QED contributions to the electron g-2

Stefano Laporta

Dipartimento di Fisica, Università di Padova, Italy

Stefano.Laporta@pd.infn.it

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Summary

- The anomalous magnetic moment of a lepton
- Experimental value of electron g-2
- Theoretical value of electron g-2
- High-precision numerical result of the mass-independent QED 4-loop coefficient
- Analytical fits
- How to calculate the g-2
- The program `SYS`

Anomalous magnetic moment of a lepton

$$\vec{\mu} = g \frac{e\hbar}{2mc} \vec{s} \qquad \text{magnetic moment } \mu \ , \quad \text{spin } s$$

g giromagnetic ratio

$$g = \begin{cases} 1 & \text{classical result} \\ 2 & \text{from Dirac equation} \\ 2.002319\ldots & \text{Quantum ElectroDynamics} \end{cases}$$

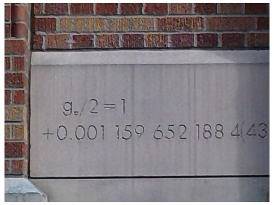
$$g = 2(1 + a)$$

$$a = \frac{g-2}{2} \quad \text{anomaly}$$

Very high experimental precision

$a_{e^-}^{exp} = 1\,159\,652\,188.4(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 198

$a_{e^+}^{exp} = 1\,159\,652\,187.9(4.3) \times 10^{-12}$ 4.3 ppb UW,Dehmelt et al 1987 (Nobel Prize 198



$a_{e^-}^{exp} = 1\,159\,652\,180.85(.76) \times 10^{-12}$ 0.66 ppb Harvard, Gabrielse 2006

$a_{e^-}^{exp} = 1\,159\,652\,180.73(.28) \times 10^{-12}$ 0.24 ppb Harvard, Gabrielse 2008

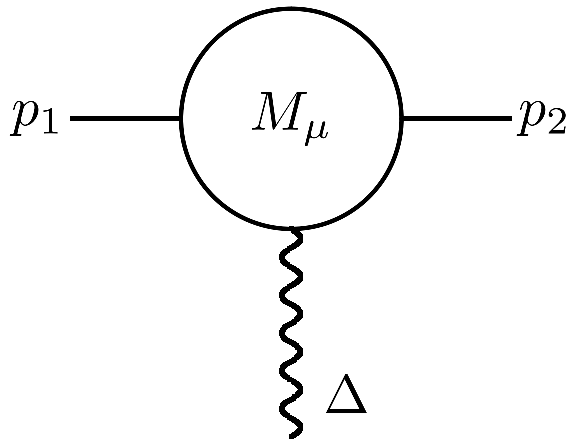
Storage of a single electron in a Penning trap (electrical quadrupole + axial B-field)

$$\omega_c = 2\frac{eB}{mc} \quad \text{cyclotron frequency}$$

$$\omega_s = g\frac{eB}{mc} \quad \text{spin precession frequency}$$

$$\omega_a = \omega_s - \omega_c \quad \text{spin flip frequency}$$

$$a = \frac{g-2}{2} = \frac{\omega_a}{\omega_c}$$ frequencies ratio → very high precision!



electron-photon vertex

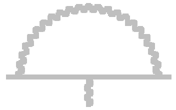
$$\bar{u}(p_2)M_\mu u(p_1) = \bar{u}(p_2)\left[F_1(-\Delta^2)\gamma_\mu - \frac{i}{4m}F_2(-\Delta^2)(\gamma_\mu\Delta_\mu - \gamma_\nu\Delta_\mu)\right]u(p_1)\ ,$$

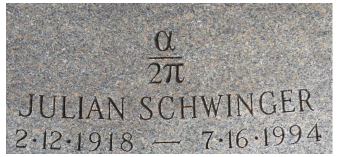
$$F_1(0) = 1\ , \qquad F_2(0) = a_e$$

Mass-independent QED contribution, 1-3 loop

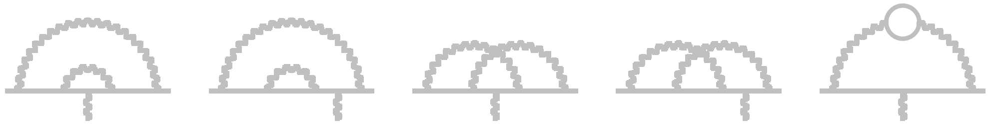
$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$$C_1 = \frac{1}{2}$$ (Schwinger 1948) *1 diagram*





$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

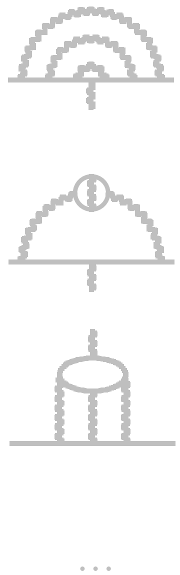


$$= -0.328\,478\,965\,579\,\dots,$$ (Petermann, Sommerfield 1957) *7 diagrams*

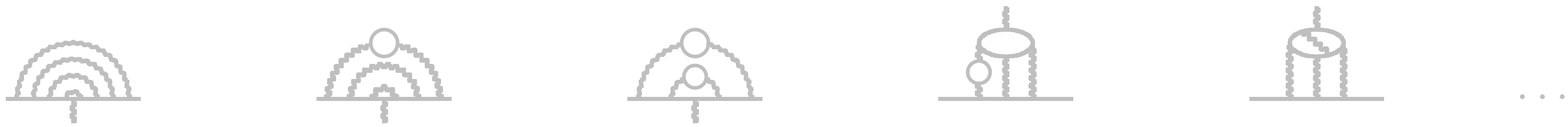
$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$= 1.181\,241\,456\,\dots,$$ (S.L., E.Remiddi 1996) *72 diagrams*



$\zeta(p)=\sum_{n=0}^{\infty} \frac{1}{n^p}$, $a_4=\sum_{n=0}^{\infty} \frac{1}{2^n n^4}$,



$C_4 \to$ **891 diagrams** obtained by inserting a photon in 104 self-mass diagrams

A few diagrams containing vacuum polarizations were known in analytical form. (Caffo, Turrini, Remiddi 1984)

Previous numerical values obtained by using MonteCarlo integration

$C_4 = -1.91298(84)$ (Kinoshita, 2014)

My new high-precision result is (S.L., 2017)

$$C_4 = -1.912245764926445574152647167439830054060873390658725345171329\ldots$$

good agreement with Kinoshita 2014 (0.9σ) $\to$ important independent check!

$C_5 \to$ **12672 diagrams**

$C_5 = 6.599(223)$ (Kinoshita et al., 2017) 3% precision

Some cronology of the values of C_n

$C_2 = -2.973\ldots$ (Karplus,Kroll 1950) analytical
$C_2 = -0.328478965579193\ldots$ (Petermann,Sommerfeld 1957) recalculation

$C_3 = 1.17611(42)$ (Kinoshita 1990)
$C_3 = 1.181259(40)$ (Kinoshita et al. 1995) 0.005164 shift due to an error
$C_3 = 1.181241456587200006\ldots$ (S.L., Remiddi 1996) analytical

$C_4 = -1.434(138)$ (Kinoshita 1990)
$C_4 = -1.5098(384)$ (Kinoshita, 1999)
$C_4 = -1.7283(35)$ (Kinoshita 2003) -0.24 shift due to the discovery of one error
$C_4 = -1.9144(35)$ (Kinoshita 2007) -0.22 shift due to the discovery of another error
$C_4 = -1.91298(84)$ (Kinoshita 2014)
$C_4 = -1.912245764926445\ldots$ (S.L. 2017) near-exact

$C_5 = 7.795(336)$ (Kinoshita et al. 2014)
$C_5 = 6.599(223)$ (Kinoshita et al. 2017) -1.2 shift due to the discovery of an error

(independent) checks seem to be crucial

dominant small terms (i.e. $\leq 3 \times 10^{-12}$)

$$a_e^{SM} = a_e^{QED} \overbrace{+ a_e^{QED}(\mu) + a_e^{QED}(\tau) + a_e^{QED}(\mu,\tau) + a_e(\text{hadr}) + a_e(\text{weak})}$$

- a_e^{QED} mass-independent:
  - 1-loop analytical, (Schwinger 1948)
  - 2-loop analytical (Petermann,Sommerfeld 1956)
  - 3-loop analytical (S.L.,Remiddi 1996)
  - 4-loop near-exact + analytical fit (S.L. 2017)
  - 5-loop numerical (3% precision) (Kinoshita 2017)
- $a_e^{QED}(X)$ mass-dependent:
  - 2-loop analytical (Elend 1966)
  - 3-loop analytical (S.L.,Remiddi 1992; S.L. 1994)
  - 4-loop analytical expansion in small mass ratio m_e/m_X, (Kurz et al 2013)
  - 5-loop numerical (10% precision) (Kinoshita 2014)
- $a_e^{QED}(X)$ small because scales as $(m_e/X)^2$ $a_e^{QED}(\mu,\tau)$ currently negligible

The electron anomaly is dominated by the QED terms. The other interactions contributes to the 10^{-12} level.

$$\alpha^{-1}(Rubidium : 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$C_1(\alpha/\pi) = 1\,161\,409\,733.631(720) \times 10^{-12}$$

$$C_2(\alpha/\pi)^2 = -\,1\,772\,305.065(3) \times 10^{-12}$$

$$C_3(\alpha/\pi)^3 = 14\,804.203 \times 10^{-12}$$

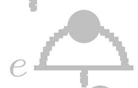
$$C_4(\alpha/\pi)^4 = -\,55.667 \times 10^{-12}$$

$$C_5(\alpha/\pi)^5 = 0.446(15) \times 10^{-12}$$

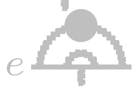
$$a_e^{QED}(\mu) = 2.738 \times 10^{-12}$$

$$a_e^{QED}(\tau) = 0.009 \times 10^{-12}$$

$$a_e(\text{hadronic v.p.,2-loop}) = 1.866(11) \times 10^{-12}$$

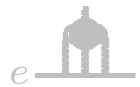


$$a_e(\text{hadronic v.p.,3-loop}) = -\,0.223(1) \times 10^{-12}$$



$$a_e(\text{hadronic v.p.,4-loop}) = 0.028(1) \times 10^{-12}$$

$$a_e(\text{hadronic l-l}) = 0.035(10) \times 10^{-12}$$



$$a_e(\text{weak}) = 0.028(1) \times 10^{-12}$$

Comparison of a_e the determination of fine structure constant

$$\alpha^{-1}(Rubidium: 2016) = 137.035\,998\,996(85) \qquad (0.62\ ppb)$$

$$a_e^{SM}(\alpha) = 1\,159\,652\,182.031\overset{C_5}{(15)}\overset{had}{(15)}\overset{\alpha}{(720)} \times 10^{-12}$$

$$a_e^{exp} = 1\,159\,652\,180.730(280) \times 10^{-12} \qquad 0.25\ ppb$$

$$a_e^{SM}(\alpha) - a_e^{exp} = 1.30(77) \times 10^{-12} \qquad 1.6\sigma \text{ agreement}$$

same order of magnitude of the hadronic contribution and muon-loop vacuum polarization

Assuming QED is correct

$$\alpha^{-1}(a_e) = 137.035\,999\,1500\overset{C_5}{(18)}\overset{had}{(18)}\overset{a_e}{(330)}$$

$$= 137.035\,999\,1500(332) \qquad (0.25\ ppb)$$

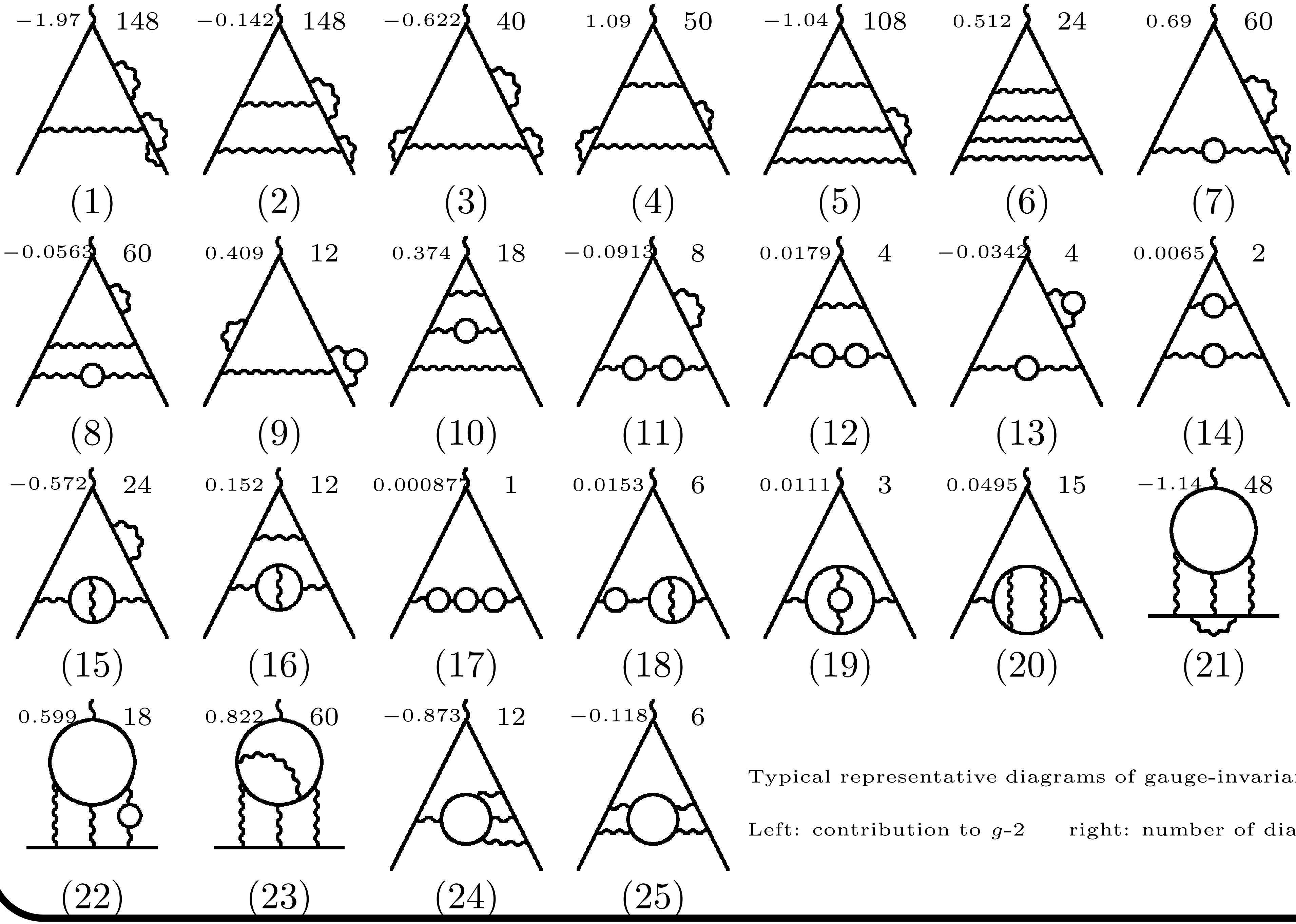
$\alpha(a_e)$ more precise than $\alpha^{exp}(Rubidium) \rightarrow \alpha(a_e)$ used in CODATA least-square adjustment of fundamental constants.

- Contributions of single diagrams may be I.R. or U.V. divergent, and depending on gauge transformation.
- It is convenient to regroup the diagrams in gauge-invariant sets
- The contribution of gauge-invariant sets are I.R. finite and invariant under gauge transformations of the internal photons.

- 2 loops: 7 diagrams arranged into 3 gauge-invariant sets
- 3 loops: 72 diagrams arranged into 9 gauge-invariant sets (Cvitanovic,1977)
- 4 loops: 891 diagrams arranged into 25 gauge-invariant sets

Calculation was performed in the Feynman gauge.

891 diagrams arranged into 25 gauge-invariant sets



Typical representative diagrams of gauge-invariant sets

Left: contribution to g-2 right: number of diagrams

1100 digits of 4-loop coefficient

$C_4 =$

-1.912245764926445574152647167439830054060873390658725345171329848006038443980651706142760892700003631583755841533147327005637851491285453919028043270502738223043455789570455627293099412966997602777822115784720339064151908166527097970867438115012155147972274322164273431927975958607405005783738496070187432831402483802519224946074229855893046350614049225266343109442400023563568812806206454940132249775943004292888367617488992369151808780869897052635785337537769641170245361960134975744943612684861751626068323871867473038315059627418780153055148794005369777983694642786843269184311758895811597435669504330483490736134265864995311638781174347538542348836408558444188223721745670687104182330743051744305573945961171550858961148995261266061246994073118403927472340023464969531735482584817998224097373710773657404645135211230912425281111372153021544537210148111211598489708842232798797204842014451228284515165852365617865945926009917330317213028654672123453405003491047007289244872006160442613254490690004319151982300474881814943110384953782994062967586787538524978194698979313216219797575067670114290489796208505...

Due to the expected analytical complexity of C_4 (triple elliptic integrals), a completely analytical calculation (similar to that of C_3) seemed out of reach. So, an alternative is:

1. compute an extremely high-precision value of (master integrals and) C_4
2. guess the right analytical ansatz (not easy!)
3. fit an analytical expression by using the "PSLQ algorithm"

- PSLQ algorithm (Ferguson and Bailey 1992)
- multi-integers extension of the Euclid algorithm for the calculation of the GCD of two integers
- it finds an integer relation between real numbers or bounds on size of coefficients.
- it requires high precision; at least number of digits of coefficients * number of real numbers
- a parallel version of the algorithm exists (Bailey and Broadhurst 1999)
  OMP fortran code available on Baileys's site

Simple example of PSLQ fit

$$G_7 = -2342.20751410602307542352254059079270988532873205655947080 7$$
$$359481483571384691680645591697318599261483194890419734356986$$
$$640536482839180927737599376306979737829110608311707671767935$$
$$983139125960766918329923883871930584868496516072868729243183$$
$$317800519694759939914751761141283435810030791136838793708071$$
$$157346099787020302357526852412095436287332846448926242430503$$
$$236449547474407307581291123637921078586418676517549877972867$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

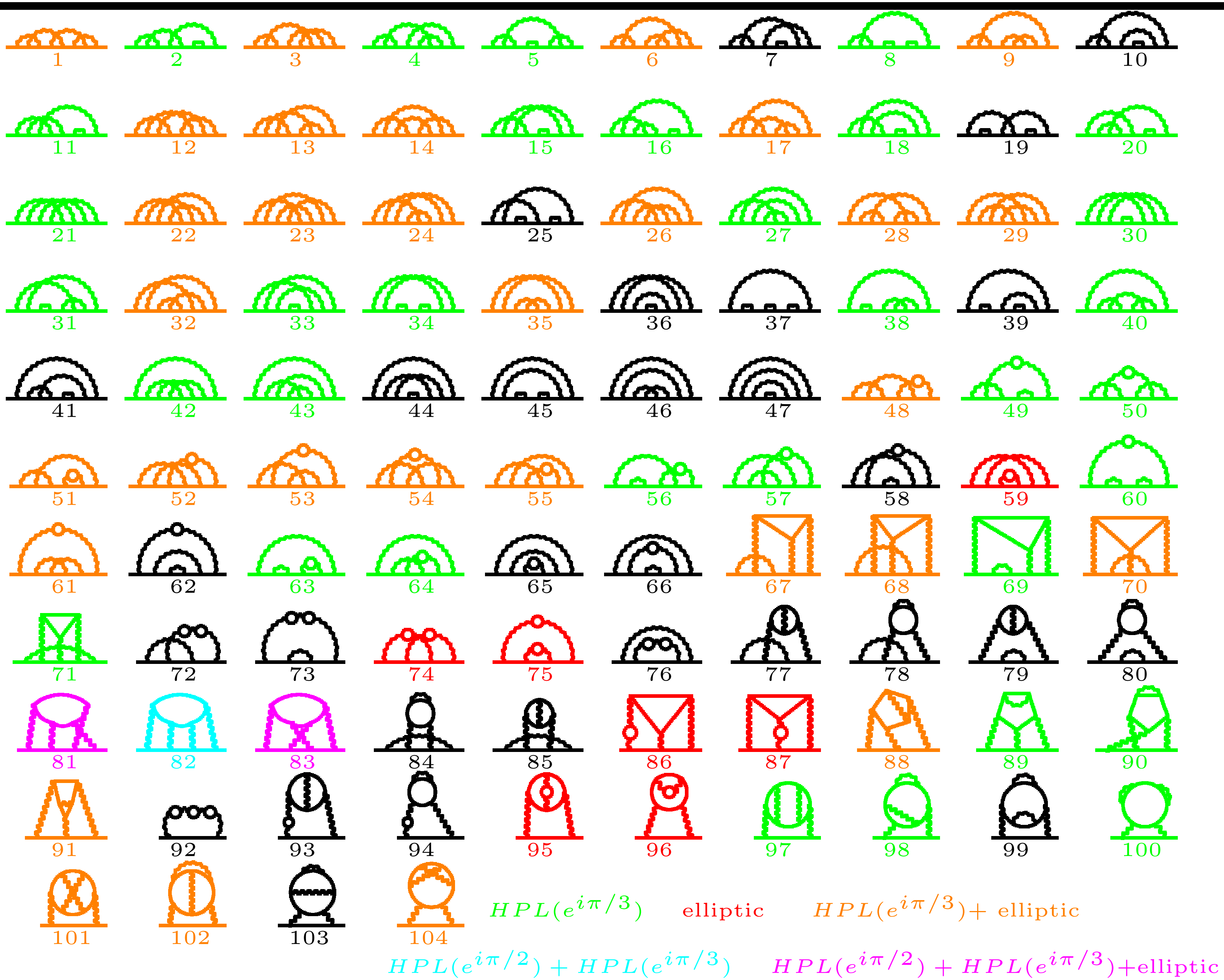
$$= \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)$$
$$- \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2$$
$$+ 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)$$

black: ansatz brown: PSLQ result

Differently from C_1, C_2, C_3, the analytical expression of C_4 is very complicated!
It can be divided in 5 parts:

1. usual constants (harmonic) polylogarithms of 1 and 1/2.
2. harmonic polylogarithms of arguments $e^{i\pi/3}$ and $e^{2i\pi/3}$ new
3. harmonic polylogarithms of arguments $e^{i\pi/2}$ new
4. elliptic constants with semi-analytic representation new
5. unknown elliptic constants

the 104 4-loop electron self-masses



analytical fit part 1

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U \qquad \text{121 terms}$$

$$\begin{aligned}T =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) \\
&+ \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
&- \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) \\
&- \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\
&- \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) \\
&+ \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 \\
&+ \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 \\
&+ \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
&- \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2\end{aligned}$$

$a_n = \mathrm{Li}_n(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&-\frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
&-\frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&-\frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

terms of weight 5 cancel out

$$\mathrm{Cl}_n(\theta) = \mathrm{ImLi_n}(\mathrm{e}^{\mathrm{i}\theta})$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+ \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&- \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi \\
&+ \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+ \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+ \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)
\end{aligned}$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned} W_b =& -\frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 \\ & +104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\ & -2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg) \end{aligned}$$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ is the Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$

analytical fit part 4

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
E_a =& \pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) \\
&+\pi\left(\frac{4715}{1944}\ln 2\; f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
&+\pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\; f_2(0,0,2) + \frac{5525}{162}\ln 2\; f_2(0,1,1)\right. \\
&-\frac{5525}{243}\ln 2\; f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
&-\frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) \\
&\left.+\frac{69245}{124416}f_2(2,1,0)\right)
\end{aligned}$$

$$\begin{aligned}
E_b =& -\frac{4715}{1458}\zeta(2)f_1(0,0,1) \\
&+\zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2) - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right) .
\end{aligned}$$

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1\text{-}x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3}\left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3}\left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})}\,{}_4F_3\left(\begin{matrix}\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925}\ {}_7F_6\left(\begin{matrix}\frac{7}{4}\ -\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ \frac{4}{3}\ \frac{3}{2}\ \frac{3}{2}\\ \frac{3}{4}\ 1\ \frac{7}{6}\ \frac{11}{6}\ \frac{13}{6}\ \frac{17}{6}\end{matrix};1\right),$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix};x\right), \qquad E_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3}\ -\frac{1}{3}\\ 1\end{matrix};x\right).$$

A_3 cancels out in the diagram contributions

A_3 and B_3 seem to be both irreducible

f_j are defined as follows:

$$f_1(i,j,k) = \int_1^9 ds\, D_1^2(s)\left(s-\frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s) \ ,$$

$$f_2(i,j,k) = \int_1^9 ds\, D_1(s)\mathrm{Re}\left(\sqrt{3}D_2(s)\right)\left(s-\frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s) \ ,$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \ ,$$

$$D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(1-\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \ ;$$

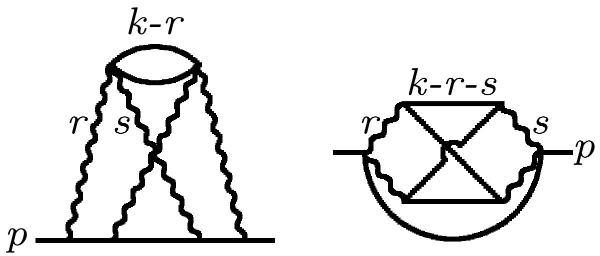
$K(x)$ is the complete elliptic integral of the first kind.

$D_1(s) \sim$ discontinuity of the 2-loop sunrise diagram with equal masses in $D=2$ dimensions.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

The term containing the ϵ^0 coefficients of the ϵ−expansion of six master integrals (see f, f', f'', g, g', g''):

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c} \; .$$



(f, f', f'') (g, g', g'')

(f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

These master integrals appear in topologies 81 and 83 (gauge-invariant sets 24 and 25, vacuum polarization diagrams containing a light-light scattering).

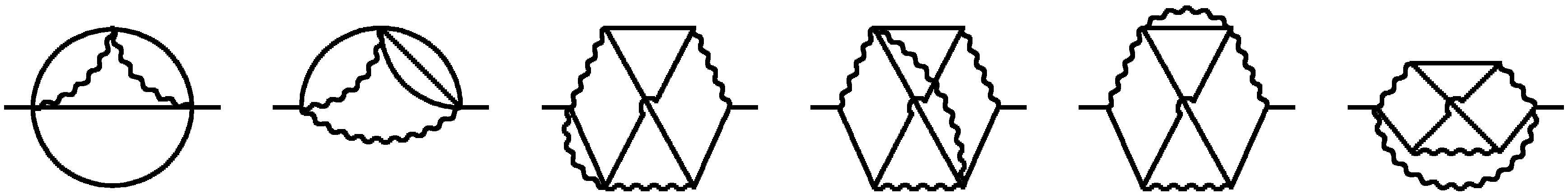
PSLQ fits

1. Calculation of M.I. with 200-300 digits
2. Recalculation of M.I. with 4096 bits (1200 digits) of precision
3. First run of PSLQ on the coefficients of the expansions of the M.I.
4. Test which coefficients are linearly independent ("unknown constants")
5. For each "unknown costant", identification of the *simplest* master integral whose coefficient are reducible to that "new" constant
6. Recalculate this M.I. with higher precision (tipically 2400, 4800, 9600 digits)
7. Analyse the M.I., ansatz
8. Identification of a basis
9. PSLQ fit: if successful, OK!. If not, enlarge the basis or revaluate the M.I. with higher precision

PSLQ fits: harmonic polylogarithms of complex arguments

- The presence of harmonic polylogarithms of complex arguments increases considerably the size of the basis needed to fit the numerical values.
- The general basis of real and imaginary parts of harmonic polylogarithms of argument of $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$ up to weight 7 has dimension $F_{17} - 1 = 1596$.
- To fit successfully quantities at weight seven, it is necessary to increase considerably the precision of the calculations, and to try various selections of the general basis.
- The fit of V_a, V_b and the master integrals involved has needed thousands of PSLQ runs with basis of ~ 500 elements calculated with 9600 digits of precision.
- The multi-pair parallel version of PSLQ algorithm has been essential to work out these difficult analytical fits in reasonable times (each one 2d on 8-core cpu).

Master Integrals with $HPL\left(e^{i\frac{m\pi}{3}}\right)$



(a) (b) (c) (d) (e) (f)

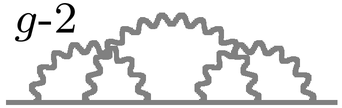
$$I(a) = \frac{7}{12\epsilon^4} + \frac{10}{3\epsilon^3} + \frac{121}{12\epsilon^2} + \left(\frac{1541}{72} + \frac{7}{6}\zeta(3)\right)\epsilon^{-1} + \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4)$$
$$+\sqrt{3}\left(6\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - 10\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + O(\epsilon^2)$$

self-mass 60

$$I(b) = \frac{5}{8\epsilon^4} + \frac{59}{16\epsilon^3} + \left(\frac{1099}{36} + 3\zeta(2)\right)\epsilon^{-2} + \left(\frac{3781}{192} + \frac{33}{2}\zeta(2) + 6\zeta(3)\right)\epsilon^{-1} + \frac{25033}{1152} - \frac{47}{4}\zeta(2)$$
$$+\frac{69}{2}\zeta(3) + \frac{411}{8}\zeta(4) - \sqrt{3}\left(9\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + 9\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + O(\epsilon^2)$$

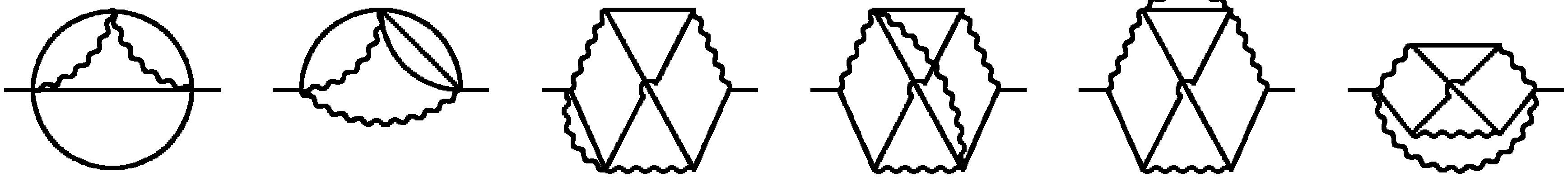
self-mass 63

(S.L. 1993) 3-loop g-2



$\pi\left(9\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + 5\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)$ appeared in intermediate results, cancelled out in final results.

Master Integrals with $HPL\left(e^{i\frac{m\pi}{3}}\right)$



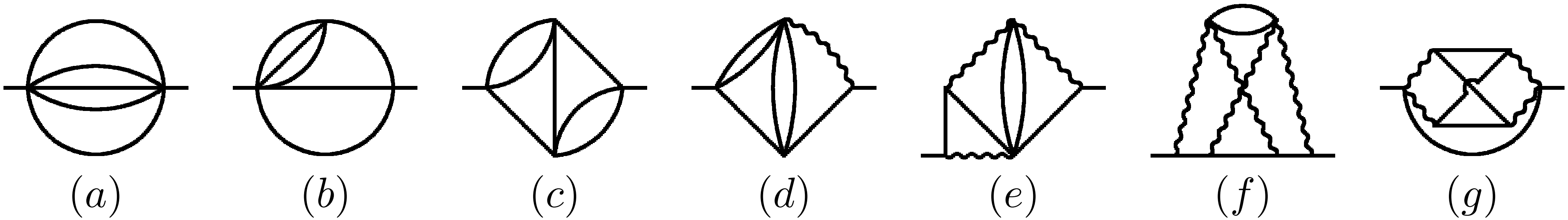
(a) (b) (c) (d) (e) (f)

$$
\begin{aligned}
I(c) =& \left(-\frac{45}{4}\zeta(5) + \frac{17}{2}\zeta(2)\zeta(3)\right)\epsilon^{-1} - \frac{14897}{96}\zeta(6) - 45\zeta(5) - \frac{7}{16}\zeta^2(3) + 84\zeta(2)a_4 + 34\zeta(2)\zeta(3) \\
&+ \frac{147}{2}\ln 2\zeta(2)\zeta(3) - \frac{105}{2}\ln^2 2\zeta(4) + \frac{7}{2}\ln^4 2\zeta(2) + \sqrt{3}\Big(4\ \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+ 4\ \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + 4\ \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{54}{13}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{207}{26}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + 7\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{2}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{36}{5}\zeta(2)\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{24}{5}\zeta(2)\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - 12\zeta(2)\ln 2\ \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&- 18\zeta(2)\ln 2\ \mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{4787}{143}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{40}{3}\mathrm{Im}H_4\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{5}{9}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 \\
&+ \frac{1765}{12}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{3469}{27}\zeta(2)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{20}{3}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{27413}{16848}\pi\zeta(5) \\
&+ \frac{1}{9}\pi\zeta(4)\ln 2 - \frac{97}{90}\pi\zeta(2)\zeta(3)\Big) + O(\epsilon);
\end{aligned}
$$

$I(d), I(e), I(f)$ similar expressions

- One finds experimentally a basis of triple integrals containing products of elliptic integrals
- 9600 digits of precision are needed.

Elliptic master integrals



$$I(a, D{=}4\text{-}2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240}(297B_3 - 1477C_3) - \frac{2320981}{20736} + O(\epsilon)$$

$$I(a, D{=}2\text{-}2\epsilon) = \sqrt{3}\pi B_3 + M_{501}\epsilon + M_{602}\epsilon^2 - M_{702}\epsilon^3 + \ldots$$

These M.I. were calculated with 5000-10000 digits.

calculation of the contribution - summary

- Generation of diagrams and extraction of the contribution to g-2
- Reduction to master integrals
- Numerical calculation of master integrals
- Renormalization: generation and calculation of the counterterms

The approach has been devised in order to maximize the number of possible consistency checks and minimize error sources, even at the expense of big increasing of computer time.

- All the 4-loop self-mass diagrams are generated with a self-made *C* program
- For each of the 104 self-mass diagrams, vertex diagrams are built by inserting a photon in all possible ways. One keeps also the non-contributing diagrams because of Furry's theorem for the sake of subsequent checks.
- The contribution to g-2 is extracted from the unrenormalized amplitude of each vertex diagram using suitable projectors (FORM). The results are expressions which typically contain 100-30000 different Feynman integrals.

Reduction to master integrals

- The total number of topologically distinct master integrals is 334.
- 82 topologies have number of master integrals $n > 1$
- For topologies with $n > 1$, one chooses scalar products as numerators, which depend on the momentum flow of the diagram.
- For the sake of subsequent checks, one processes separately the contributions from each one of the 104 self-mass diagrams.
- The master integrals are calculated separately for each self-mass diagram. The total number of of master integrals to be calculated increases to 4607.
- Because of differences in the numerators, topologies with $n > 1$ will have a slightly different set of master integrals in different self-mass diagrams. This provides a lot of useful checks.

Reduction to master integrals

- The reduction (exact in D) is performed by generating a large system of integration-by-parts identities and solving it with the algorithm (L.S.,2001) implemented in SYS.
- The number of necessary identities is $4 \times 10^6 - 50 \times 10^6$. Disk sizes of systems are $4GB - 100GB$.
- Reduction was repeated on two different machines, with 32bit and 64bit versions of SYS. In addition, the principal self-mass diagrams were reprocessed using a different momentum flow, checking that reduction to master integrals remained the same (after converting different sets of master integrals).

Numerical calculation of master integrals

- I used combinations of the difference methods (S.L.,2001) and differential equations methods (Kotikov,1991), (Remiddi,1997), (Remiddi,Gehrmann 2000).
- An approach consists in calculating with difference equations (inserting the exponent n on the first electron propagator) the integral obtained putting the photon mass λ equal to the electron mass m, and integrating a differential equation in λ from $\lambda = m$ to $\lambda = 0$.
- An alternative approach consists in calculating with difference equations the diagram obtained by putting the the external momentum $p = 0$, and integrating a differential equation in p^2 from $p^2 = 0$ to $p^2 = -m^2$.
- The systems difference and differential equations for master integrals are obtained by building systems of suitable integration-by-parts identities and solving them by using the algorithm (S.L.,2001), using rational arithmetic in D.
- The sizes of the systems of difference or differential equations to be numerically solved are in the range $1MB - 3GB$.

Numerical calculation of master integrals

- Difference equations are solved using the Laplace transformation method (integral representation of solutions and differential equation of the integrand).

$$F(n) = \int_0^1 dt\, v(t)\, t^{n-1} \tag{1}$$

$F(n)$ satisfies a difference equationin n, $v(t)$ a differential equation in t. In this way we need only one high-precision solver for differential equations (less code, less errors).

- Maximum order of differential equations solved: 9
- Differential equations are solved numerically, by using series expansion with truncated expansions in $\epsilon = (4 - D)/2$ as coefficients.
- The minimum number of terms of the expansion in ϵ in 9 ($e^{-4} \ldots\ e^{4}$).
- There are cancellations of ϵ terms in intermediate steps; no care is used avoiding cancelations, as the corresponding numerical zeroes are extremely useful checks. In the worst cases 37 terms of expansions are needed.

Numerical calculation of master integrals

- The standard precision of calculations is 4096bit (1232 digits). About 130 digits are lost due to cancellations, so $1232 - 130 = 1100$.
- The fit of some selected master integrals required a precision much higher, up to 16kbits (9864 digits).
- Typical execution times:
  - self-mass diagram 22, 548 differential equations, 195 master integrals, 1200-digits precision: 7 months on 128 cores.
  - self-mass diagram 37, 10 differential equations, 6 master integrals, 1200-digits precision: 13 minutes on 8 cores.

- Renormalization counterterms are generated with two self-made procedures *C* and FORM.
- Their expressions are reduced to master integrals, already known in analytical form.
- The calculation is performed in the Feynman gauge
- I checked explicitly the (internal) gauge invariance of the expressions for arbitrary gauge of the photon line going into 1-,2- or 3-loop vacuum polarization diagrams.

In order to perform this calculation, in 1995 I begun writing a C program, `SYS`, containing all the necessary ingredients:

- a symplified fast algebraic (invoking repeatedly FORM, that I had successfully used for C_3, has a not negligible time cost)
- a numerical solver of systems of difference and differential equations
- a library of arbitrary precision mathematical routines, integer and floating point (in mid-1990 the GMP library was still in its infancy).

The program SYS

- C program, about 23000 lines.
- The program automatically determines the master integrals of a diagram, it builds and solves the systems of difference or differential equations.
- Input: description of the diagram, number of terms of the expansion in $D-4$.
- The program contains a simplified algebraic manipulator, used to solve systems of identities among integrals with this kind of coefficients: arbitrary precision integers, rationals, ratios of polynomials in one and two variables (for example D and x) with integer coefficients.
- Efficient management of systems of identities of size up to the limit of disk space (tested up to 500 million of identities).
- Numerical solution of systems of difference and differential equations up to 900 equations, using arbitrary precision floating point complex numbers and truncated series in ϵ.
- All the coefficients of the expansions in ϵ are worked out in numerical form,

The program SYS

even those of divergent terms.

- Floating number precision: up to 9800 digits (essentially one sums expansions in *one* variable).
- Arithmetic libraries which deal with operations on arbitrary precision integers, polynomials, rationals, arbitrary precision floating point numbers and truncated series in ϵ were written on purpose by the author. *Independent* of all other available libraries.
- Several Multicore/multinode parallel versions of the program were written on purpose.
- Sistematic protection of large buffers, I/O with crc/checksums. Found several subtle corruptions in the years, like marginal coupling of non-ECC RAM modules (*bit flipping*, 1 bit changed per week), failing RAID systems (corrupted blocks of 64KBytes), etc. . . .)

- 1100-digits value of C_4 allows a successful analytical fit with (relatively) small coefficients
- with the recent update of C_5, now experimental and theoretical values of a_e agree at level of 1.6σ
- the ultimate limit of the precision of the theoretical value is the error in the hadronic contribution $\approx 10^{-14}$
- that corresponds to an amount of 0.15 in C_5 or 64 in C_6
- C_6 is currently not needed
- historically checks with experiments or independent theoretical results have often highlighted inconsistences in QED calculations
- for this reason an independent calculation of C_5 (a gargantuan task), would be important

The End