Hadronic light-by-light contribution to $(g-2)_\mu$: a dispersive approach

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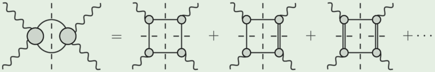

Based on:

JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)

in collab. with M. Hoferichter, M. Procura and P. Stoffer and

PLB738(2014)6 .... +B. Kubis

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Volume 118, Number 23

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
- Gauge invariance and crossing symmetry
- Master Formula

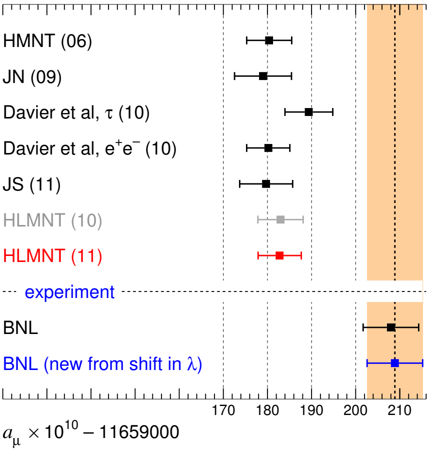
A dispersion relation for HLbL

Numerics
- - Pion box contribution
- - Pion rescattering contribution

Outlook and Conclusions

Status of $(g-2)_\mu$, experiment vs SM

cf. M. Knecht's talk



Hagiwara et al. 2012

Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ "$(g-2)_\mu$ Theory Initiative" lead by A. El-Khadra and C. Lehner

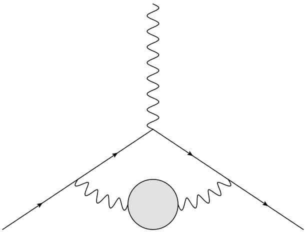
Status of $(g-2)_\mu$, experiment vs SM

cf. M. Knecht's talk

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 11]	6 949.	43.
HVP (NLO) [Hagiwara et al. 11]	−98.	1.
HLbL [Jegerlehner-Nyffeler 09]	116.	40.
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.	2.
theory	116 591 855.	59.

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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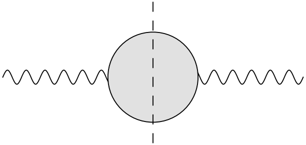
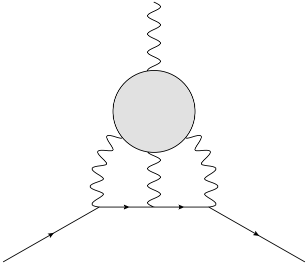


- basic principles: unitarity and analyticity
- direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
  $\to$ A. Denig and F. Ignatov talk
- alternative approach: lattice
  $\to$ M. Marinkovich talk

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



- 4-point fct. of em currents in QCD
- "*it cannot be expressed in terms of measurable quantities*"
- up to now, only model calculations
- lattice QCD is making fast progress

→ C. Lehner & A. Nyffeler talk

Analytic Approaches to Hadronic light-by-light

- Model calculations
  - ENJL — Bijnens, Pallante, Prades (95-96)
  - NJL and hidden gauge — Hayakawa, Kinoshita, Sanda (95-96)
  - nonlocal χQM — Dorokhov, Broniowski (08)
  - AdS/CFT — Cappiello, Cata, D'Ambrosio (10)
  - Dyson-Schwinger — Goecke, Fischer, Williams (11)
  - constituent χQM — Greynat, de Rafael (12)
  - resonances in the narrow-width limit — Pauk, Vanderhaeghen (14)
- Impact of rigorously derived constraints
  - high-energy constraints taken into account in several models above
    addressed specifically by — Knecht, Nyffeler (01)
  - high-energy constraints related to the axial anomaly — Melnikov, Vainshtein (04) and Nyffeler (09)
  - sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen (12)
    see also: workshop MesonNet (13)
  - low-energy constraints–pion polarizabilities — Engel, Ramsey-Musolf (13)
- Lattice — RBC/UKQCD, Mainz, $\to$ C. Lehner & A. Nyffeler talk

Different evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- heavier single-particle poles decreasingly important

Our approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

- model independent ⇒ rely on dispersion relations
- as data-driven as possible
- takes into account high-energy constraints
  [OPE, perturbative QCD]
  (work in progress, not discussed here)

Alternative dispersive approach for the μ-FF Pauk-Vanderhaeghen (14)

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Hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle = \left(q_\mu q_\nu - g_{\mu\nu} q^2 \right) \Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^\infty dt \frac{\mathrm{Im}\bar{\Pi}(t)}{t(t - q^2)}$$

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Easy!

The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz \, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

The HLbL tensor (much less easy...)

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$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent

Eichmann *et al.* (14)

The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0|T\{j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\}|0\rangle$$

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with: GC, Hoferichter, Procura, Stoffer (2015)

- 43 basis tensors (BT) in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta} q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma} q_{4\,\delta}\,,\\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right),\\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right),\\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right),\\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right),\\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\left(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right)\right.\\
&\qquad\left. + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\right),\\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma\left(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right)\right.\\
&\qquad\left. + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu}\right)\right)\\
&\quad - q_4^\lambda\left(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right)\right.\\
&\qquad\left. + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma}\right)\right)\\
&\quad + q_3\cdot q_4\Big(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\Big).
\end{aligned}
$$

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- 43 basis tensors (BT) — in $d=4$: 41=no. of helicity amplitudes
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- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to a_μ: $m := m_\mu$

$$a_\mu = \frac{-1}{48m} \mathrm{Tr}\Big\{ (\not p + m)[\gamma^\rho, \gamma^\sigma](\not p + m)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) \Big\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\gamma^\mu (\not p + \not q_1 + m)\gamma^\lambda (\not p - \not q_2 + m)\gamma^\nu}{((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)} \times$$
$$\times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\Big|_{k=0}$$

BTT basis (no kin. singularities!) $\Rightarrow$ limit $k_\mu \to 0$ unproblematic

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

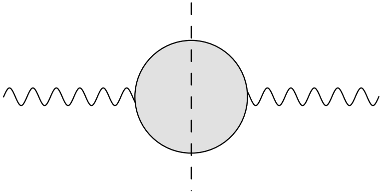
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

Outline

Setting up the dispersive calculation

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \text{hadrons})$$

cf. D. Nomura's talk

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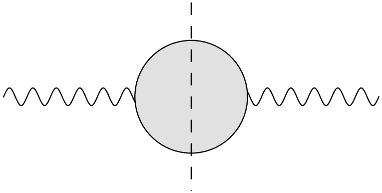


$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \text{hadrons})$$

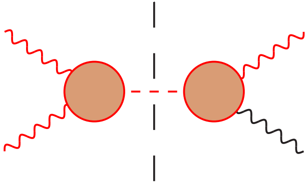
cf. D. Nomura's talk

For HLbL things are more complicated

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
Input: pion transition form factor Hoferichter, Kubis, Leupold, Niecknig, Schneider (14)
First results of direct lattice calculations Gerardin-Mayer-Nyffeler (16)

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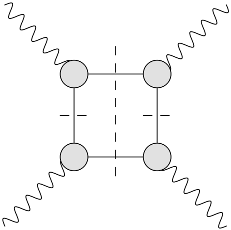
π-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole

– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

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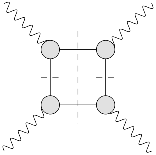
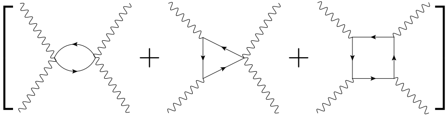


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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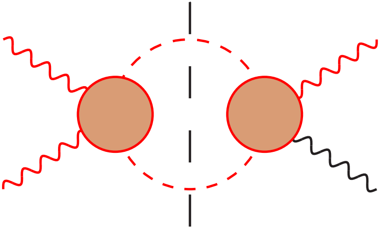


The "rest" with 2π intermediate states has cuts only in one channel and will be
calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will not be discussed here

Pion Pole contribution to $\gamma^*\gamma^* \to \pi\pi$

BTT basis for $\gamma^*\gamma^* \to \pi\pi$:

$$\tilde{T}_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$
$$\tilde{T}_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$
$$\tilde{T}_3^{\mu\nu} = q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - \frac{1}{2}(t-u) q_1^2 g^{\mu\nu} + \frac{1}{2}(t-u) q_1^\mu q_1^\nu,$$
$$\tilde{T}_4^{\mu\nu} = q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu + \frac{1}{2}(t-u) q_2^2 g^{\mu\nu} - \frac{1}{2}(t-u) q_2^\mu q_2^\nu,$$
$$\tilde{T}_5^{\mu\nu} = q_1 \cdot q_2 q_3^\mu q_3^\nu - \frac{1}{4}(t-u)^2 g^{\mu\nu} + \frac{1}{2}(t-u)\left(q_3^\mu q_1^\nu - q_2^\mu q_3^\nu\right),$$
$$\tilde{T}_6^{\mu\nu} = q_1^2 q_2^2 q_3^\mu q_3^\nu + \frac{1}{2}(t-u)\left(q_1^2 q_3^\mu q_2^\nu - q_2^2 q_1^\mu q_3^\nu\right) - \frac{1}{4}(t-u)^2 q_1^\mu q_2^\nu$$

Pion Pole contribution to $\gamma^*\gamma^* \to \pi\pi$

BTT basis for $\gamma^*\gamma^* \to \pi\pi$:

$$W_{\mu\nu} = \sum_{i=1}^{5} T^i_{\mu\nu} A_i = \sum_{i=1}^{6} \tilde{T}^i_{\mu\nu} B_i,$$

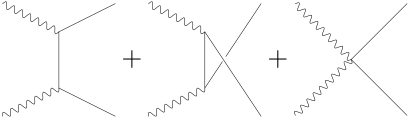
where

$$\begin{aligned}
T_1^{\mu\nu} &:= \tilde{T}_1^{\mu\nu}, \\
T_2^{\mu\nu} &:= \tilde{T}_2^{\mu\nu}, \\
T_3^{\mu\nu} &:= (t-u)(\tilde{T}_3^{\mu\nu} - \tilde{T}_4^{\mu\nu}), \\
T_4^{\mu\nu} &:= \tilde{T}_5^{\mu\nu}, \\
T_5^{\mu\nu} &:= \tilde{T}_6^{\mu\nu},
\end{aligned}$$

Pion Pole contribution to $\gamma^*\gamma^* \to \pi\pi$

sQED calculation:

$W^{\mu\nu}_{\text{Born}} =$



$$= (2p_1^\mu - q_1^\mu)(2p_2^\nu - q_2^\nu)\frac{1}{t - M_\pi^2} + (2p_2^\mu - q_1^\mu)(2p_1^\nu - q_2^\nu)\frac{1}{u - M_\pi^2} + 2g^{\mu\nu}$$

$\Rightarrow$ read off the Born values of the scalar functions:

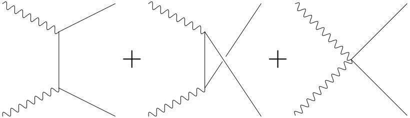
$$A_1^{\text{Born}} = -\left(\frac{1}{t - M_\pi^2} + \frac{1}{u - M_\pi^2}\right)$$

$$A_4^{\text{Born}} = -\frac{2}{s - q_1^2 - q_2^2}\left(\frac{1}{t - M_\pi^2} + \frac{1}{u - M_\pi^2}\right)$$

$$A_2^{\text{Born}} = A_3^{\text{Born}} = A_5^{\text{Born}} = 0$$

Pion Pole contribution to $\gamma^*\gamma^* \to \pi\pi$

sQED calculation:

$W^{\mu\nu}_{\text{Born}} =$ 

$$= (2p_1^\mu - q_1^\mu)(2p_2^\nu - q_2^\nu)\frac{1}{t - M_\pi^2} + (2p_2^\mu - q_1^\mu)(2p_1^\nu - q_2^\nu)\frac{1}{u - M_\pi^2} + 2g^{\mu\nu}$$

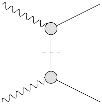
Multiply by $F_\pi^V(q_i^2)$ to get the correct off-shell behaviour:

$$A_1^{\pi,FsQED} = -F_\pi^V(q_1^2)F_\pi^V(q_2^2)\left(\frac{1}{t - M_\pi^2} + \frac{1}{u - M_\pi^2}\right)$$

$$A_4^{\pi,FsQED} = -F_\pi^V(q_1^2)F_\pi^V(q_2^2)\frac{2}{s - q_1^2 - q_2^2}\left(\frac{1}{t - M_\pi^2} + \frac{1}{u - M_\pi^2}\right)$$

$$A_2^{\pi,FsQED} = A_3^{\pi,FsQED} = A_5^{\pi,FsQED} = 0$$

Pion Pole contribution to $\gamma^*\gamma^* \to \pi\pi$



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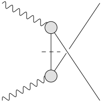
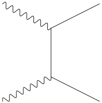


$$= F_\pi^V(q_1^2) F_\pi^V(q_2^2) \times \Bigg($$



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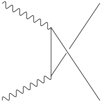

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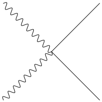

$$\Bigg)$$

Partial wave expansion for 2π contributions

To complete the program of writing down a dispersion relation for two-pion contributions **is not easy**:

- unitarity relations are diagonal in a helicity amplitude basis;
- the helicity basis relevant for $(g-2)_\mu$ is the one with one on-shell photon, which has 27 elements;
- in the limit $q_4^2, q_4^\sigma \to 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- there is freedom in the choice of this subset (singly-on-shell basis);
- the arbitrariness in the choice of the 27 elements of the singly-on-shell basis does not influence the final result because of sum rules
- these sum rules follow from the assumption that the HLbL tensor has a uniform behaviour at short distances
- Pascalutsa, Pauk, Vanderhaeghen (12) forward-kinematics sum-rules are a special case of our general sum rules

S-wave 2π contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s' \,\mathrm{Im} h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t' \,\mathrm{Im} h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u' \,\mathrm{Im} h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

Analogous expressions for the D, G and all higher waves have been derived but are too long to be shown

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- - Pion rescattering contribution

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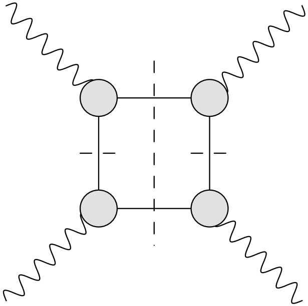
Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

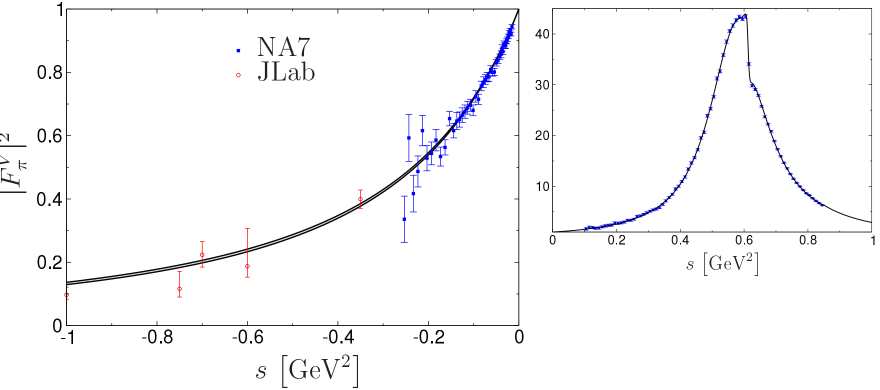
where

$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion box contribution



Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion box contribution

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

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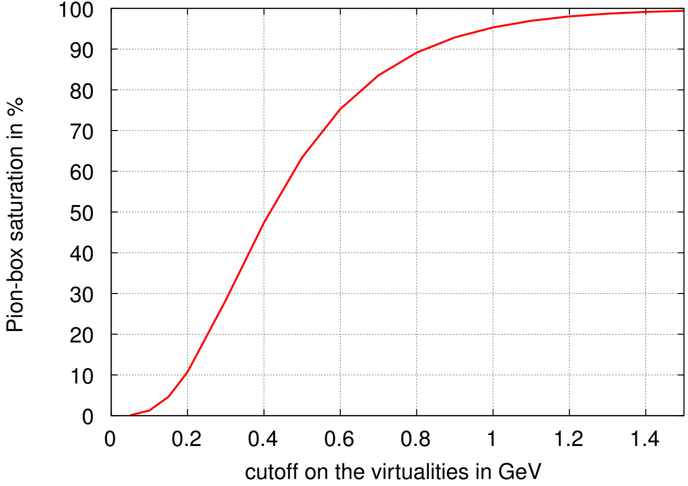
Pion-box saturation with photon virtualities



Check of the partial-wave formalism

Comparison partial-wave expansion of the pion-box vs. full result

J_{max}	$\delta_{J_{\text{max}}}$	$\Delta_{J_{\text{max}}}$
0	29.2%	55.4%
2	10.4%	20.9%
4	4.3%	11.0%
6	2.4%	6.2%
8	1.5%	3.7%
10	1.0%	2.4%
12	0.7%	1.6%
14	0.6%	1.1%

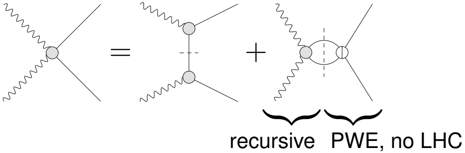
where

$$\delta_{J_{\text{max}}} := 1 - \frac{a_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}} \qquad \Delta_{J_{\text{max}}} := \frac{\left| \mathbf{a}_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}} - \mathbf{a}_\mu^{\pi\text{-box}} \right|}{\left| \mathbf{a}_\mu^{\pi\text{-box}} \right|}$$

Convergence for real helicity amplitudes should be much better

First evaluation of *S*- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



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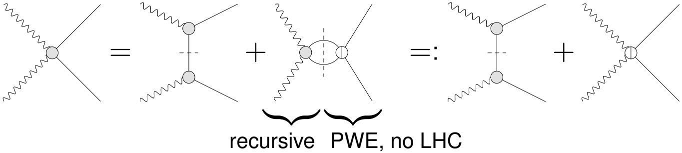

First evaluation of *S*- wave 2π-rescattering

Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

S-wave contributions: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

a_μ^{HLbL} in 10^{-11} units

cutoff	1 GeV	1.5 GeV	2 GeV	∞
$I = 0$	-9.2	-9.5	-9.3	-8.8
$I = 2$	2.0	1.3	1.1	0.9
sum	-7.3	-8.3	-8.3	-7.9

First evaluation of S- wave 2π-rescattering

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Recall π-Box: $a_\mu^{\pi-\text{box}} = -15.9(2) \cdot 10^{-11}$

First evaluation of S- wave 2π-rescattering

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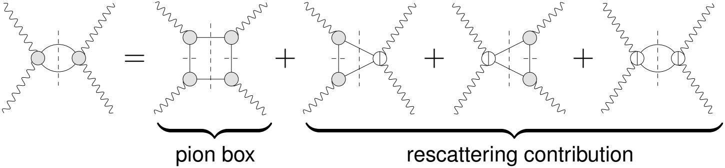
Our first numerical result

Two-pion contributions to HLbL:



Our first numerical result

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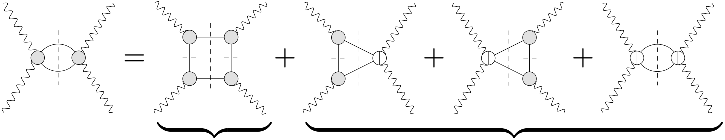

$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

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Conclusions

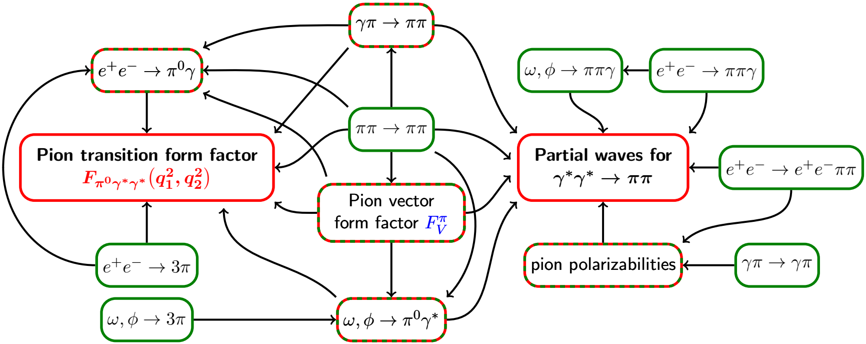
- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes
- I have presented results for the pion-box and the *S*-wave pion-rescattering contributions:
  model independence $=$ much reduced uncertainties

Outlook

- More work is needed to complete the evaluation of contributions of 2π intermediate states
  - take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
  - estimate the dependence on the q^2 of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
  - $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- short-distance constraints need to be derived and imposed

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists

Backup Slides

Detour: the subprocess $\gamma^*\gamma^* \to \pi\pi$

Consider $\gamma^*(q_1, \lambda_1)\gamma^*(q_2, \lambda_2) \to \pi^a(p_1)\pi^b(p_2)$:

$$W_{ab}^{\mu\nu}(p_1, p_2, q_1) = i \int d^4x\, e^{-iq_1\cdot x} \langle \pi^a(p_1)\pi^b(p_2)| T\{j_{\rm em}^\mu(x) j_{\rm em}^\nu(0)\}|0\rangle$$

General tensor decomposition (q_i, $i = 1, \ldots, 3$, $q_3 = p_2 - p_1$):

$$W^{\mu\nu} = g^{\mu\nu} W_1 + \sum_{i,j} q_i^\mu q_j^\nu W_2^{ij}$$

gives ten independent scalar functions.
Gauge invariance requires:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

Gauge invariance: Bardeen-Tung-Tarrach approach

Consider the projector

Bardeen, Tung (68)

$$I^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

which satisfies

$$I_\mu{}^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}, \qquad q_1^\mu I_{\mu\nu} = q_2^\nu I_{\mu\nu} = 0$$

and contract it twice with $W_{\mu\nu}$, leaving it invariant:

$$W_{\mu\nu} = I_{\mu\mu'} I_{\nu'\nu} W^{\mu'\nu'} = \sum_{i=1}^{5} \bar{T}^i_{\mu\nu} \bar{A}_i = \sum_{i=1}^{5} T^i_{\mu\nu} A_i$$

The $\bar{A}_i$ are free of kinematic singularities, but have zeros. To remove the zeros from the $\bar{A}_i$ $\Rightarrow$ remove the poles from the $\bar{T}_i^{\mu\nu}$

Gauge invariance: Bardeen-Tung-Tarrach approach

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$
$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$
$$T_3^{\mu\nu} = q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu,$$
$$T_4^{\mu\nu} = q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu,$$
$$T_5^{\mu\nu} = q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,$$

This is a basis of gauge-invariant tensors, but for $q_1 \cdot q_2 = 0$ it becomes degenerate: need one more structure: Tarrach (75)

$$T_6^{\mu\nu} = \left(q_1^2 q_3^\mu - q_1 \cdot q_3 q_1^\mu\right)\left(q_2^2 q_3^\nu - q_2 \cdot q_3 q_2^\nu\right)$$

Inverse-amplitude method's input