cdots D_n(\mathbf{z})}$$

$$\mathbf{z} = \{x_1, x_2, x_3, x_4, \mu^2\}$$

$$\mathcal{N}_{1\cdots n}(\mathbf{z}) = \sum_{\vec{j}\in J_5(n)} \alpha_{\vec{j}} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5}$$

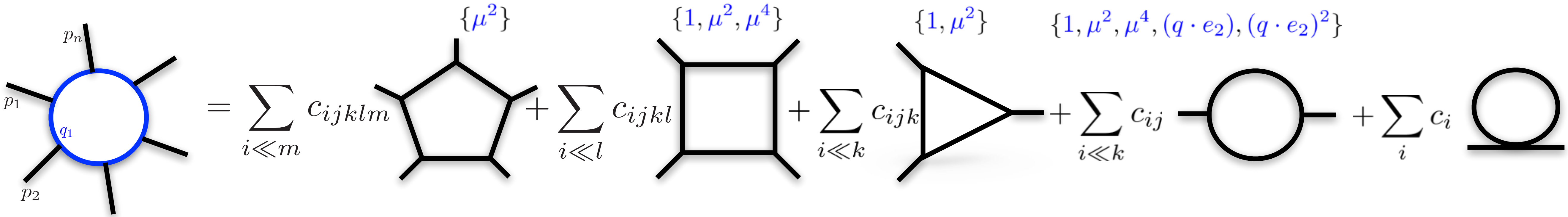
- ❖ **Universal residues** for renormalizable theories

Ossola, Papadopoulos, Pittau (07)
Ellis, Giele, Kunszt, Melnikov (08)

$$\begin{aligned}
\Delta_{ijklm} &= c_0\mu^2 \\
\Delta_{ijkl} &= c_0 + c_1x_4 + c_2\mu^2 + c_3x_4\mu^2 + c_4\mu^4 \\
\Delta_{ijk} &= c_0 + c_1x_4 + c_2x_4^2 + c_3x_4^3 + c_4x_3 + c_5x_3^2 + c_6x_3^3 + c_7\mu^2 + c_8x_4\mu^2 + c_9x_3\mu^2 \\
\Delta_{ij} &= c_0 + c_1x_1 + c_2x_1^2 + c_3x_4 + c_4x_4^2 + c_5x_3 + c_6x_3^3 + c_7x_1x_4 + c_8x_1x_3 + c_9\mu^2 \\
\Delta i &= c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4
\end{aligned}$$

- ❖ 4-dimensional contributions are **spurious** $\Delta_{i_1\cdots i_k} = \Delta_{i_1\cdots i_k} + \Delta^{\text{spurious}}_{i_1\cdots i_k}$

$$\int \frac{d\mathbf{z}\,\Delta^{\text{spurious}}_{i\cdots i_n}}{D_1(\mathbf{z})\cdots D_n(\mathbf{z})} = 0$$



- ❖ Use integral level relation to reduce to rank 0

Bern, Morgan (95)
Tarasov (96), Lee (10)

Longitudinal and Transverse space

- What is the best choice of variables $\mathbf{z}$?
- The n external particles span a longitudinal space with $\boxed{d_\parallel = n-1 \leq 4}$

Collins(84)
van Neerven and Vermaseren (84)
Kreimer (92)

- Choose a basis with the maximum number of transverse vectors

$$q^\alpha_{i\,[4]} = x_{1\,i}e^\alpha_1 + x_{2\,i}e^\alpha_2 + x_{3\,i}e^\alpha_3 + x_{4\,i}e^\alpha_4$$

with

$$\begin{aligned} e_i\cdot p_j &= 0, & i &> d_\parallel \\ e_i\cdot e_j &= \delta_{ij}, & i,j &> d_\parallel \end{aligned}$$

$$\boxed{\begin{aligned} g^{\mu\nu}_d &= \begin{pmatrix} g^{\mu\nu}_{d_\parallel} & \mathbf{0} \\ \mathbf{0} & g^{\mu\nu}_{d_\perp} \end{pmatrix} \\ q^\alpha_i &= q^\alpha_{\parallel\, i} + \lambda^\alpha_i \end{aligned}}$$

$$\Longrightarrow$$

$$\lambda^\alpha_i = x_{n\,i}e^\alpha_n + \cdots + x_{4\,i}e^\alpha_4 + \mu^\alpha_i$$

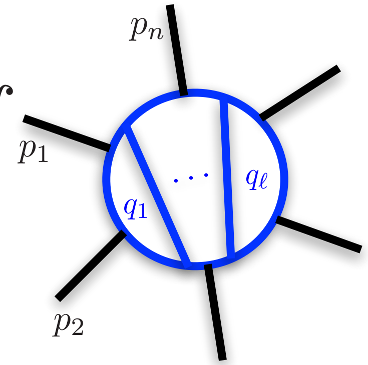
$$\boxed{\begin{aligned} q_i\cdot q_j &= q_{\parallel\, i}\cdot q_{\parallel\, j} + \lambda_{ij} \\ q_i\cdot p_j &= q_{\parallel\, i}\cdot p_j \end{aligned}}$$

- Choose $\mathbf{z} = \{x_{1i},\ldots,x_{n-1\,i}, x_{ni},\ldots,x_{4i},\lambda_{ij}\}$
- Numerators can still depend on $\vec{x}_\perp$: e.g. $\lambda_i\cdot\varepsilon(p_j) \propto (x_n,\ldots,x_4) \neq 0$
- Denominators don't depend on $\vec{x}_\perp$
- The integrand are rational in the longitudinal vars but they are polynomial in the transverse vars
- We always know how to integrate polynomials!

Transverse space integrals

- Use longitudinal/transverse vars to parametrize Feynman integrals

$$\int$$



$$= \int \prod_i d^{d_\parallel} q_{\parallel\, i} \int \prod_{1\leq i\leq j\leq \ell} d\lambda_{ij} G(\lambda_{ij})^{\frac{d_\perp -1-\ell}{2}} \int d\mathbf{\Theta}_\perp \frac{\mathcal{N}(q_{\parallel,i},\lambda_{ij},\mathbf{\Theta}_\perp)}{D_1(q_{\parallel,i},\lambda_{ij})\cdots D_m(q_{\parallel,i},\lambda_{ij})}$$

Mastrolia, Peraro, A.P. (16)

- The integrand is a polynomial in $\sin(\theta_{\perp\, i})$ and $\cos(\theta_{\perp\, i})$
- The integration domains are ℓ hyperspheres with dimensions ranging from $(d_\perp - 1)$ to $(d_\perp - \ell)$

$$\int d\,\mathbf{\Theta}_\perp = \int_{-1}^{1} \prod_{i=1}^{4-d_\parallel} \prod_{j=1}^{\ell} d\cos\theta_{i+j-1\, j} (\sin\theta_{i+j-1\, j})^{d_\perp - i - j - 1}$$

- Expand in Gegenbauer polynomials $C_n^{(\alpha)}$ and use

$$\int_{-1}^{1} d\cos\theta (\sin\theta)^{2\alpha-1} C_n^{(\alpha)}(\cos\theta) C_m^{(\alpha)}(\cos\theta) = \delta_{mn} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}$$

Spurious terms are killed, non spurious reduced to λ_{ij} (~ PaVe in vacuum)

Example 1

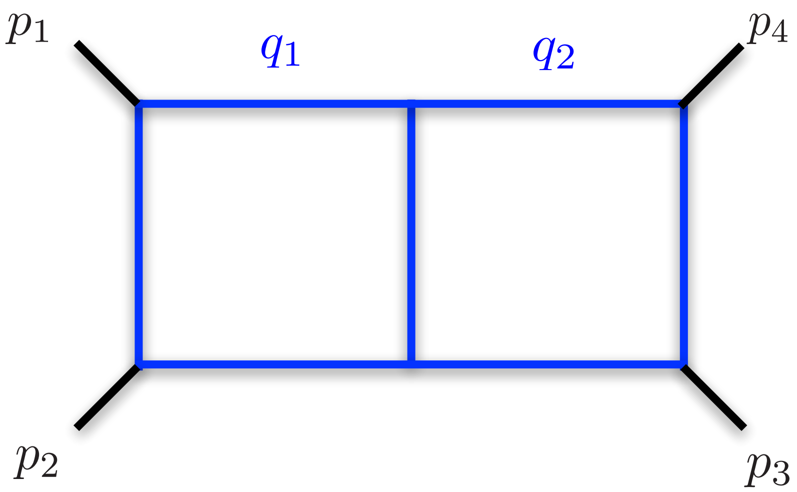
- **Decompose q_i^α in long/transv components:**

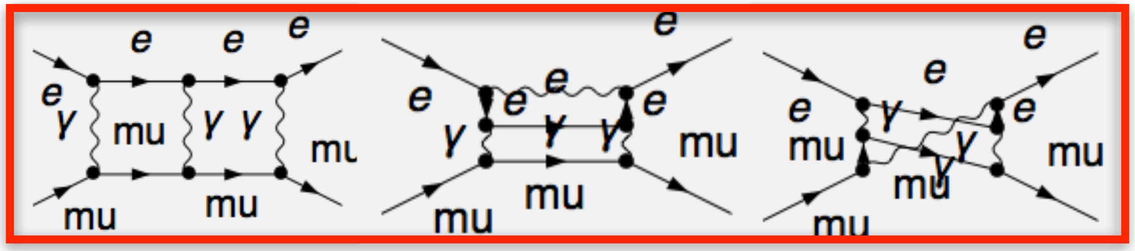
$$\boxed{d_{\|} = 3 \to e_4 \cdot p_i = 0}$$

$$q_{i\,\|}^{\alpha} = x_{i1} e_1^\alpha + x_{i2} e_2^\alpha + x_{i3} e_3^\alpha$$

$$\lambda_i^\alpha = x_{4i} e_4^\alpha + \mu_i^\alpha$$

$$D_i = l_{\|\,i}^2 + \sum_{j,l} \alpha_{ij} \alpha_{il} \, \lambda_{jl} + m_i^2$$





- **Parametrise the integral as**

$$\boxed{\begin{aligned} I_4^{d\,(2)}[\mathcal{N}] = \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int d^3 q_{1\,\|} \int d^3 q_{2\,\|} \int d\lambda_{11} d\lambda_{22} d\lambda_{12} [G(\lambda_{ij})]^{\frac{d-6}{2}} \\ \times \int_{-1}^{1} d\cos\theta_{11} d\cos\theta_{22} (\sin\theta_{11})^{d-6} (\sin\theta_{11})^{d-7} \frac{\mathcal{N}}{D_1 \cdots D_7} \end{aligned}}$$

with

$$G(\lambda_{ij}) = \lambda_{11}\lambda_{22} - \lambda_{12}^2 \qquad \begin{cases} x_{41} = \sqrt{\lambda_{11}} \cos\theta_{11} \\ x_{42} = \sqrt{\lambda_{22}} (\cos\theta_{11} \cos\theta_{12} + \sin\theta_{12} \cos\theta_{22}) \end{cases}$$

- **Integrate away transverse directions**

$$I_4^{d\,(2)}[\, x_{41}^{\alpha_4} x_{42}^{\beta_4} \,] = 0 \qquad \alpha_4 + \beta_4 = 2n+1 \qquad I_4^{d\,(2)}[\, x_{42}^3 x_{41}^3 \,] = \frac{3}{(d-3)(d-1)(d+1)} I_4^{d\,(2)}[\lambda_{12}(2\lambda_{12}^2 + 3\lambda_{11}\lambda_{22})]$$

$$I_4^{d\,(2)}[\, x_{41}^2 x_{42}^2 \,] = \frac{3}{(d-3)(d-1)} I_4^{d\,(2)}[\, 2\lambda_{12}^2 + \lambda_{11}\lambda_{22}] \qquad \ldots$$

Example 2

- Decompose q_i^α in long/transv components:

$$\boxed{d_{\parallel} = 2 \to e_{3\,,4} \cdot p_{1\,,2} = 0}$$

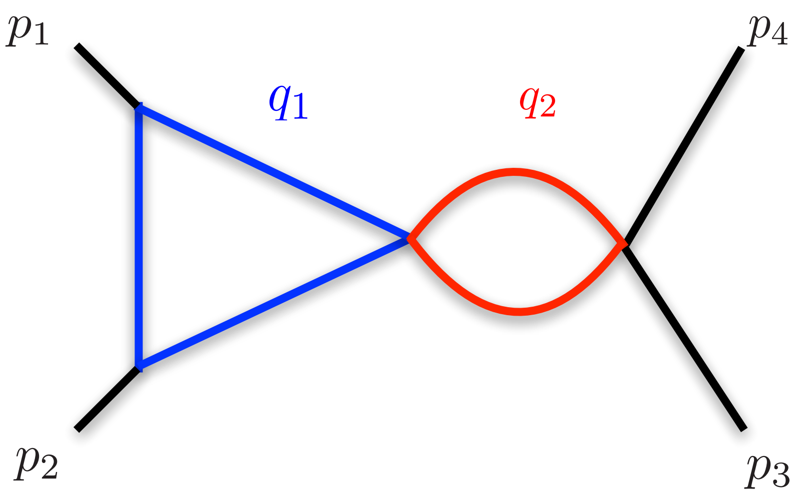
$$q_{1\,\parallel}^\alpha = x_{11} e_1^\alpha + x_{12} e_2^\alpha$$

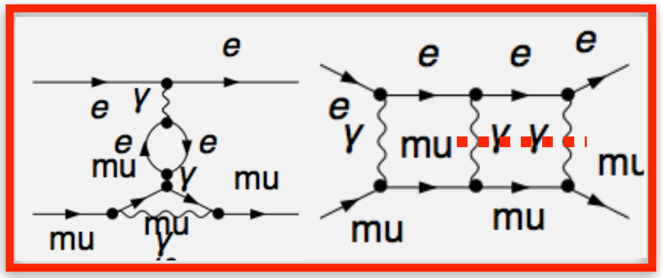
$$\lambda_1^\alpha = x_{13} e_3^\alpha + x_{14} e_4^\alpha + \mu_1^\alpha$$

$$\boxed{d_{\parallel} = 1 \to e_{2\,,3\,,4} \cdot (p_3 + p_4) = 0}$$

$$q_{2\,\parallel}^\alpha = x_{21} \hat{e}_1^\alpha$$

$$\lambda_2^\alpha = x_{22} \hat{e}_2^\alpha + x_{23} \hat{e}_3^\alpha + x_{24} \hat{e}_4^\alpha + \mu_2^\alpha$$





$$\mathcal{N}(q_1, q_2) = (\mu_{12})^\alpha \, \mathcal{N}(q_{1\,[4]}, \mu_{11}) \mathcal{N}(q_{2\,[4]}, \mu_{22})$$

- Parametrise the integral as

$$\boxed{\begin{aligned} I_4^{d\,(2)}[\mathcal{N}] = \Omega_d \int d^2 q_{1\,\parallel} \int d\lambda_{11} [\lambda_{11}]^{\frac{d-4}{2}} \int dc_{\theta_{11}} dc_{\theta_{12}} (s_{\theta_{11}})^{d-5} (s_{\theta_{12}})^{d-6} \frac{\mathcal{N}_1}{D_1 D_2 D_3} \\ \times \int dq_{2\,\parallel} \int d\lambda_{22} [\lambda_{22}]^{\frac{d-3}{2}} \int dc_{\theta_{21}} dc_{\theta_{22}} dc_{\theta_{23}} (s_{\theta_{21}})^{d-4} (s_{\theta_{22}})^{d-5} (s_{\theta_{23}})^{d-6} \frac{\mathcal{N}_2}{D_4 D_5} \end{aligned}}$$

with

$$\begin{cases} x_{13} = & \sqrt{\lambda_{11}} c_{\theta_{11}} \\ x_{14} = & \sqrt{\lambda_{11}} s_{\theta_{11}} c_{\theta_{12}} \end{cases} \qquad \begin{cases} x_{22} = & \sqrt{\lambda_{22}} c_{\theta_{21}} \\ x_{23} = & \sqrt{\lambda_{22}} s_{\theta_{21}} c_{\theta_{22}} \\ x_{24} = & \sqrt{\lambda_{22}} s_{\theta_{21}} s_{\theta_{22}} c_{\theta_{23}} \end{cases}$$

Adaptive Integrand Decomposition

Mastrolia, Peraro, A.P. (16)

- Adapt integrand decomposition to longitudinal/transverse space of each integrand

- At every step of the algorithm
  - Denominators depend on a minimal set of vars
  - Cut-conditions are always linearised
  - Polynomial division reduced to an algebraic substitution rule
  - extra-dim vars are always reducible

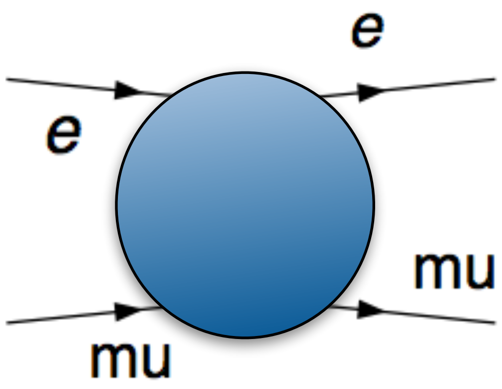
- Recipe in 3 steps
  1) Divide and get $\Delta(q_{\parallel\, i}, \boldsymbol{\Theta}_\perp)$
  2) Integrate away transverse vars
  3) Divide again and get rid of λ_{ij}

- Final decomposition in terms of Irreducible Scalar Products $q_i \cdot p_j$
  - IBPs-friendly output
  - no need for tensor decomposition
  - works for helicity amplitudes
  - at 1 loop no need for any integral identity

$$\begin{tikzcd}
\dfrac{\mathcal{N}(q_{\parallel\, i}, \boldsymbol{\Theta}_\perp, \lambda_{ij})}{D_1(q_{\parallel\, i}, \lambda_{ij}) \cdots D_n(q_{\parallel\, i}, \lambda_{ij})} \arrow[d, "1)"'] \\
\dfrac{\Delta(q_{\parallel\, i}, \boldsymbol{\Theta}_\perp)}{D_1(q_{\parallel\, i}, \lambda_{ij}) \cdots D_n(q_{\parallel\, i}, \lambda_{ij})} \arrow[d, "2)"'] \\
\dfrac{\Delta^{\text{int}}(q_{\parallel\, i}, \lambda_{ij})}{D_1(q_{\parallel\, i}, \lambda_{ij}) \cdots D_n(q_{\parallel\, i}, \lambda_{ij})} \arrow[d, "3)"'] \\
\dfrac{\Delta^{'}(q_{\parallel\, i})}{D_1(q_{\parallel\, i}, \lambda_{ij}) \cdots D_n(q_{\parallel\, i}, \lambda_{ij})}
\end{tikzcd}$$

AID for muon-electron scattering

- In the massless electron limit, 4-point process depending on 3 scales

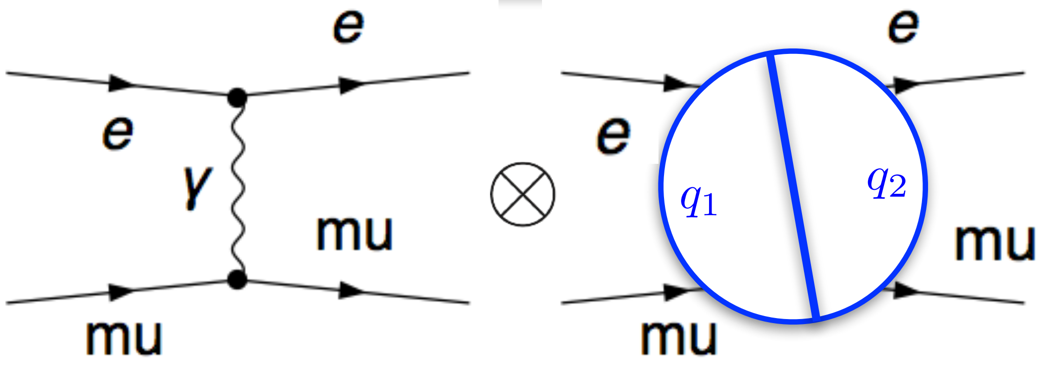


$$e(p_1) + \mu(p_4) \to e(-p_2) + \mu(-p_3)$$

$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2$$

$$m_e^2 \simeq 0 \qquad u = -s - t + 2m^2$$

- NNLO virtual contribution with adaptive integrand decomposition



$$= \sum_{k=2}^{7} \sum_{i_1 \cdots i_k} \frac{\Delta_{i_1 \cdots i_k}(q_i)}{D_{i_1} \cdots D_{i_k}}$$

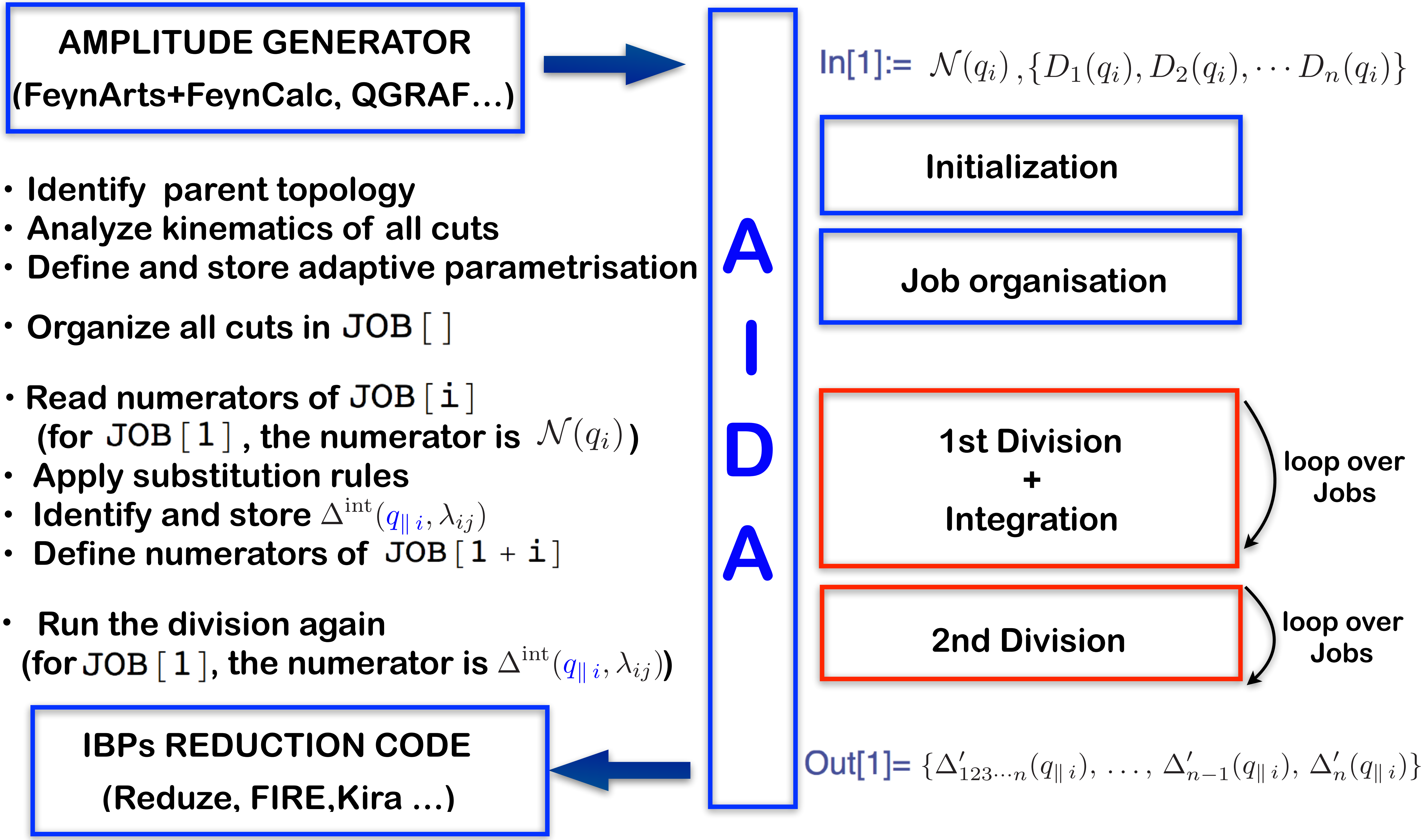
- Residue coefficients are rational functions in 4 variables

$$\Delta_{i_1 \cdots i_k}(q_i) = c_{i_1 \cdots i_k}(s, t, m^2, d) \prod_{i,j} (q_i \cdot p_j)^{\alpha_{ij}}$$

- The calculation requires the automation of the algorithm

AIDA: a Mathematica implementation

Mastrolia, Peraro, A.P., Torres Bobadilla



AMPLITUDE GENERATOR
(FeynArts+FeynCalc, QGRAF…)

- Identify parent topology
- Analyze kinematics of all cuts
- Define and store adaptive parametrisation
- Organize all cuts in `JOB[ ]`
- Read numerators of `JOB[i]` (for `JOB[1]`, the numerator is $\mathcal{N}(q_i)$)
- Apply substitution rules
- Identify and store $\Delta^{\rm int}(q_{\parallel\, i}, \lambda_{ij})$
- Define numerators of `JOB[1 + i]`
- Run the division again (for `JOB[1]`, the numerator is $\Delta^{\rm int}(q_{\parallel\, i}, \lambda_{ij})$)

IBPs REDUCTION CODE
(Reduze, FIRE,Kira …)

In[1]:= $\mathcal{N}(q_i)\,, \{D_1(q_i), D_2(q_i), \cdots D_n(q_i)\}$

Initialization

Job organisation

1st Division + Integration

loop over Jobs

2nd Division

loop over Jobs

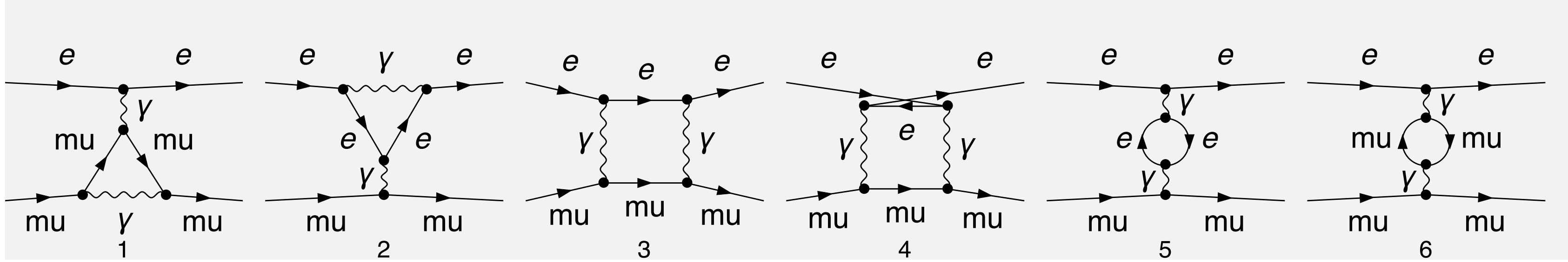
Out[1]= $\{\Delta'_{123\cdots n}(q_{\parallel\, i}),\ \ldots,\ \Delta'_{n-1}(q_{\parallel\, i}),\ \Delta'_{n}(q_{\parallel\, i})\}$

- The code is designed for general one- and two-loop numerical and analytical calculations

Initialization

- **Identify parent topologies from Feynman graphs**

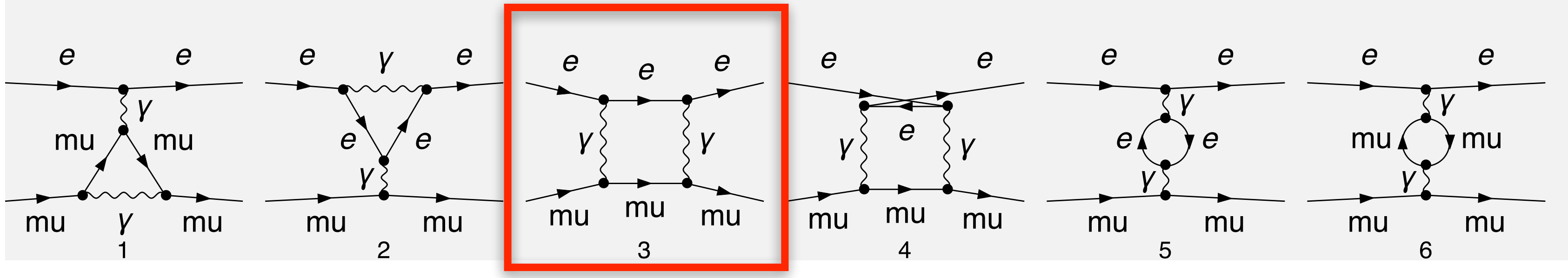
e.g. 1 Loop



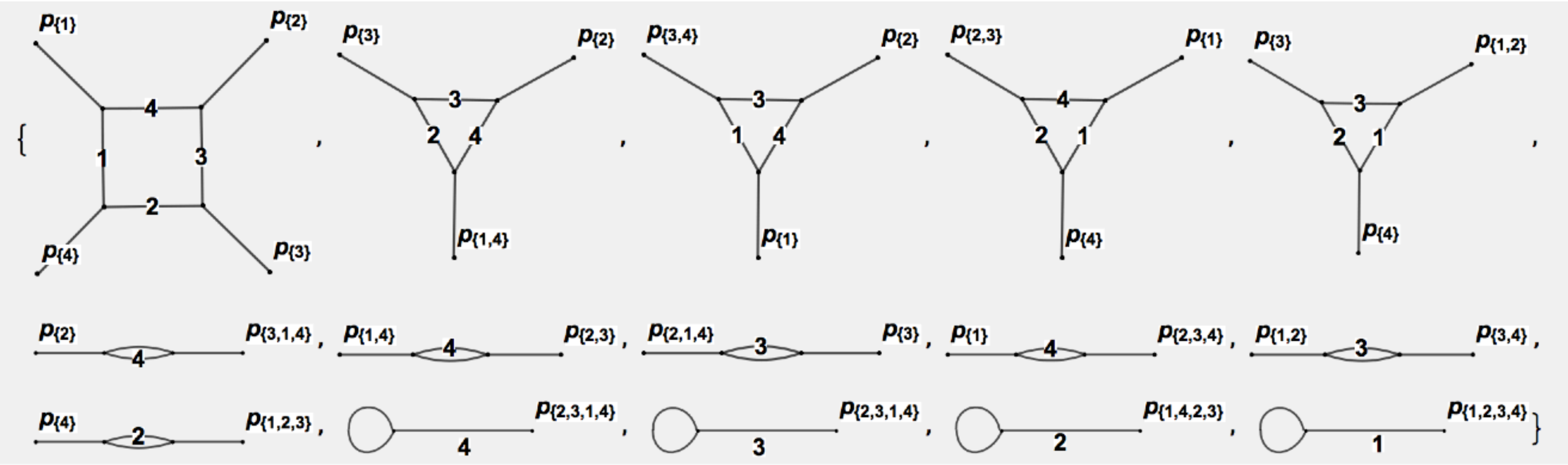
Initialization

- **Identify parent topologies from Feynman graphs**

e.g. 1 Loop



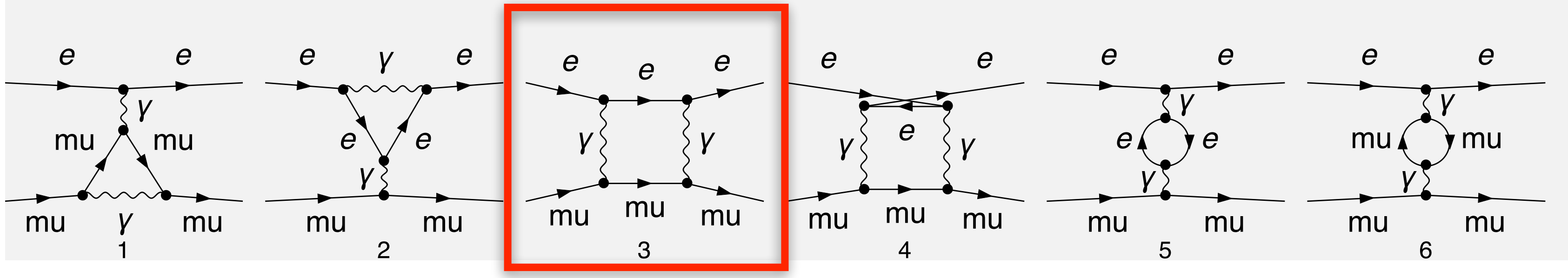
- **Generate all cuts and analyze their kinematics**



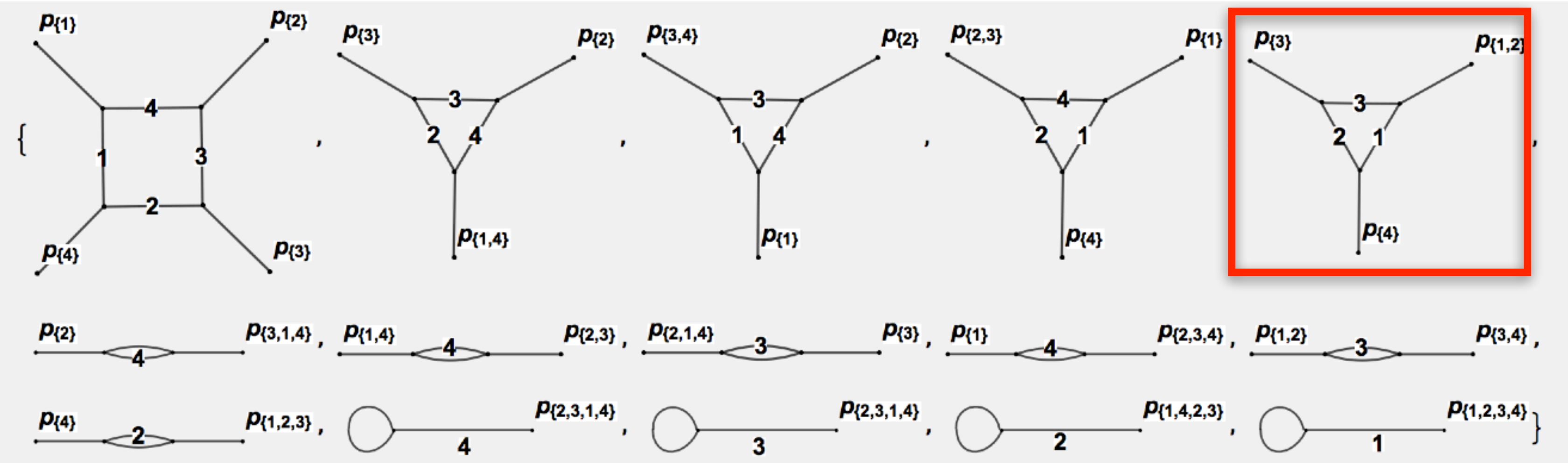
Initialization

- **Identify parent topologies from Feynman graphs**

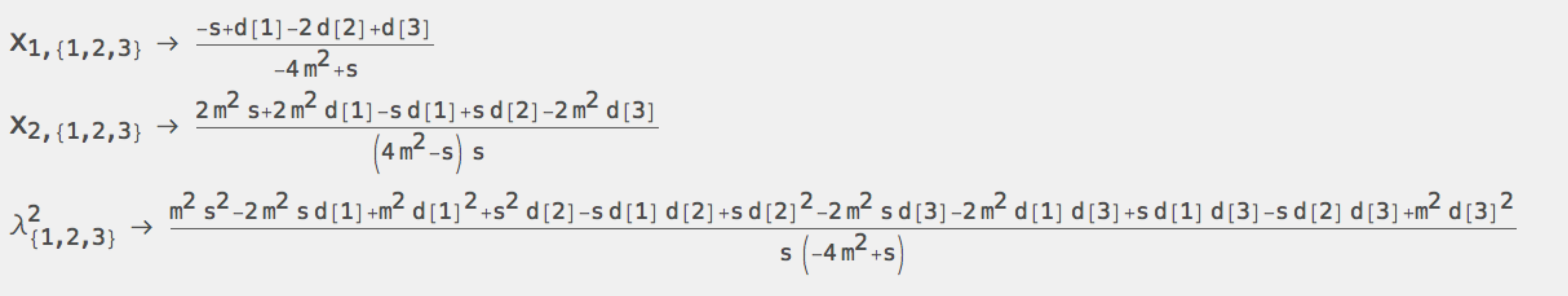
e.g. 1 Loop



- **Generate all cuts and analyse their kinematics**



- **Define adaptive variables and prepare substitution rules for all cuts**



$$x_{1,\{1,2,3\}} \to \frac{-s+d[1]-2\,d[2]+d[3]}{-4\,m^2+s}$$

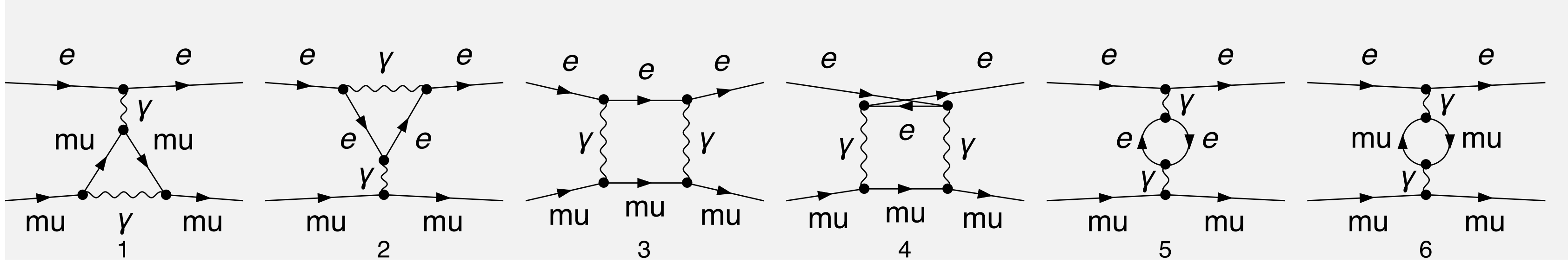
$$x_{2,\{1,2,3\}} \to \frac{2\,m^2\,s+2\,m^2\,d[1]-s\,d[1]+s\,d[2]-2\,m^2\,d[3]}{\left(4\,m^2-s\right)\,s}$$

$$\lambda^2_{\{1,2,3\}} \to \frac{m^2\,s^2-2\,m^2\,s\,d[1]+m^2\,d[1]^2+s^2\,d[2]-s\,d[1]\,d[2]+s\,d[2]^2-2\,m^2\,s\,d[3]-2\,m^2\,d[1]\,d[3]+s\,d[1]\,d[3]-s\,d[2]\,d[3]+m^2\,d[3]^2}{s\,\left(-4\,m^2+s\right)}$$

JOB structure

- **Group diagrams belonging to the same parent topology**

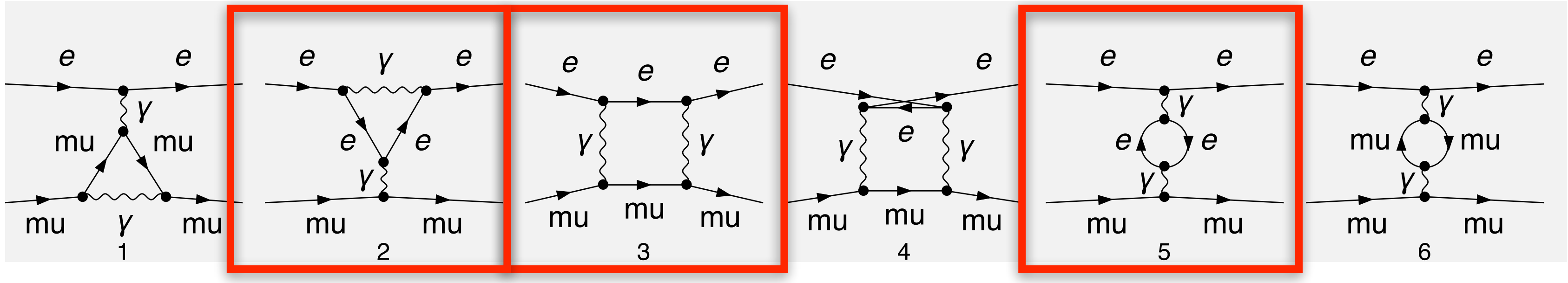
e.g. 1 Loop



JOB structure

- **Group diagrams belonging to the same parent topology**

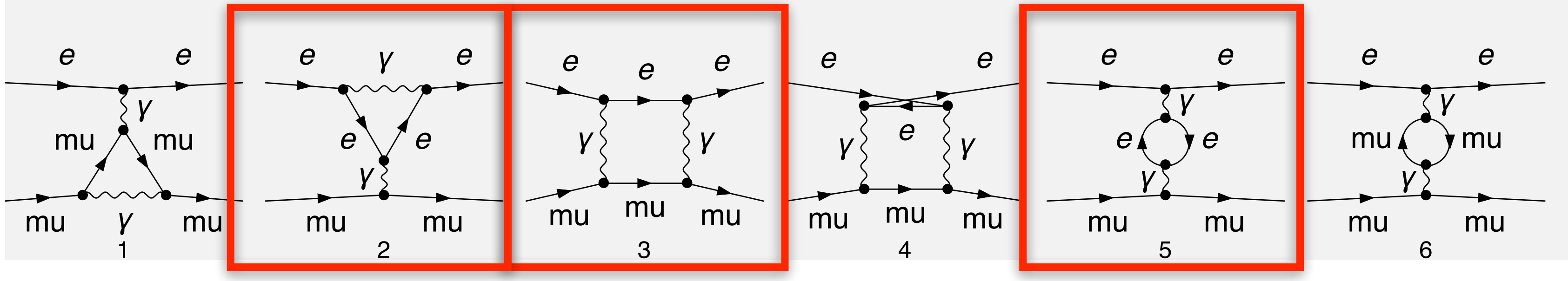
e.g. 1 Loop



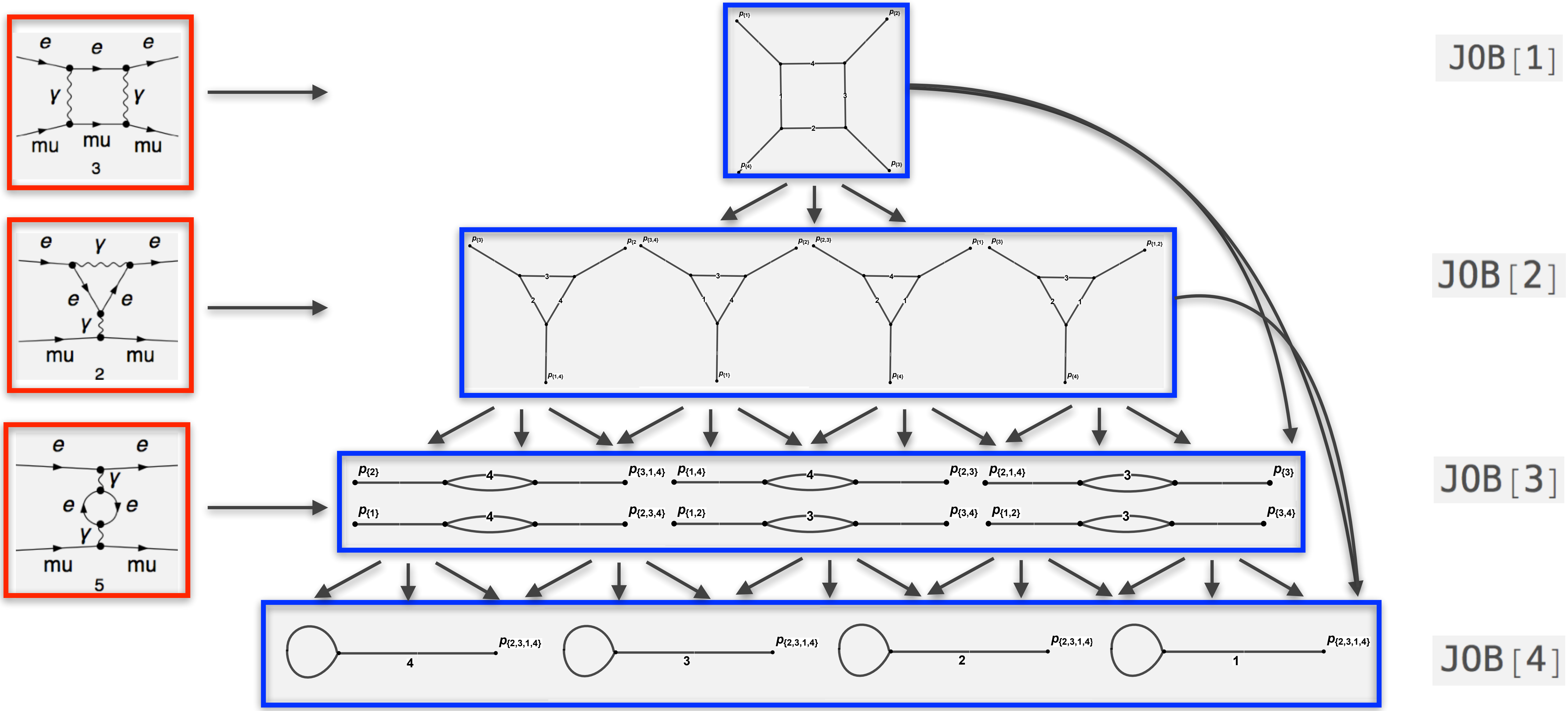
JOB structure

- Group diagrams belonging to the same parent topology

e.g. 1 Loop



- Organize all cuts of the parent topology in Jobs



JOB[1]

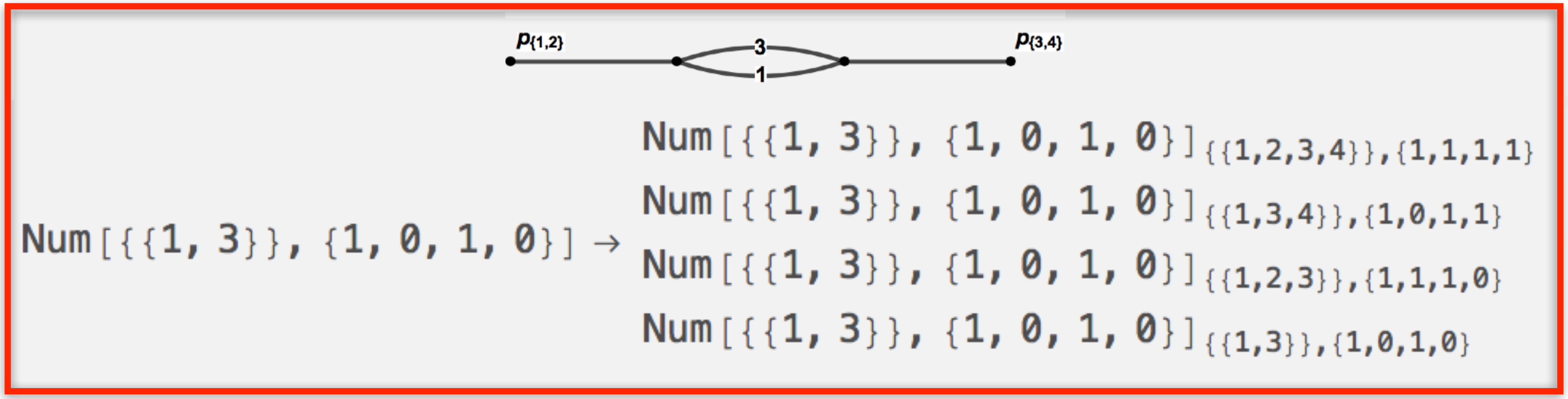
JOB[2]

JOB[3]

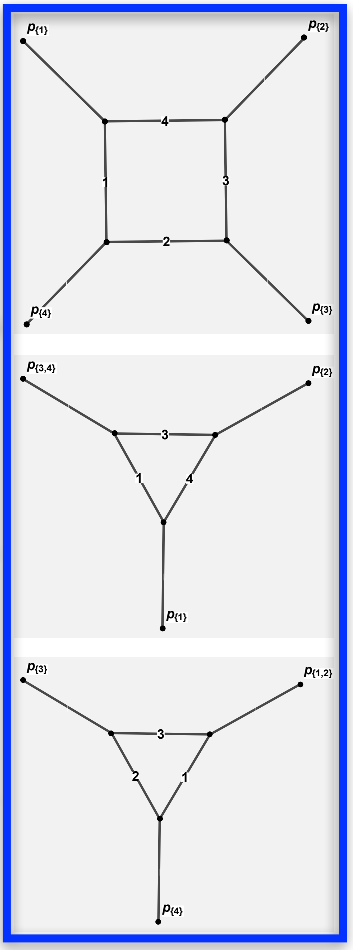
JOB[4]

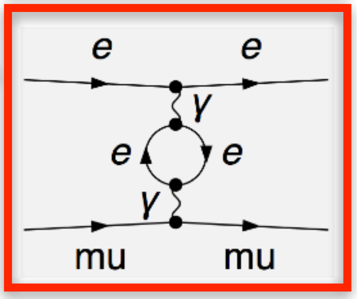
Divide-Integra-Divide

- For every Job, build the numerators of the corresponding cuts

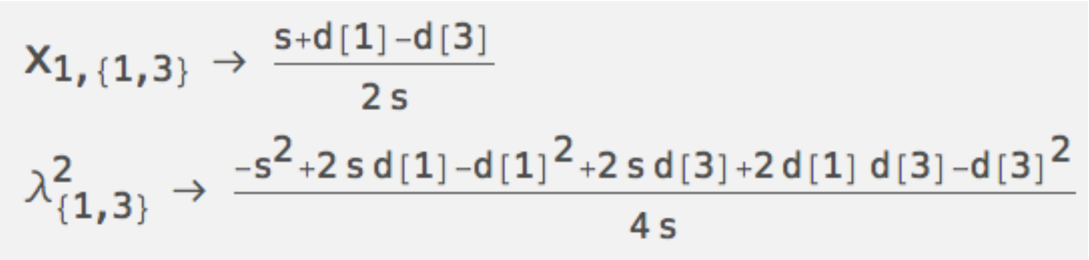


$$\text{Num}[\{\{1,3\}\},\{1,0,1,0\}] \to \begin{array}{l} \text{Num}[\{\{1,3\}\},\{1,0,1,0\}]_{\{\{1,2,3,4\}\},\{1,1,1,1\}} \\ \text{Num}[\{\{1,3\}\},\{1,0,1,0\}]_{\{\{1,3,4\}\},\{1,0,1,1\}} \\ \text{Num}[\{\{1,3\}\},\{1,0,1,0\}]_{\{\{1,2,3\}\},\{1,1,1,0\}} \\ \text{Num}[\{\{1,3\}\},\{1,0,1,0\}]_{\{\{1,3\}\},\{1,0,1,0\}} \end{array}$$





- Apply substitution rules to the numerator



$$X_{1,\{1,3\}} \to \frac{s+d[1]-d[3]}{2\,s}$$

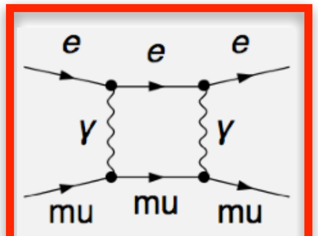
$$\lambda^{2}_{\{1,3\}} \to \frac{-s^{2}+2\,s\,d[1]-d[1]^{2}+2\,s\,d[3]+2\,d[1]\,d[3]-d[3]^{2}}{4\,s}$$

- By collecting powers of denominators read off residue and numerators of lower cuts

- Integrate(=substitute) transverse vars appearing the the residues

- Run the division through all Jobs again, using as input numerators the residues!

Input numerators



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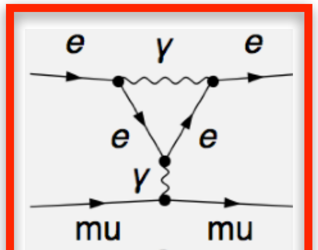

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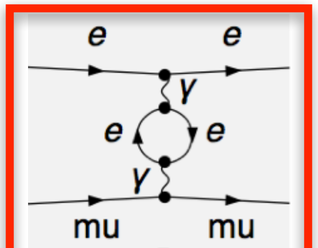

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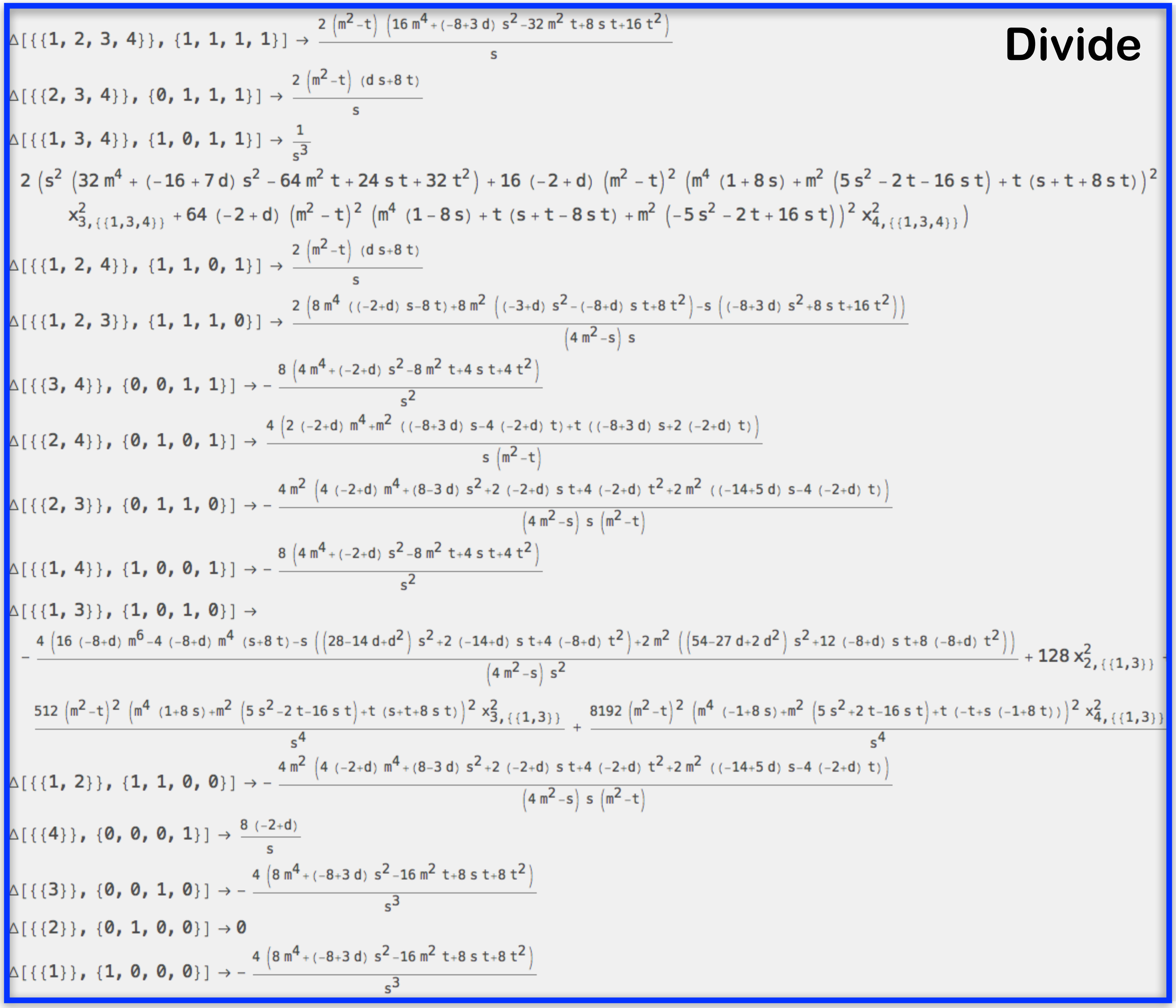


Divide

$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16\,m^4+(-8+3\,d)\,s^2-32\,m^2\,t+8\,s\,t+16\,t^2\right)}{s}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{1}{s^3}$$

$$2\left(s^2\left(32\,m^4+(-16+7\,d)\,s^2-64\,m^2\,t+24\,s\,t+32\,t^2\right)+16\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1+8\,s)+m^2\left(5\,s^2-2\,t-16\,s\,t\right)+t\,(s+t+8\,s\,t)\right)^2 x^2_{3,\{\{1,3,4\}\}}+64\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1-8\,s)+t\,(s+t-8\,s\,t)+m^2\left(-5\,s^2-2\,t+16\,s\,t\right)\right)^2 x^2_{4,\{\{1,3,4\}\}}\right)$$

$$\Delta[\{\{1,2,4\}\},\{1,1,0,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,2,3\}\},\{1,1,1,0\}] \to \frac{2\left(8\,m^4\,((-2+d)\,s-8\,t)+8\,m^2\left((-3+d)\,s^2-(-8+d)\,s\,t+8\,t^2\right)-s\left((-8+3\,d)\,s^2+8\,s\,t+16\,t^2\right)\right)}{\left(4\,m^2-s\right)s}$$

$$\Delta[\{\{3,4\}\},\{0,0,1,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{2,4\}\},\{0,1,0,1\}] \to \frac{4\left(2\,(-2+d)\,m^4+m^2\,((-8+3\,d)\,s-4\,(-2+d)\,t)+t\,((-8+3\,d)\,s+2\,(-2+d)\,t)\right)}{s\left(m^2-t\right)}$$

$$\Delta[\{\{2,3\}\},\{0,1,1,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

$$\Delta[\{\{1,4\}\},\{1,0,0,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{1,3\}\},\{1,0,1,0\}] \to$$

$$-\frac{4\left(16\,(-8+d)\,m^6-4\,(-8+d)\,m^4\,(s+8\,t)-s\left(\left(28-14\,d+d^2\right)s^2+2\,(-14+d)\,s\,t+4\,(-8+d)\,t^2\right)+2\,m^2\left(\left(54-27\,d+2\,d^2\right)s^2+12\,(-8+d)\,s\,t+8\,(-8+d)\,t^2\right)\right)}{\left(4\,m^2-s\right)s^2}+128\,x^2_{2,\{\{1,3\}\}}$$

$$\frac{512\left(m^2-t\right)^2\left(m^4\,(1+8\,s)+m^2\left(5\,s^2-2\,t-16\,s\,t\right)+t\,(s+t+8\,s\,t)\right)^2 x^2_{3,\{\{1,3\}\}}}{s^4}+\frac{8192\left(m^2-t\right)^2\left(m^4\,(-1+8\,s)+m^2\left(5\,s^2+2\,t-16\,s\,t\right)+t\,(-t+s\,(-1+8\,t))\right)^2 x^2_{4,\{\{1,3\}\}}}{s^4}$$

$$\Delta[\{\{1,2\}\},\{1,1,0,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

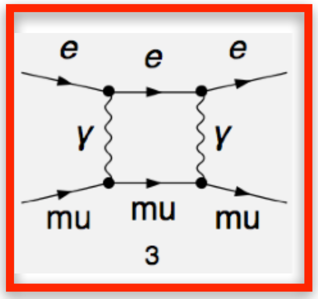
$$\Delta[\{\{4\}\},\{0,0,0,1\}] \to \frac{8\,(-2+d)}{s}$$

$$\Delta[\{\{3\}\},\{0,0,1,0\}] \to -\frac{4\left(8\,m^4+(-8+3\,d)\,s^2-16\,m^2\,t+8\,s\,t+8\,t^2\right)}{s^3}$$

$$\Delta[\{\{2\}\},\{0,1,0,0\}] \to 0$$

$$\Delta[\{\{1\}\},\{1,0,0,0\}] \to -\frac{4\left(8\,m^4+(-8+3\,d)\,s^2-16\,m^2\,t+8\,s\,t+8\,t^2\right)}{s^3}$$

Input numerators



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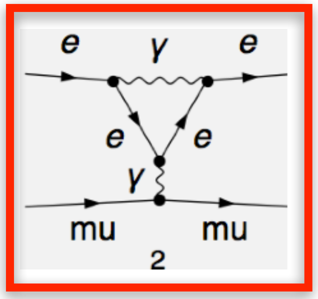

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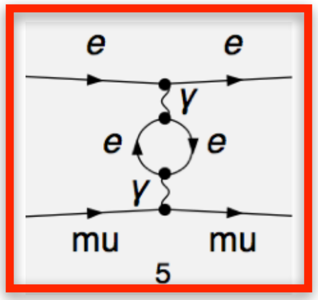

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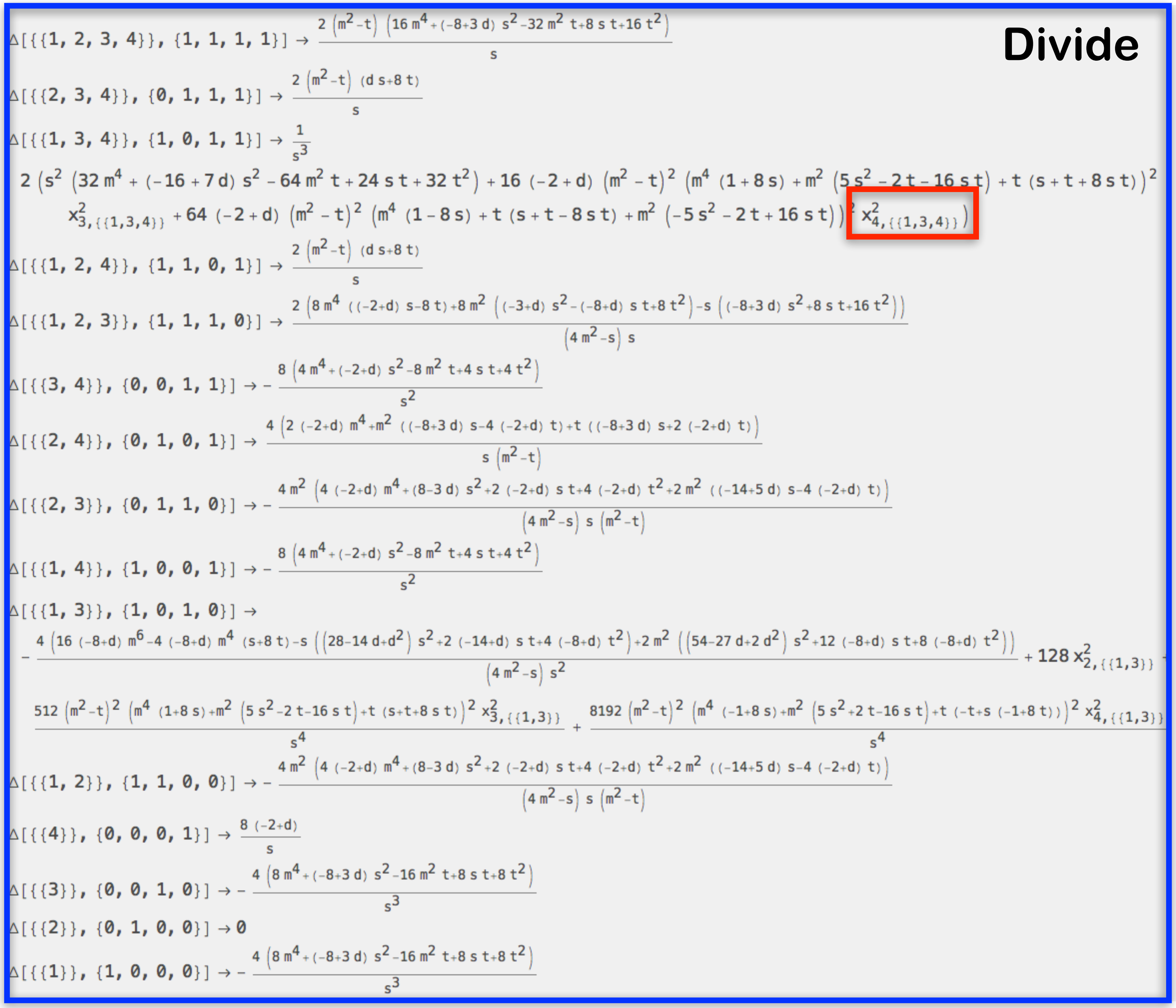


Divide

$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16\,m^4+(-8+3\,d)\,s^2-32\,m^2\,t+8\,s\,t+16\,t^2\right)}{s}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{1}{s^3}$$

$$2\left(s^2\left(32\,m^4+(-16+7\,d)\,s^2-64\,m^2\,t+24\,s\,t+32\,t^2\right)+16\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1+8\,s)+m^2\left(5\,s^2-2\,t-16\,s\,t\right)+t\,(s+t+8\,s\,t)\right)^2 x^2_{3,\{\{1,3,4\}\}}+64\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1-8\,s)+t\,(s+t-8\,s\,t)+m^2\left(-5\,s^2-2\,t+16\,s\,t\right)\right)^2 \boxed{x^2_{4,\{\{1,3,4\}\}}\,}\right)$$

$$\Delta[\{\{1,2,4\}\},\{1,1,0,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,2,3\}\},\{1,1,1,0\}] \to \frac{2\left(8\,m^4\,((-2+d)\,s-8\,t)+8\,m^2\left((-3+d)\,s^2-(-8+d)\,s\,t+8\,t^2\right)-s\left((-8+3\,d)\,s^2+8\,s\,t+16\,t^2\right)\right)}{\left(4\,m^2-s\right)s}$$

$$\Delta[\{\{3,4\}\},\{0,0,1,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{2,4\}\},\{0,1,0,1\}] \to \frac{4\left(2\,(-2+d)\,m^4+m^2\,((-8+3\,d)\,s-4\,(-2+d)\,t)+t\,((-8+3\,d)\,s+2\,(-2+d)\,t)\right)}{s\left(m^2-t\right)}$$

$$\Delta[\{\{2,3\}\},\{0,1,1,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

$$\Delta[\{\{1,4\}\},\{1,0,0,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{1,3\}\},\{1,0,1,0\}] \to$$

$$-\frac{4\left(16\,(-8+d)\,m^6-4\,(-8+d)\,m^4\,(s+8\,t)-s\left(\left(28-14\,d+d^2\right)s^2+2\,(-14+d)\,s\,t+4\,(-8+d)\,t^2\right)+2\,m^2\left(\left(54-27\,d+2\,d^2\right)s^2+12\,(-8+d)\,s\,t+8\,(-8+d)\,t^2\right)\right)}{\left(4\,m^2-s\right)s^2}+128\,x^2_{2,\{\{1,3\}\}}$$

$$\frac{512\left(m^2-t\right)^2\left(m^4\,(1+8\,s)+m^2\left(5\,s^2-2\,t-16\,s\,t\right)+t\,(s+t+8\,s\,t)\right)^2 x^2_{3,\{\{1,3\}\}}}{s^4}+\frac{8192\left(m^2-t\right)^2\left(m^4\,(-1+8\,s)+m^2\left(5\,s^2+2\,t-16\,s\,t\right)+t\,(-t+s\,(-1+8\,t))\right)^2 x^2_{4,\{\{1,3\}\}}}{s^4}$$

$$\Delta[\{\{1,2\}\},\{1,1,0,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

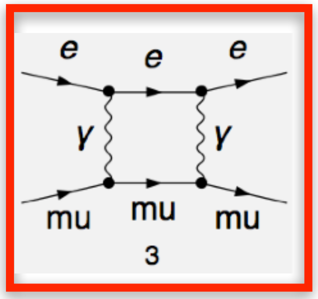
$$\Delta[\{\{4\}\},\{0,0,0,1\}] \to \frac{8\,(-2+d)}{s}$$

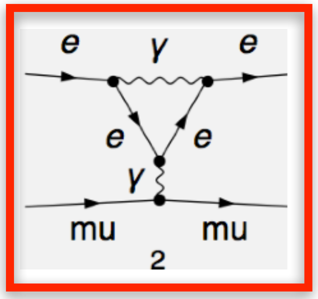
$$\Delta[\{\{3\}\},\{0,0,1,0\}] \to -\frac{4\left(8\,m^4+(-8+3\,d)\,s^2-16\,m^2\,t+8\,s\,t+8\,t^2\right)}{s^3}$$

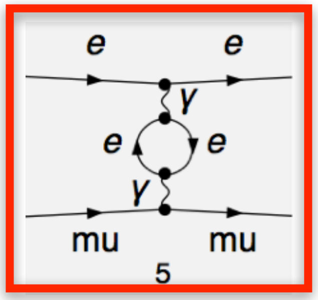
$$\Delta[\{\{2\}\},\{0,1,0,0\}] \to 0$$

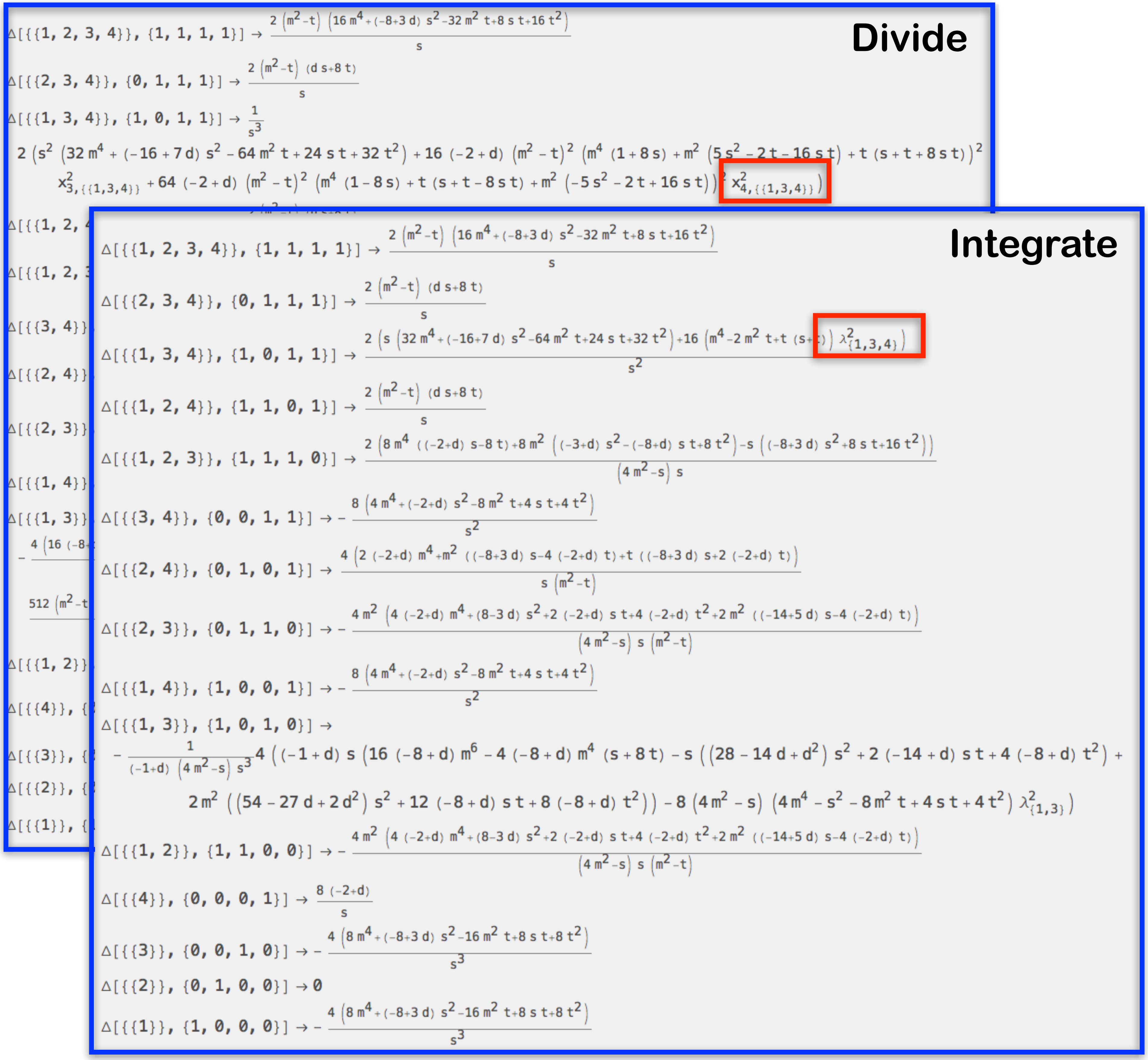
$$\Delta[\{\{1\}\},\{1,0,0,0\}] \to -\frac{4\left(8\,m^4+(-8+3\,d)\,s^2-16\,m^2\,t+8\,s\,t+8\,t^2\right)}{s^3}$$

Input numerators









Divide

$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16m^4+(-8+3d)s^2-32m^2t+8st+16t^2\right)}{s}$

$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(ds+8t)}{s}$

$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{1}{s^3}$

$$2\left(s^2\left(32m^4+(-16+7d)s^2-64m^2t+24st+32t^2\right)+16(-2+d)\left(m^2-t\right)^2\left(m^4(1+8s)+m^2\left(5s^2-2t-16st\right)+t(s+t+8st)\right)^2 x^2_{3,\{\{1,3,4\}\}}+64(-2+d)\left(m^2-t\right)^2\left(m^4(1-8s)+t(s+t-8st)+m^2\left(-5s^2-2t+16st\right)\right)^2 \boxed{x^2_{4,\{\{1,3,4\}\}}}\right)$$

Integrate

$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16m^4+(-8+3d)s^2-32m^2t+8st+16t^2\right)}{s}$

$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(ds+8t)}{s}$

$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{2\left(s\left(32m^4+(-16+7d)s^2-64m^2t+24st+32t^2\right)+16\left(m^4-2m^2t+t(s+t)\right)\boxed{\lambda^2_{\{1,3,4\}}}\right)}{s^2}$

$\Delta[\{\{1,2,4\}\},\{1,1,0,1\}] \to \frac{2\left(m^2-t\right)(ds+8t)}{s}$

$\Delta[\{\{1,2,3\}\},\{1,1,1,0\}] \to \frac{2\left(8m^4((-2+d)s-8t)+8m^2\left((-3+d)s^2-(-8+d)st+8t^2\right)-s\left((-8+3d)s^2+8st+16t^2\right)\right)}{\left(4m^2-s\right)s}$

$\Delta[\{\{3,4\}\},\{0,0,1,1\}] \to -\frac{8\left(4m^4+(-2+d)s^2-8m^2t+4st+4t^2\right)}{s^2}$

$\Delta[\{\{2,4\}\},\{0,1,0,1\}] \to \frac{4\left(2(-2+d)m^4+m^2((-8+3d)s-4(-2+d)t)+t((-8+3d)s+2(-2+d)t)\right)}{s\left(m^2-t\right)}$

$\Delta[\{\{2,3\}\},\{0,1,1,0\}] \to -\frac{4m^2\left(4(-2+d)m^4+(8-3d)s^2+2(-2+d)st+4(-2+d)t^2+2m^2((-14+5d)s-4(-2+d)t)\right)}{\left(4m^2-s\right)s\left(m^2-t\right)}$

$\Delta[\{\{1,4\}\},\{1,0,0,1\}] \to -\frac{8\left(4m^4+(-2+d)s^2-8m^2t+4st+4t^2\right)}{s^2}$

$\Delta[\{\{1,3\}\},\{1,0,1,0\}] \to$

$$-\frac{1}{(-1+d)\left(4m^2-s\right)s^3}4\left((-1+d)s\left(16(-8+d)m^6-4(-8+d)m^4(s+8t)-s\left(\left(28-14d+d^2\right)s^2+2(-14+d)st+4(-8+d)t^2\right)+2m^2\left(\left(54-27d+2d^2\right)s^2+12(-8+d)st+8(-8+d)t^2\right)\right)-8\left(4m^2-s\right)\left(4m^4-s^2-8m^2t+4st+4t^2\right)\lambda^2_{\{1,3\}}\right)$$

$\Delta[\{\{1,2\}\},\{1,1,0,0\}] \to -\frac{4m^2\left(4(-2+d)m^4+(8-3d)s^2+2(-2+d)st+4(-2+d)t^2+2m^2((-14+5d)s-4(-2+d)t)\right)}{\left(4m^2-s\right)s\left(m^2-t\right)}$

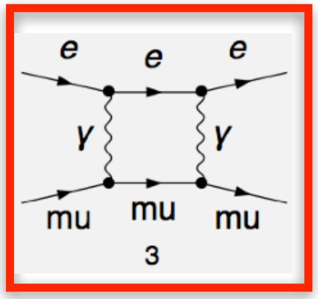
$\Delta[\{\{4\}\},\{0,0,0,1\}] \to \frac{8(-2+d)}{s}$

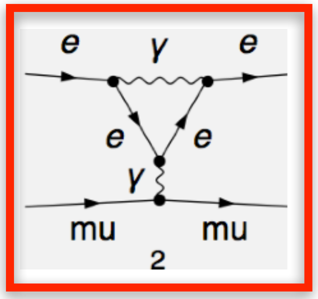
$\Delta[\{\{3\}\},\{0,0,1,0\}] \to -\frac{4\left(8m^4+(-8+3d)s^2-16m^2t+8st+8t^2\right)}{s^3}$

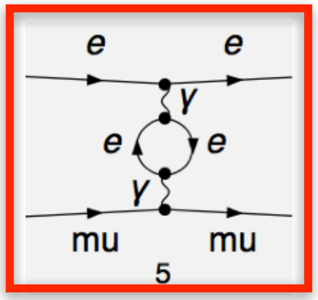
$\Delta[\{\{2\}\},\{0,1,0,0\}] \to 0$

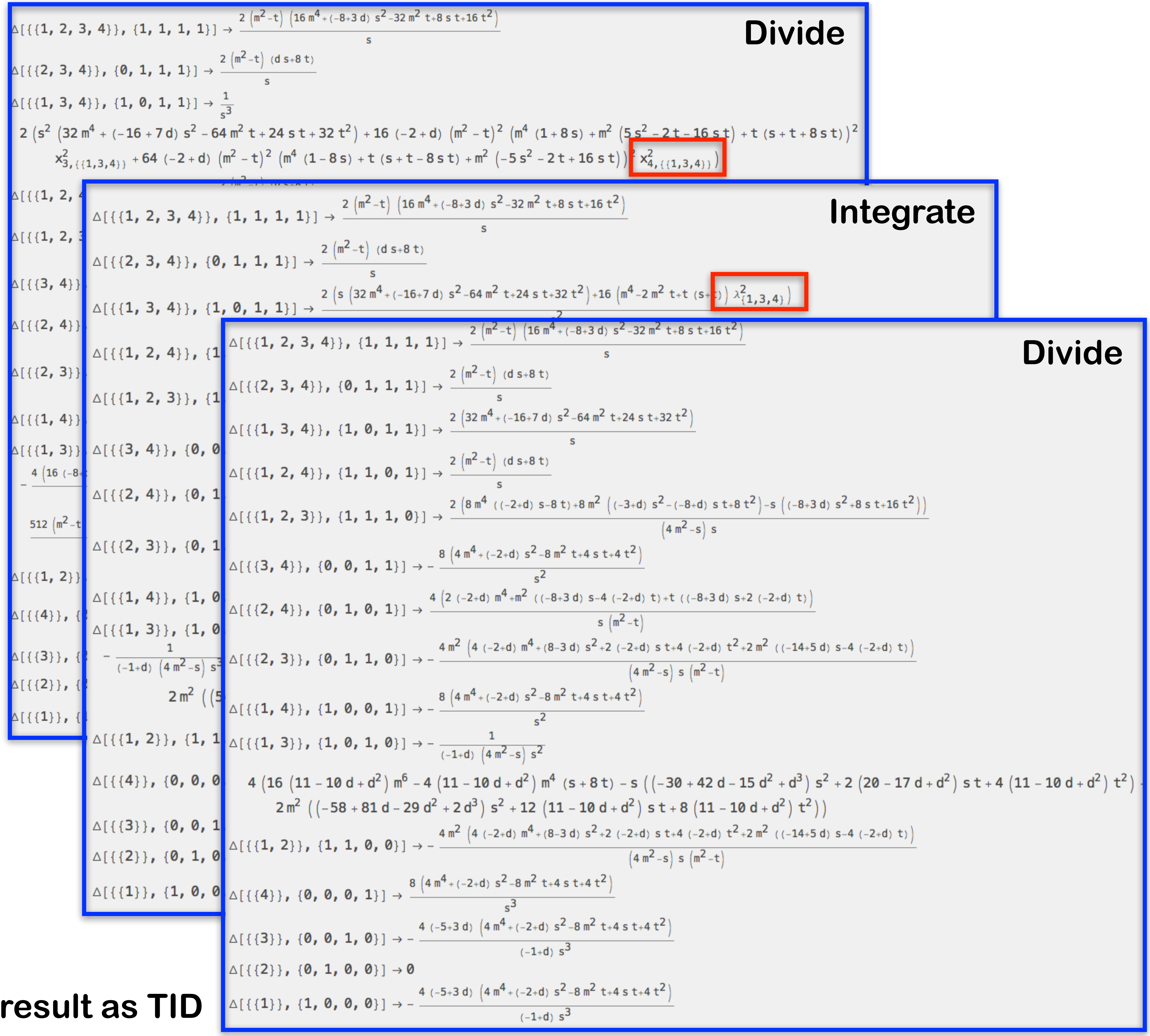
$\Delta[\{\{1\}\},\{1,0,0,0\}] \to -\frac{4\left(8m^4+(-8+3d)s^2-16m^2t+8st+8t^2\right)}{s^3}$

Input numerators









$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16\,m^4+(-8+3\,d)\,s^2-32\,m^2\,t+8\,s\,t+16\,t^2\right)}{s}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{1}{s^3}$$

$$2\left(s^2\left(32\,m^4+(-16+7\,d)\,s^2-64\,m^2\,t+24\,s\,t+32\,t^2\right)+16\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1+8\,s)+m^2\left(5\,s^2-2\,t-16\,s\,t\right)+t\,(s+t+8\,s\,t)\right)^2\right.$$
$$\left.x^2_{3,\{\{1,3,4\}\}}+64\,(-2+d)\left(m^2-t\right)^2\left(m^4\,(1-8\,s)+t\,(s+t-8\,s\,t)+m^2\left(-5\,s^2-2\,t+16\,s\,t\right)\right)^2\boxed{x^2_{4,\{\{1,3,4\}\}}}\right)$$

$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16\,m^4+(-8+3\,d)\,s^2-32\,m^2\,t+8\,s\,t+16\,t^2\right)}{s}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to 2\left(s\left(32\,m^4+(-16+7\,d)\,s^2-64\,m^2\,t+24\,s\,t+32\,t^2\right)+16\left(m^4-2\,m^2\,t+t\,(s+t)\right)\boxed{\lambda^2_{\{1,3,4\}}}\right)$$

$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}] \to \frac{2\left(m^2-t\right)\left(16\,m^4+(-8+3\,d)\,s^2-32\,m^2\,t+8\,s\,t+16\,t^2\right)}{s}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}] \to \frac{2\left(32\,m^4+(-16+7\,d)\,s^2-64\,m^2\,t+24\,s\,t+32\,t^2\right)}{s}$$

$$\Delta[\{\{1,2,4\}\},\{1,1,0,1\}] \to \frac{2\left(m^2-t\right)(d\,s+8\,t)}{s}$$

$$\Delta[\{\{1,2,3\}\},\{1,1,1,0\}] \to \frac{2\left(8\,m^4\,((-2+d)\,s-8\,t)+8\,m^2\left((-3+d)\,s^2-(-8+d)\,s\,t+8\,t^2\right)-s\left((-8+3\,d)\,s^2+8\,s\,t+16\,t^2\right)\right)}{\left(4\,m^2-s\right)s}$$

$$\Delta[\{\{3,4\}\},\{0,0,1,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{2,4\}\},\{0,1,0,1\}] \to \frac{4\left(2\,(-2+d)\,m^4+m^2\,((-8+3\,d)\,s-4\,(-2+d)\,t)+t\,((-8+3\,d)\,s+2\,(-2+d)\,t)\right)}{s\left(m^2-t\right)}$$

$$\Delta[\{\{2,3\}\},\{0,1,1,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

$$\Delta[\{\{1,4\}\},\{1,0,0,1\}] \to -\frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^2}$$

$$\Delta[\{\{1,3\}\},\{1,0,1,0\}] \to -\frac{1}{(-1+d)\left(4\,m^2-s\right)s^2}$$

$$4\left(16\left(11-10\,d+d^2\right)m^6-4\left(11-10\,d+d^2\right)m^4\,(s+8\,t)-s\left(\left(-30+42\,d-15\,d^2+d^3\right)s^2+2\left(20-17\,d+d^2\right)s\,t+4\left(11-10\,d+d^2\right)t^2\right)\right.$$
$$\left.2\,m^2\left(\left(-58+81\,d-29\,d^2+2\,d^3\right)s^2+12\left(11-10\,d+d^2\right)s\,t+8\left(11-10\,d+d^2\right)t^2\right)\right)$$

$$\Delta[\{\{1,2\}\},\{1,1,0,0\}] \to -\frac{4\,m^2\left(4\,(-2+d)\,m^4+(8-3\,d)\,s^2+2\,(-2+d)\,s\,t+4\,(-2+d)\,t^2+2\,m^2\,((-14+5\,d)\,s-4\,(-2+d)\,t)\right)}{\left(4\,m^2-s\right)s\left(m^2-t\right)}$$

$$\Delta[\{\{4\}\},\{0,0,0,1\}] \to \frac{8\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{s^3}$$

$$\Delta[\{\{3\}\},\{0,0,1,0\}] \to -\frac{4\,(-5+3\,d)\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{(-1+d)\,s^3}$$

$$\Delta[\{\{2\}\},\{0,1,0,0\}] \to 0$$

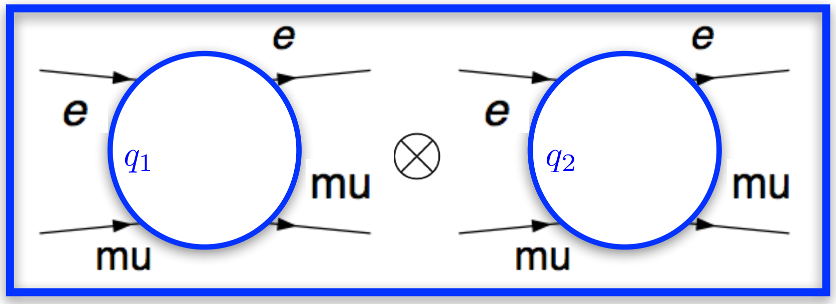
$$\Delta[\{\{1\}\},\{1,0,0,0\}] \to -\frac{4\,(-5+3\,d)\left(4\,m^4+(-2+d)\,s^2-8\,m^2\,t+4\,s\,t+4\,t^2\right)}{(-1+d)\,s^3}$$

@ 1 Loop same result as TID

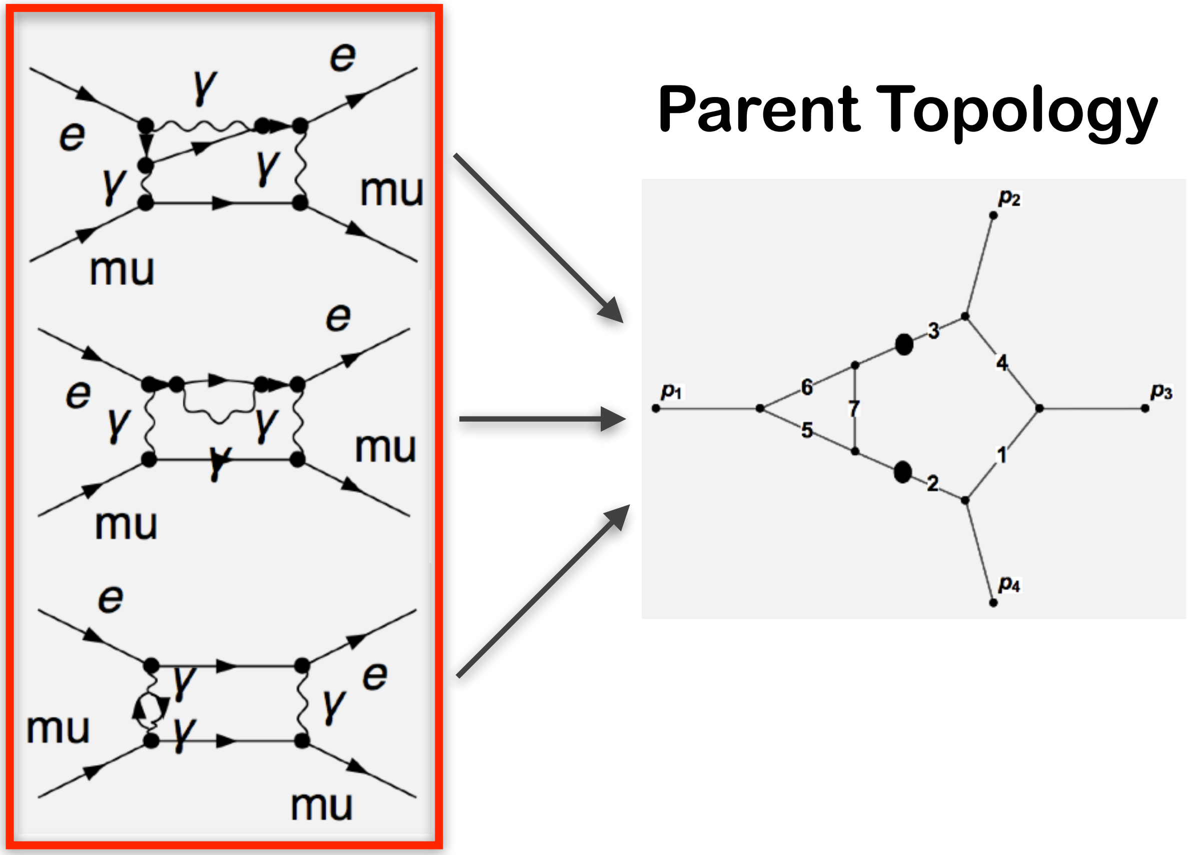
Two loop features

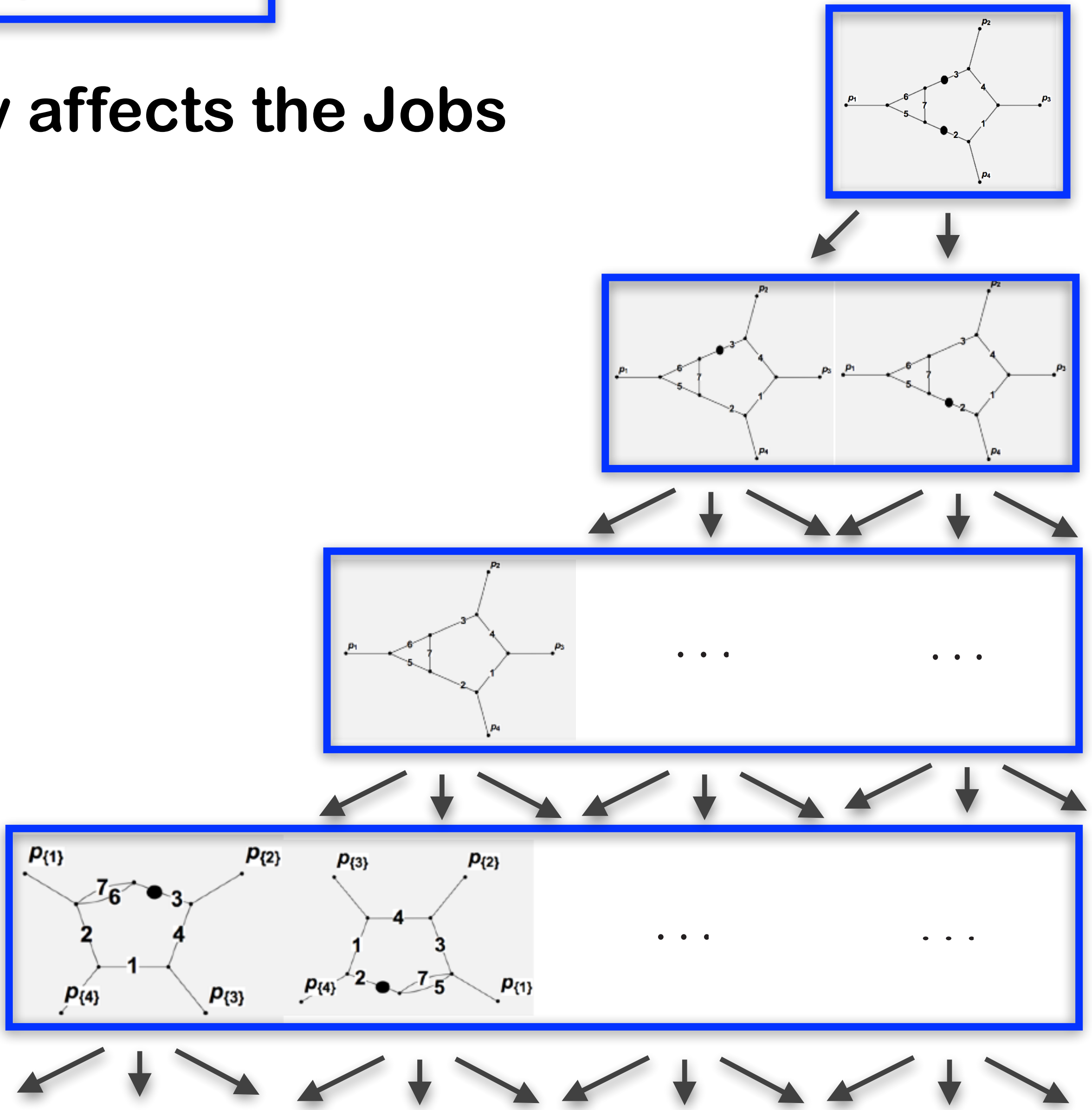
- During initialisation factorized cuts are detected and treated accordingly

same for
1 L x 1L

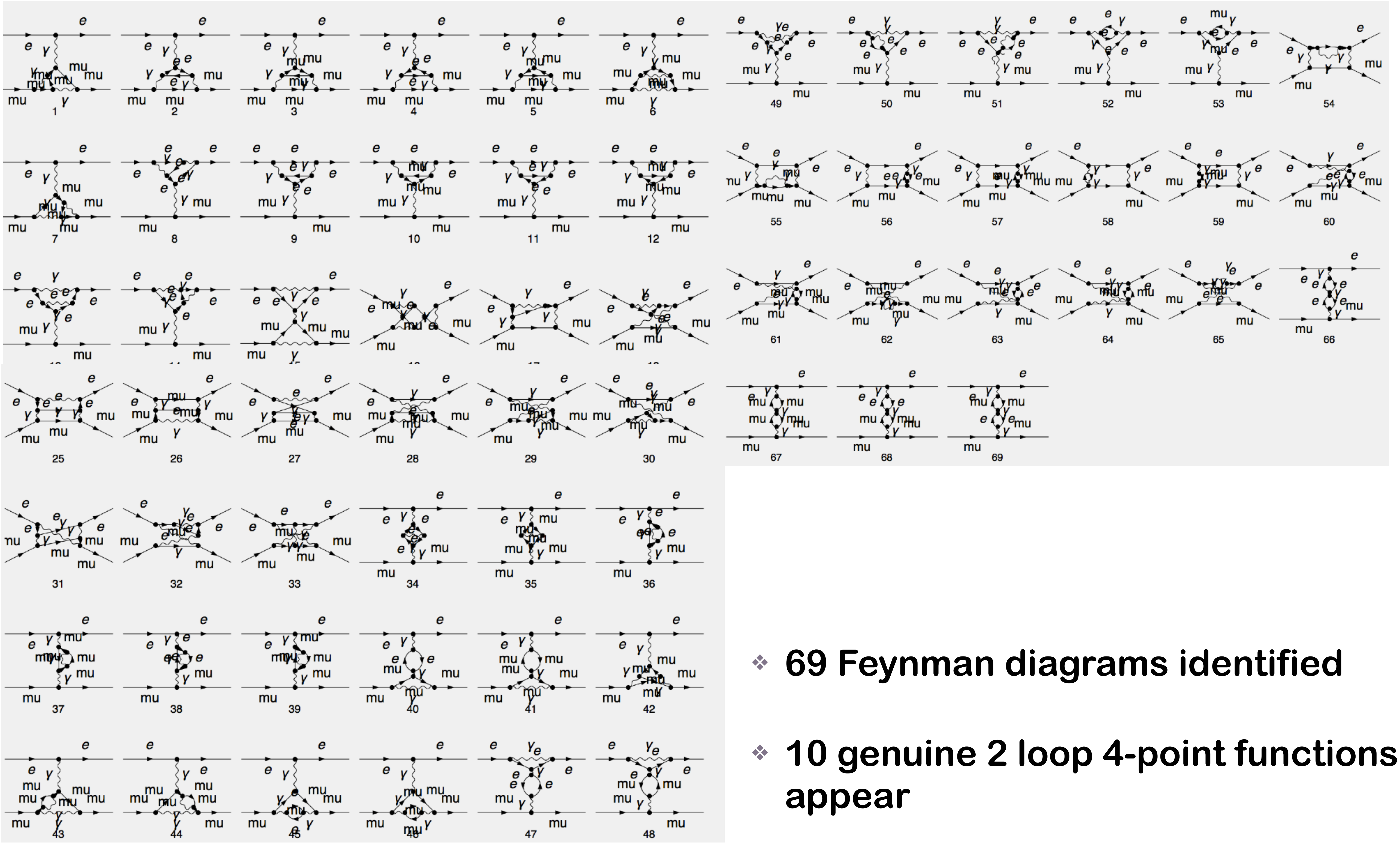


- The presence of squared propagators only affects the Jobs organisation


Parent Topology

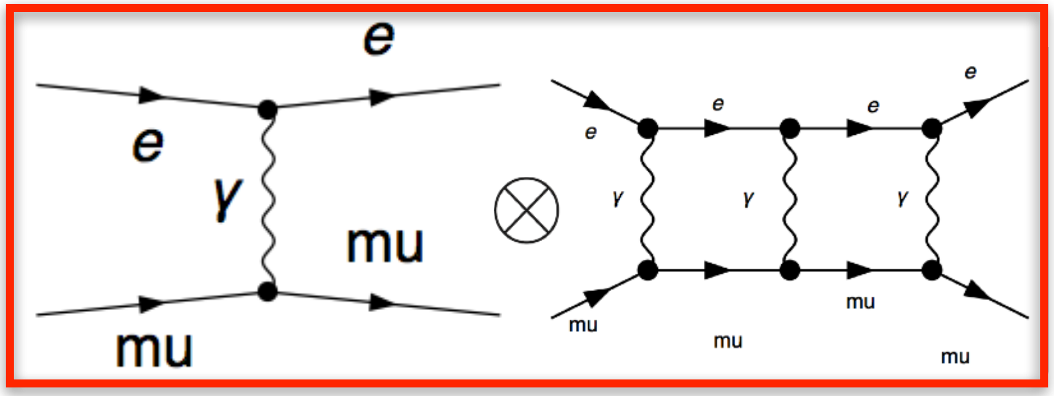


2 loop preliminary results

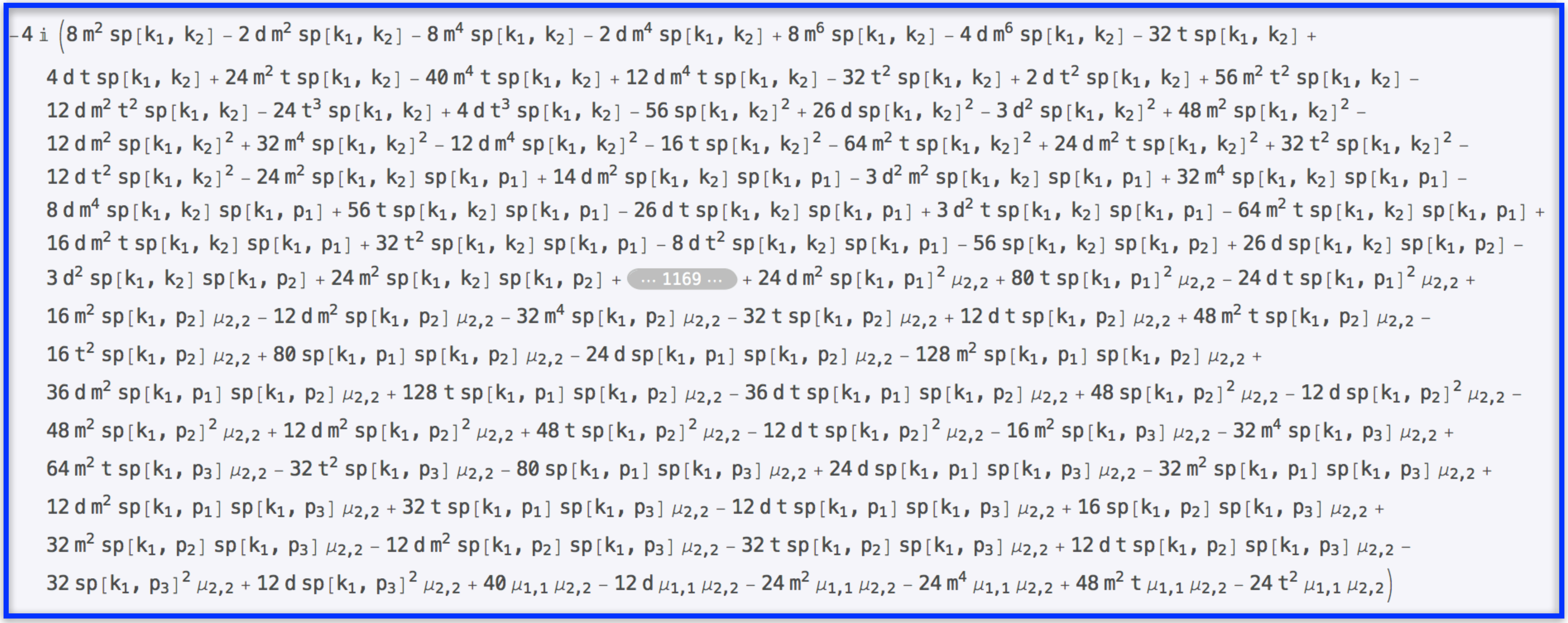


- 69 Feynman diagrams identified
- 10 genuine 2 loop 4-point functions appear

2 loop preliminary results

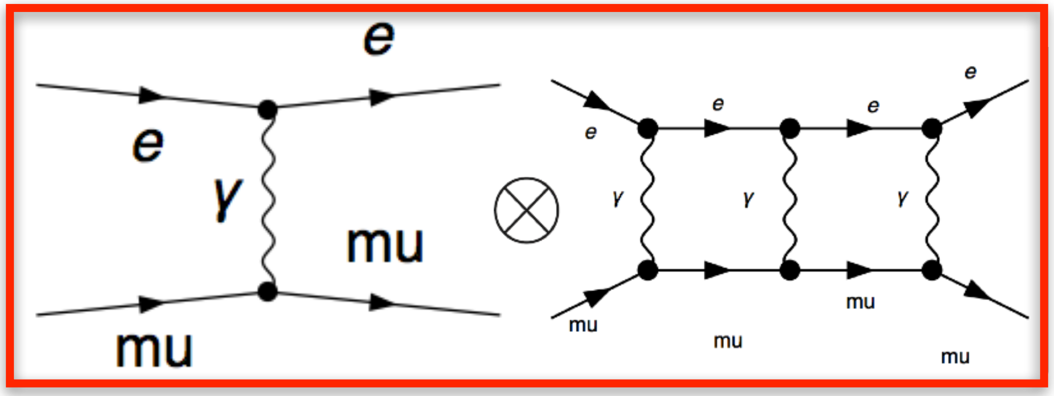


- **Input : rank 4 numerator with 108 monomials in the loop variables**

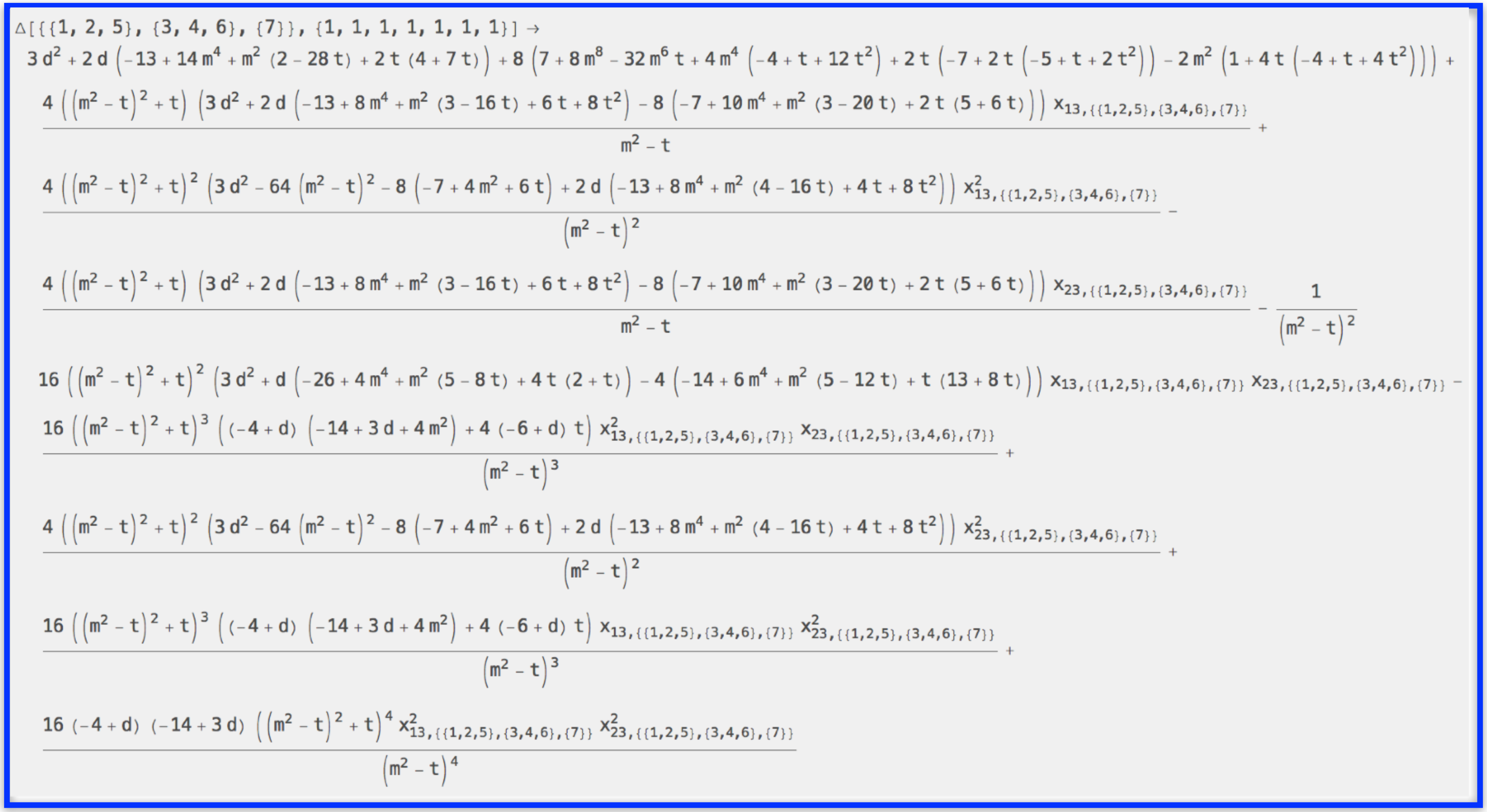


$$\begin{aligned}
&-4\,i\,\big(8\,m^2\,\mathrm{sp}[k_1,k_2]-2\,d\,m^2\,\mathrm{sp}[k_1,k_2]-8\,m^4\,\mathrm{sp}[k_1,k_2]-2\,d\,m^4\,\mathrm{sp}[k_1,k_2]+8\,m^6\,\mathrm{sp}[k_1,k_2]-4\,d\,m^6\,\mathrm{sp}[k_1,k_2]-32\,t\,\mathrm{sp}[k_1,k_2]+\\
&4\,d\,t\,\mathrm{sp}[k_1,k_2]+24\,m^2\,t\,\mathrm{sp}[k_1,k_2]-40\,m^4\,t\,\mathrm{sp}[k_1,k_2]+12\,d\,m^4\,t\,\mathrm{sp}[k_1,k_2]-32\,t^2\,\mathrm{sp}[k_1,k_2]+2\,d\,t^2\,\mathrm{sp}[k_1,k_2]+56\,m^2\,t^2\,\mathrm{sp}[k_1,k_2]-\\
&12\,d\,m^2\,t^2\,\mathrm{sp}[k_1,k_2]-24\,t^3\,\mathrm{sp}[k_1,k_2]+4\,d\,t^3\,\mathrm{sp}[k_1,k_2]-56\,\mathrm{sp}[k_1,k_2]^2+26\,d\,\mathrm{sp}[k_1,k_2]^2-3\,d^2\,\mathrm{sp}[k_1,k_2]^2+48\,m^2\,\mathrm{sp}[k_1,k_2]^2-\\
&12\,d\,m^2\,\mathrm{sp}[k_1,k_2]^2+32\,m^4\,\mathrm{sp}[k_1,k_2]^2-12\,d\,m^4\,\mathrm{sp}[k_1,k_2]^2-16\,t\,\mathrm{sp}[k_1,k_2]^2-64\,m^2\,t\,\mathrm{sp}[k_1,k_2]^2+24\,d\,m^2\,t\,\mathrm{sp}[k_1,k_2]^2+32\,t^2\,\mathrm{sp}[k_1,k_2]^2-\\
&12\,d\,t^2\,\mathrm{sp}[k_1,k_2]^2-24\,m^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+14\,d\,m^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-3\,d^2\,m^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+32\,m^4\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-\\
&8\,d\,m^4\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+56\,t\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-26\,d\,t\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+3\,d^2\,t\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-64\,m^2\,t\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+\\
&16\,d\,m^2\,t\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]+32\,t^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-8\,d\,t^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_1]-56\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_2]+26\,d\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_2]-\\
&3\,d^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_2]+24\,m^2\,\mathrm{sp}[k_1,k_2]\,\mathrm{sp}[k_1,p_2]+\cdots 1169 \cdots+24\,d\,m^2\,\mathrm{sp}[k_1,p_1]^2\,\mu_{2,2}+80\,t\,\mathrm{sp}[k_1,p_1]^2\,\mu_{2,2}-24\,d\,t\,\mathrm{sp}[k_1,p_1]^2\,\mu_{2,2}+\\
&16\,m^2\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-12\,d\,m^2\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-32\,m^4\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-32\,t\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+12\,d\,t\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+48\,m^2\,t\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-\\
&16\,t^2\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+80\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-24\,d\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-128\,m^2\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+\\
&36\,d\,m^2\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+128\,t\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}-36\,d\,t\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_2]\,\mu_{2,2}+48\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}-12\,d\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}-\\
&48\,m^2\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}+12\,d\,m^2\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}+48\,t\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}-12\,d\,t\,\mathrm{sp}[k_1,p_2]^2\,\mu_{2,2}-16\,m^2\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-32\,m^4\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+\\
&64\,m^2\,t\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-32\,t^2\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-80\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+24\,d\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-32\,m^2\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+\\
&12\,d\,m^2\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+32\,t\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-12\,d\,t\,\mathrm{sp}[k_1,p_1]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+16\,\mathrm{sp}[k_1,p_2]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+\\
&32\,m^2\,\mathrm{sp}[k_1,p_2]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-12\,d\,m^2\,\mathrm{sp}[k_1,p_2]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-32\,t\,\mathrm{sp}[k_1,p_2]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}+12\,d\,t\,\mathrm{sp}[k_1,p_2]\,\mathrm{sp}[k_1,p_3]\,\mu_{2,2}-\\
&32\,\mathrm{sp}[k_1,p_3]^2\,\mu_{2,2}+12\,d\,\mathrm{sp}[k_1,p_3]^2\,\mu_{2,2}+40\,\mu_{1,1}\,\mu_{2,2}-12\,d\,\mu_{1,1}\,\mu_{2,2}-24\,m^2\,\mu_{1,1}\,\mu_{2,2}-24\,m^4\,\mu_{1,1}\,\mu_{2,2}+48\,m^2\,t\,\mu_{1,1}\,\mu_{2,2}-24\,t^2\,\mu_{1,1}\,\mu_{2,2}\big)
\end{aligned}$$

2 loop preliminary results



- **Output : 9 double-box integrals + 62 subtopology contributions**



$$
\begin{aligned}
&\Delta[\{\{1,2,5\},\{3,4,6\},\{7\}\},\{1,1,1,1,1,1,1\}]\to\\
&3d^2+2d\left(-13+14m^4+m^2(2-28t)+2t(4+7t)\right)+8\left(7+8m^8-32m^6t+4m^4\left(-4+t+12t^2\right)+2t\left(-7+2t\left(-5+t+2t^2\right)\right)-2m^2\left(1+4t\left(-4+t+4t^2\right)\right)\right)+\\
&\frac{4\left(\left(m^2-t\right)^2+t\right)\left(3d^2+2d\left(-13+8m^4+m^2(3-16t)+6t+8t^2\right)-8\left(-7+10m^4+m^2(3-20t)+2t(5+6t)\right)\right)x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}}{m^2-t}+\\
&\frac{4\left(\left(m^2-t\right)^2+t\right)^2\left(3d^2-64\left(m^2-t\right)^2-8\left(-7+4m^2+6t\right)+2d\left(-13+8m^4+m^2(4-16t)+4t+8t^2\right)\right)x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2}{\left(m^2-t\right)^2}-\\
&\frac{4\left(\left(m^2-t\right)^2+t\right)\left(3d^2+2d\left(-13+8m^4+m^2(3-16t)+6t+8t^2\right)-8\left(-7+10m^4+m^2(3-20t)+2t(5+6t)\right)\right)x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}}{m^2-t}-\frac{1}{\left(m^2-t\right)^2}\\
&16\left(\left(m^2-t\right)^2+t\right)^2\left(3d^2+d\left(-26+4m^4+m^2(5-8t)+4t(2+t)\right)-4\left(-14+6m^4+m^2(5-12t)+t(13+8t)\right)\right)x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}\,x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}-\\
&\frac{16\left(\left(m^2-t\right)^2+t\right)^3\left((-4+d)\left(-14+3d+4m^2\right)+4(-6+d)t\right)x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2\,x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}}{\left(m^2-t\right)^3}+\\
&\frac{4\left(\left(m^2-t\right)^2+t\right)^2\left(3d^2-64\left(m^2-t\right)^2-8\left(-7+4m^2+6t\right)+2d\left(-13+8m^4+m^2(4-16t)+4t+8t^2\right)\right)x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2}{\left(m^2-t\right)^2}+\\
&\frac{16\left(\left(m^2-t\right)^2+t\right)^3\left((-4+d)\left(-14+3d+4m^2\right)+4(-6+d)t\right)x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}\,x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2}{\left(m^2-t\right)^3}+\\
&\frac{16(-4+d)(-14+3d)\left(\left(m^2-t\right)^2+t\right)^4x_{13,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2\,x_{23,\{\{1,2,5\},\{3,4,6\},\{7\}\}}^2}{\left(m^2-t\right)^4}
\end{aligned}
$$

Conclusions

- Integrand decomposition is an effective tool for computation beyond NLO
- Adaptive parametrization of Feynman integrands streamlines the algorithm
- The first working code for integrand decomposition at 2 loops is available (still some work to do!)
- Muon-electron scattering at NNLO is at hand, more processes to come in the future