Feynman Integrals beyond multiple polylogarithms

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TTP - KIT, Karlsruhe

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Based on collaboration with *A. von Manteuffel*, *A. Primo*, *E. Remiddi*

[arXiv:1602.01481], [arXiv:1610.08397], [arXiv:1701.05905], [arXiv:1704.05465], [arXiv:17yy.xxxxx]

Differential equations method

[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,..., C. Papadopoulos '14]

$\Downarrow$

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, \ldots, N$.

$\Downarrow$

Differentiating the masters and using the **IBPs** we get a system of N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

Let's look more in detail - *we should recall* that equations are in block form

$$I_j(d;x_k) = (m_j(d;x_k)\,,\quad sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is MAIN source of complexity - whether differential equations are coupled

$$\Downarrow$$

No way to solve this in general...
We must use some other "physical" insight...

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non-homogeneous piece is the second source of complexity – we must integrate over it!

$$\Downarrow$$

Can be symplified using differential equations and dispersion relations [E.Remiddi, LT '16]

We are interested in computing the integrals as Laurent series in $(d-4)$, which requires integrating iteratively on the homogeneous solution and on the non-homogeneous piece.

There are **two possibilities**

1- The differential equations are not coupled for $d \to 4$. Order by order we need to solve first order equations with rational coefficients

2- The differential equations are coupled for $d \to 4$. Case 2×2 under study since some time, example is **two loop massive Sunrise**

$$\Downarrow$$

Of course, higher order couplings are possible, $n \times n$ with $n > 2$

First case is "simple" (conceptually, often not in practice!), solution **naturally expressed** in terms of so-called **multiple polylogarithms** [E.Remiddi, J.Vermaseren '99; T. Gehrmann, E.Remiddi '00; Goncharov et al '00; Duhr, Gangl, Rhodes '13; ...]

$$G(0;x) = \ln(x)\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right) \quad \text{for} \quad a \neq 0$$

$$G(\underbrace{0,\ldots,0}_{n};x) = \frac{1}{n!}\ln^n(x)\,, \qquad G(a,\vec{w};x) = \int_0^x \frac{dy}{y-a}\, G(\vec{w};y)\,.$$

$$\Downarrow$$

Multiple polylogarithms are *special*:
they become simpler under differentiation → it decreases weight!

$$\frac{d}{dx} G(a,\vec{w};x) = \frac{1}{x-a} G(\vec{w};x) \qquad \rightarrow \qquad \frac{d}{dx} G(x) = \frac{d}{dx} 1 = 0 \ !$$

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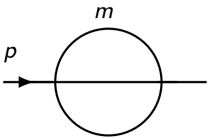
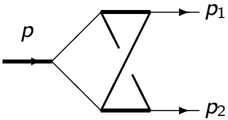
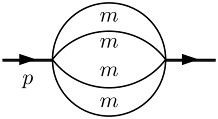
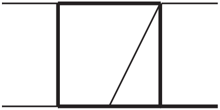
What lies beyond? → We know a couple of examples now









What all these examples have **in common** is a bulk 2×2 (or 3×3) irreducible system of differential equations

What do we have to do?

1- Solve the **homogeneous equations** in the limit $d \to 4$ (or $d \to 2n$, $n \in \mathbb{N}$)

$$\frac{d}{dx}\vec{I}(d;x) = A(x)\vec{I}(d;x) + (d-4)B(x)\vec{I}(d;x) + \mathcal{O}(d-4)^2\,,$$

with $A(x)$ $n \times n$, non-triangular!

Find $n \times n$ matrix homogeneous solutions $G(x)$, with

$$\frac{d}{dx}G(x) = A(x)G(x)\,, \qquad \to \qquad \vec{I}(d;x) = G(x)\vec{m}(d;x)$$

then

$$\frac{d}{dx}\vec{m}(d;x) = (d-4)G^{-1}(x)\,B(x)G(x)\,\vec{m}(d;x) + \mathcal{O}(d-4)^2\,,$$

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2- Solution given by iterative integrals over complicated kernels that contain **products of the homogeneous solutions**, and previous orders

By expanding in $(d-4)$:

$$\vec{m}^{\,[n]}(x) = \int^x dy\, G^{-1}(y)\, B(y) G(y)\, \vec{m}^{\,[n-1]}(y) \; + \; \text{simpler terms}\,,$$

Or equivalently for the original functions

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How to solve the homogeneous equation?

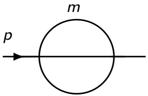
Given the equations, there is no general way... this **was** a bottleneck

Solution:

- Take an older idea by [S.Laporta, E.Remiddi '04]

- Generalize it to all cases [A.Primo, L.Tancredi '16, '17]

$$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right)$$



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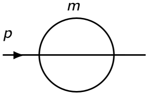

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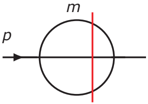
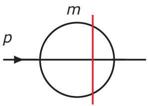
Cut $\rightarrow$ $$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right)$$



$$= 0$$

<u>Maximal cut</u> solves homogeneous differential equations

[A.Primo, L.Tancredi '16, '17]



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind.

$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-x\,t^2)}}$$

Computation of the maximal cut can be simplified in **Baikov representation**
[Papadopoulos, Frellesvig '17; Bosma, Sogaard, Zhang '17;
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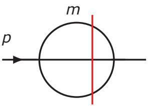

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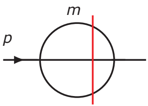

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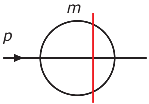
How do we get the other solutions?

There is **not only one independent contour**! Other solutions found *integrating on the other independent contours*!
[Bosma, Sogaard, Zhang '17; Tancredi, Primo '17; Harley, Moriello, Schabinger '17]



$$= \oint_C \mathfrak{D}^d k\, \mathfrak{D}^d l\, \delta(k^2 - m^2)\, \delta(l^2 - m^2)\, \delta((k - l - p)^2 - m^2)$$

where C is *some contour* on the complex plane.

$$\mathrm{sol}_1(p, m) = \oint_{\mathcal{C}_1} = \frac{\mathrm{K}\,(w)}{\sqrt{(3m - \sqrt{s})(\sqrt{s} + m)^3}}\,,$$

$$\mathrm{sol}_2(p, m) = \oint_{\mathcal{C}_2} = \frac{\mathrm{K}\,(1 - w)}{\sqrt{(3m - \sqrt{s})(\sqrt{s} + m)^3}}\,, \qquad w = \frac{16m^3\sqrt{s}}{(3m - \sqrt{s})(\sqrt{s} + m)^3}$$

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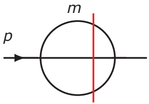

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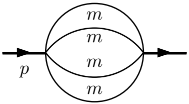
It is very general. A nice tool to simplify the solution of differential equations, particularly useful when solution requires **elliptic integrals**, but not only!

$\Downarrow$

Application to a 3×3 example, "**beyond just elliptic integrals**"



Has three master integrals : $\mathcal{I}_1(\epsilon; s)$ $\mathcal{I}_2(\epsilon; s)$ $\mathcal{I}_3(\epsilon; s)$

$$\frac{d}{dx}\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} = B(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \epsilon\, D(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -\frac{1}{2(4x-1)}\end{pmatrix}$$

where $B(x)$ and $D(x)$ are 3×3 matrices, with $x = 4m^2/p^2$

$$B(x) = \begin{pmatrix} \frac{1}{x} & \frac{4}{x} & 0 \\ -\frac{1}{4(x-1)} & \frac{1}{x}-\frac{2}{x-1} & \frac{3}{x}-\frac{3}{x-1} \\ \frac{1}{8(x-1)}-\frac{1}{8(4x-1)} & \frac{1}{x-1}-\frac{3}{2(4x-1)} & \frac{1}{x}-\frac{6}{4x-1}+\frac{3}{2(x-1)} \end{pmatrix}$$

$$D(x) = \begin{pmatrix} \frac{3}{x} & \frac{12}{x} & 0 \\ -\frac{1}{x-1} & \frac{2}{x}-\frac{6}{x-1} & \frac{6}{x}-\frac{6}{x-1} \\ \frac{1}{2(x-1)}-\frac{1}{2(4x-1)} & \frac{3}{x-1}-\frac{9}{2(4x-1)} & \frac{1}{x}-\frac{12}{4x-1}+\frac{3}{x-1} \end{pmatrix}$$

We need to find now three independent solutions, i.e. a matrix

$$G(x) = \begin{pmatrix} H_1(x) & J_1(x) & I_1(x) \\ H_2(x) & J_2(x) & I_2(x) \\ H_3(x) & J_3(x) & I_3(x) \end{pmatrix} \qquad \to \qquad \frac{d}{dx} G(x) = B(x)\, G(x)\,.$$

Or, if our idea is correct, there should exists **three independent** integration contours $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ such that (for $\epsilon = 0$)

$$G(x) = \begin{pmatrix} \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_1(x)) \\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_2(x)) \\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_3(x)) \end{pmatrix}$$

Interestingly enough, with some effort, and following:
[Bailey, Borwein, Broadhurst '08]

$$H_1(x) = x\, \mathrm{K}\left(k_+^2\right) \mathrm{K}\left(k_-^2\right), \qquad J_1(x) = x\, \mathrm{K}\left(k_+^2\right) \mathrm{K}\left(1-k_-^2\right),$$
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Remaining rows of the matrix $G(x)$ can be obtained by differentiation.

Result expected from studies of Joyce '73 on cubic lattice Green functions
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We have powerful method to write **iterated integral representations** for solutions. How do we handle the functions now?

1- In the sunrise case, progress on **Elliptic polylogarithms** [Brown, Levin '11; Bloch, Vanhove '13,'14 ; Weinzierl et al, '14,'15,'16...]

2- Can we say "something general", which applies to all cases and allows us to handle these functions? [Remiddi, Tancredi '17 (soon...?)]

What is the difference with Polylogs?

$$\frac{d}{dx} G^{[n+1]}(a, \vec{w}, x) = \frac{1}{x-a} G^{[n]}(\vec{w}, x) \,, \qquad G^{[0]}(x) = 1 \quad \to \quad \frac{d}{dx} G^{[0]}(x) = 0$$

We can see them as **iterative integrations** over rational functions with the solution of the homogeneous equation which, properly normalized, is a trivial kernel $K = 1$. The fundamental property is $d/dx\, K = 0$.

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This *generalizes* as follows:

New kernel is again $\sim$ **solution of the homogeneous equation**, for 2×2

$$G(u) = \begin{pmatrix} I_0(u) & J_0(u) \\ I_2(u) & J_2(u) \end{pmatrix} \quad \to \quad \left(\frac{d}{du} - B(u)\right) G(u) = 0$$

Or alternatively rephrased as

$$D\left(\frac{d}{du}, u\right) I_0(u) = \left(\frac{d^2}{du^2} + A_1(u)\frac{d}{du} + A_2(u)\right) I_0(u) = 0\,.$$

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(And in principle similarly for higher order equations)

New functions at iteration n, let's call them for now $\mathrm{EI}_k^{[n]}(\vec{w}, u)$, $k = 0, 2$

At first sight, they **do not have a simple concept of weight** since

$$\frac{d}{du}\mathrm{EI}_k^{[n]}(\vec{w}, u) = \sum_{j=0,2} c_j(u)\,\mathrm{EI}_j^{[n]}(\vec{w}, u)$$

But one finds (a posteriori it is obvious...)

$$D\left(\frac{d}{du}, u\right)\mathrm{EI}_k^{[n]}(\vec{w}, u) = \sum_{j=0,2} c_j^{[n-1]}(u)\,\mathrm{EI}_j^{[n-1]}(\vec{w}, u) + \sum_{j=0,2} c_j^{[n-2]}(u)\,\mathrm{EI}_j^{[n-2]}(\vec{w}, u)$$

They **do have** a concept of weight w.r.t. the second order operator $D(d/du, u)$
It lowers their weight!
It can be used to study them **bottom up**, like polylogs (find relations, rewrite them in terms of other functions, etc)!

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Let's see an example of these functions

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db}{\sqrt{R_4(b,u)}} G^{[n]}(\vec{w};b)$$

with $\quad R_4(b,u) = b(b-4m^2)(b-(\sqrt{u}-m)^2)(b-(\sqrt{u}+m)^2)$
alphabet $\vec{w}$ drawn from roots of $R_4(b,u)$

(a subset appears in **imaginary part** of two-loop sunrise graph)

At **weight zero**

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db}{\sqrt{R_4(b,u)}} = I_0(u)$$

At **weight one**

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db}{\sqrt{R_4(b,u)}} \left\{ \begin{array}{c} \ln(b) \\ \ln(b-4m^2) \\ \ln(b-(\sqrt{u}-m)^2) \\ \ln(b-(\sqrt{u}+m)^2) \end{array} \right\} = ??$$

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We know everything about **weight zero**, $I_0(u) = K(x)$, elliptic integrals, ...

Discover relations at **weight one:**

$$\mathrm{EI}_0^{[1]}(0,u) = \int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db\,\ln b}{\sqrt{R_4(b,u)}}$$

$$D\left(\frac{d}{du},u\right)\ \mathrm{EI}_0^{[1]}(0,u) = \frac{1}{m^2}\left(-\frac{8}{9u}+\frac{3}{4(u-m^2)}+\frac{5}{36(u-9m^2)}-\frac{4m^2}{3(u-m^2)^2}\right)I_0(u)$$
$$+\frac{1}{m^6}\left(\frac{2}{9u}-\frac{7}{32(u-m^2)}-\frac{1}{288(u-9m^2)}+\frac{4m^2}{(u-m^2)^2}\right)I_2(u)$$

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This implies a sort of (half-) **shuffle relation**

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db\, \ln b}{\sqrt{R_4(b,u)}} = \frac{2}{3}\ln\left(u-m^2\right) I_0(u) + c_1 I_0(u) + c_2 J_0(u)\,.$$

Fixing the **boundary conditions** we finally have

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db\, \ln b}{\sqrt{R_4(b,u)}} = \frac{2}{3}\ln\left(u-m^2\right) I_0(u)\,.$$

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This sort of **shuffles** at weight one is not an accident! Similarly we find:

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \ln(b-4m^2) = \left(\frac{1}{2}\ln(u-9m^2) + \frac{1}{6}\ln(u-m^2)\right) I_0(u) - \frac{1}{2}\pi J_0(u)$$

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This approach can be used:

1- **Iteratively**, at **higher weights** and for more general polylogarithms

$$\int_{4m^2}^{(\sqrt{u}-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \, \mathrm{Li}_2\left(\frac{4m^2}{b}\right)$$

At higher weights, the identities are more complicated, but the highest transcendental piece follows the same pattern

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There are good indications that many Feynman integrals beyond multiple polylogarithms can be expressed as combinations of

$$\int_{b_i}^{b_j} \frac{db}{\sqrt{(b-b_1)(b-b_2)(b-b_3)(b-b_4)}} G(\vec{w}, b)$$

for **more general alphabets** of polylogs (beyond only roots of the 4-th order polynomial!)

1- Approach well suited to be generalized in this case

2- It allows to find simple and compact representation for the result

3- Can be (in principle) equally applied for higher order differential equations

CONCLUSIONS

1- Until recently no tools to study Feynman integrals beyond multiple polylogarithms

2- First issue, being able to solve **higher order differential equations**.

3- **Maximal cut** provides general solution to this problem
it allows to write integral representations for the solutions

4- Second issues, who are these functions? A lot of progress in studying properties of **elliptic multiple polylogarithms**

5- We propose a way to **classify them** and **study their properties** based on a concept of weight w.r.t to their (higher order) differential equations.

More to come soon...

THANKS!