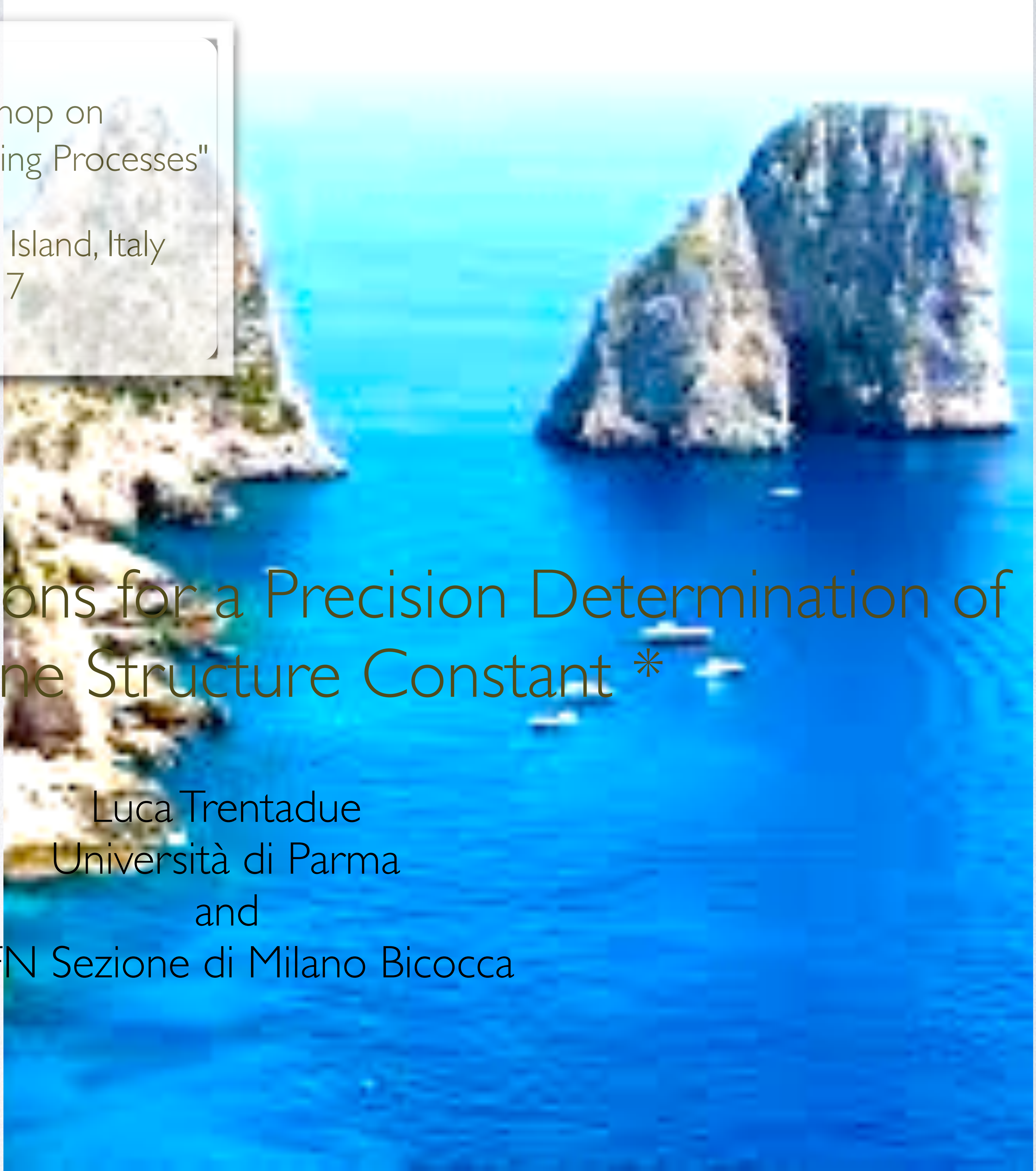


2nd International Workshop on
"Flavour Changing and Conserving Processes"
(FCCP2017)
Villa Orlandi, Anacapri, Capri Island, Italy
7-9 September 2017

Radiative Corrections for a Precision Determination of the Fine Structure Constant *

Luca Trentadue
Università di Parma
and
INFN Sezione di Milano Bicocca

This talk is dedicated to Lev Lipatov 1940-2017

The title of this talk might be:

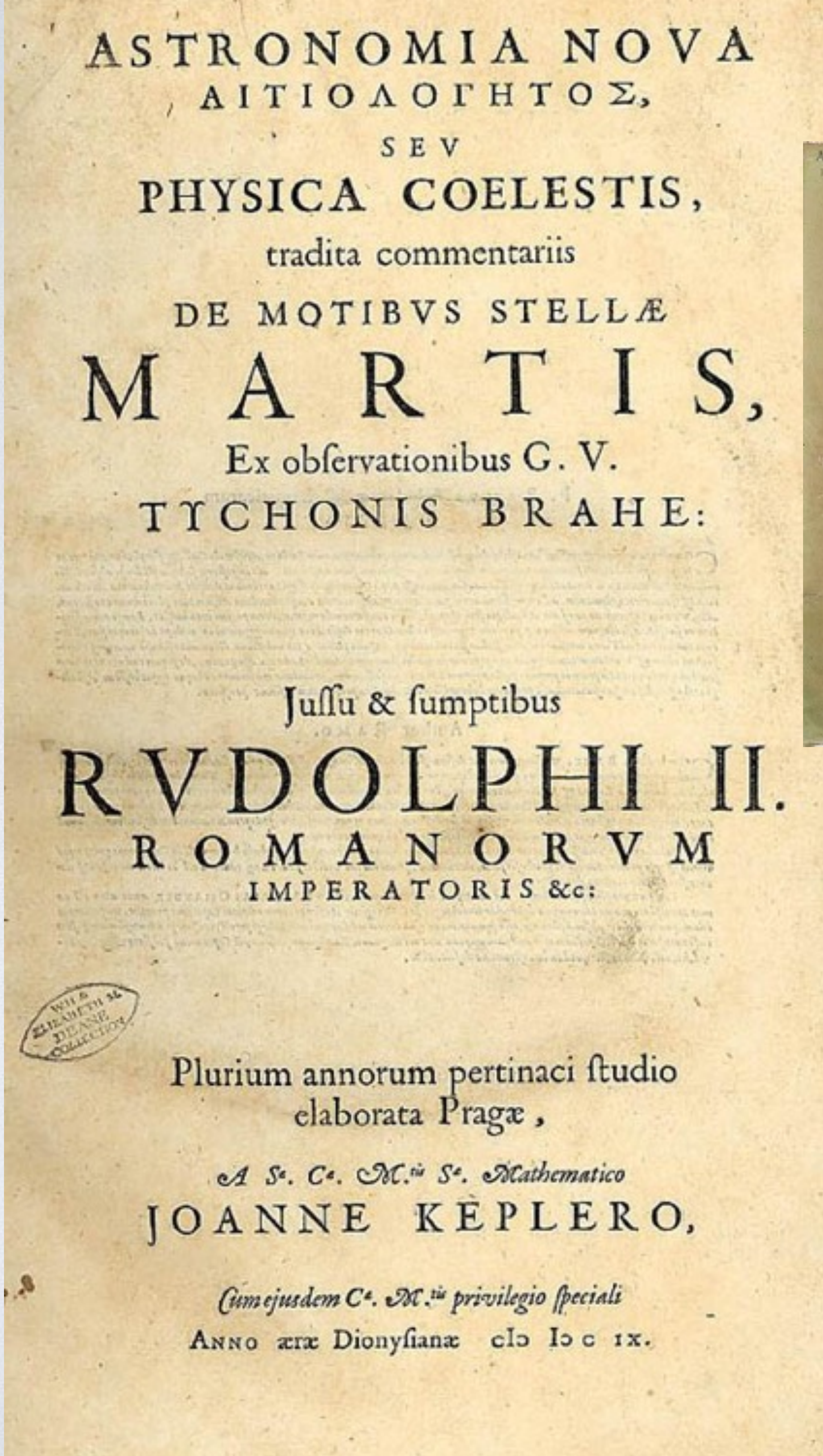
Why physicists make such complex , lenghty and cumbersome calculations ?

or, as in Cicero's "De Oratore II 36",

Historia magistra vitae (est)

("history is life's teacher")

Historia vero testis temporum, lux veritatis, vita memoriae, magistra vitae, nuntia vetustatis, qua voce alia nisi oratoris immortalitati commendatur?
Cicero, De Oratore, II, 36
(By what other voice, too, than that of the orator, is history, the witness of time, the light of truth, the life of memory, the directress of life, the herald of antiquity, committed to immortality?)

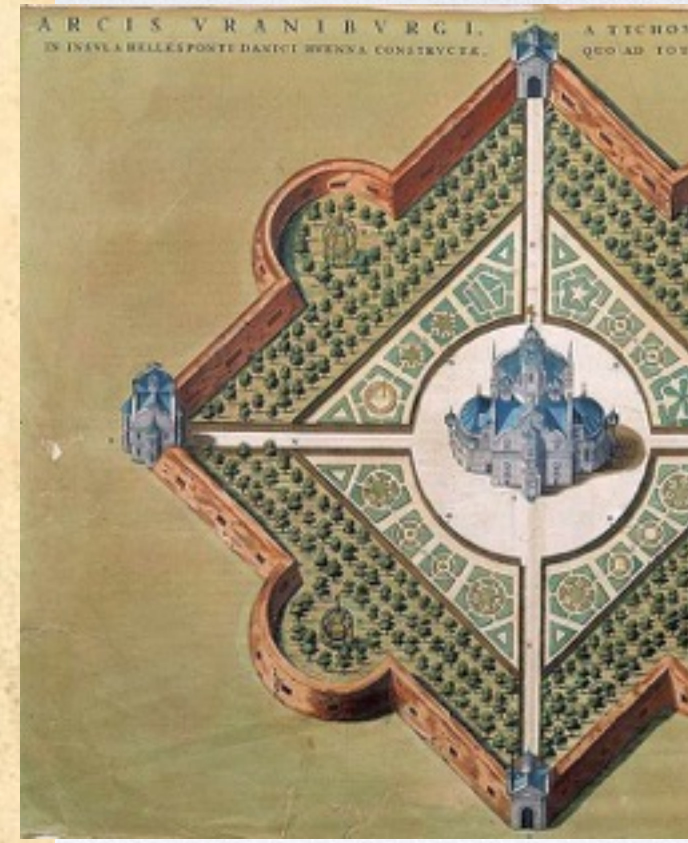

ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

Plurium annorum pertinaci ſtudio
elaborata Pragæ,

A S^e. C^e. M.^tis S^e. Mathematico
JOANNE KEPLERO,

Cum ejusdem C^e. M.^tis privilegio ſpeciali
ANNO æræ Dionyſianæ cIↃ IↃ c IX.



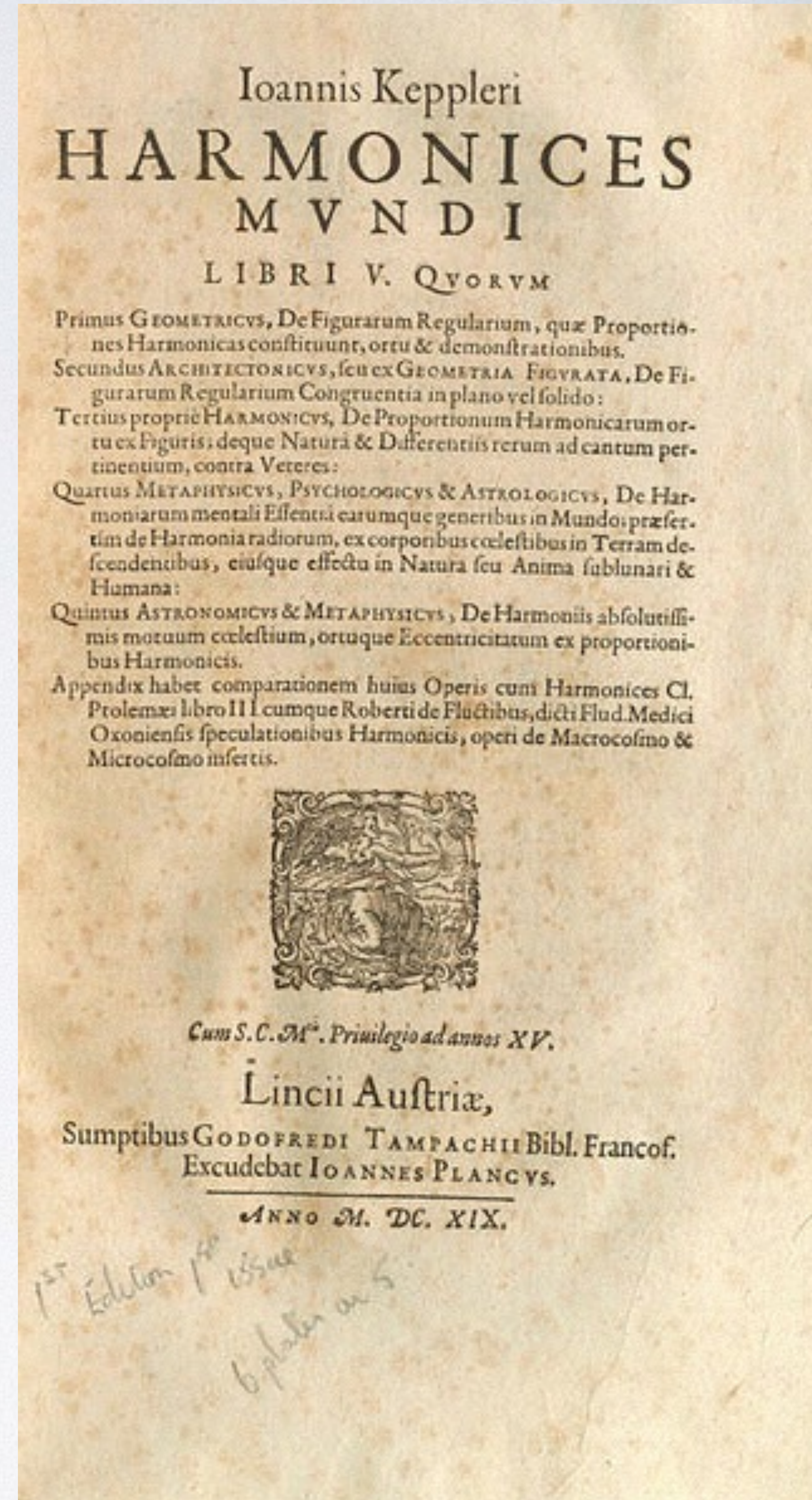
Brahe collected a huge amount of observation (at naked eye) data

Brahe was not a “copernican” nevertheless

Kepler’s Work

- Tycho Brahe led a team which collected data on the position of the planets (1580-1600 with no telescopes).
- Mathematician Johannes Kepler was hired by Brahe to analyze the data.

- He took 20 years of data on position and relative distance.
- No calculus, no graph paper, no log tables.
- Both Ptolemy and Copernicus were wrong.
- He determined 3 laws of planetary motion (1600-1630).



Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.
Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:
Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:
Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:
Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.
Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III cumque Roberti de Fluctibus, dicti Flud. Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

Cum S. C. Mᵗⁱˢ. Privilegio ad annos XV.

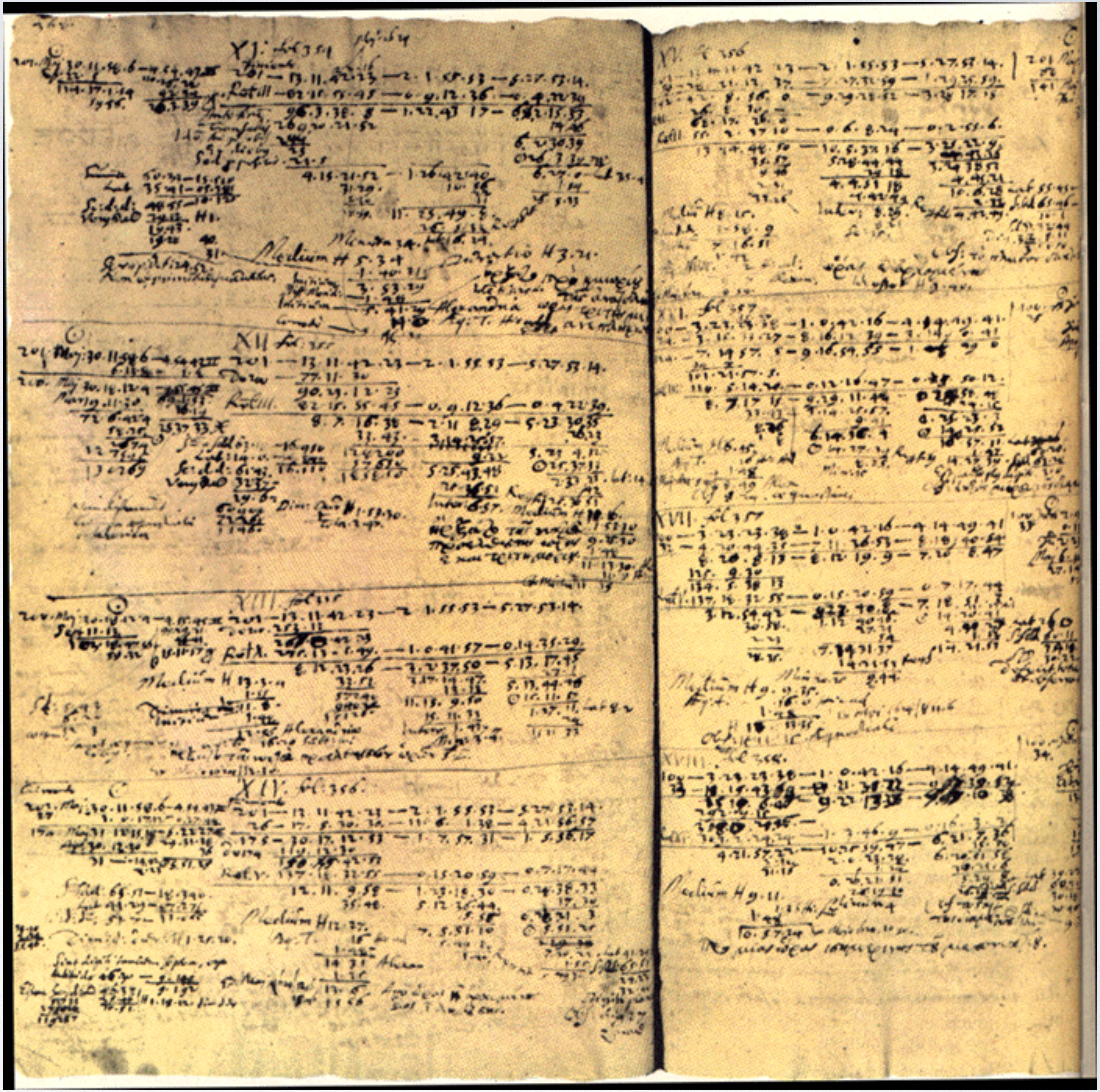
Lincii Austriæ,

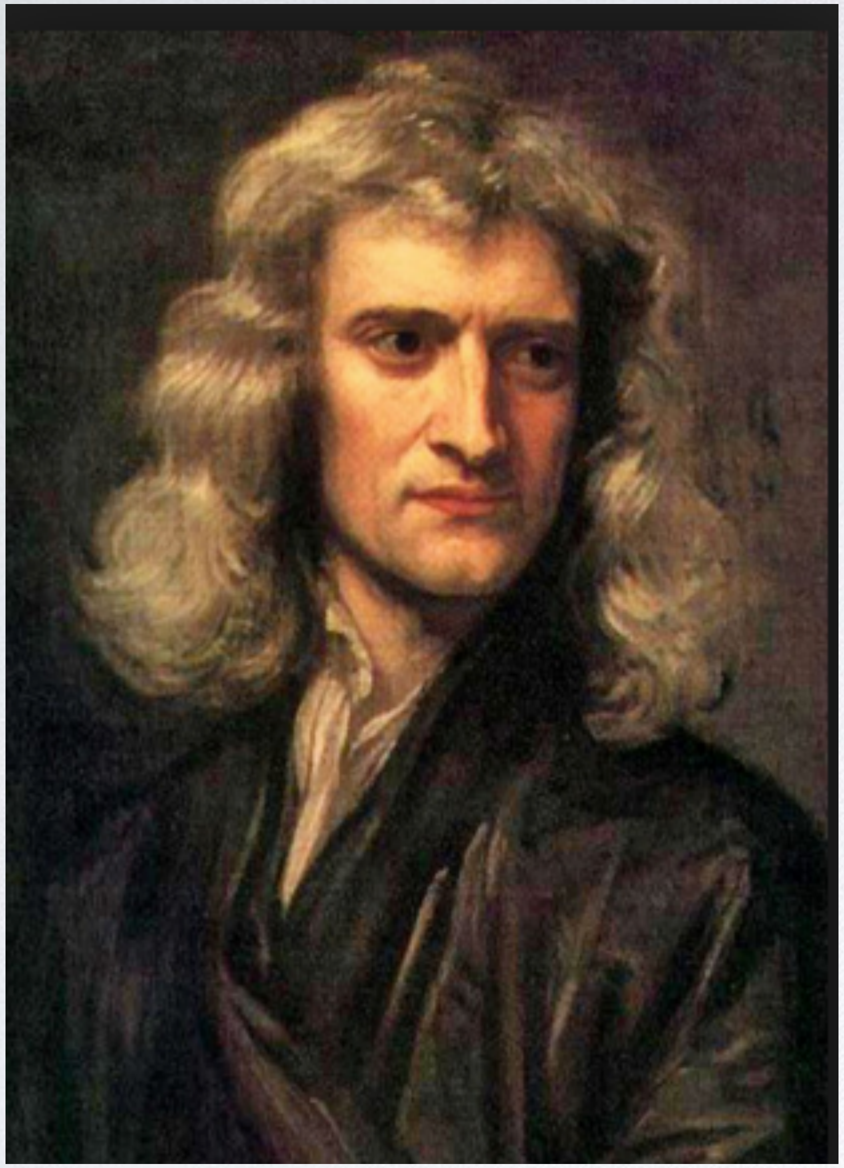
Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.

ANNO M. DC. XIX.

Keplers laws (starting in 1618)

Two pages from the Kepler's logbook





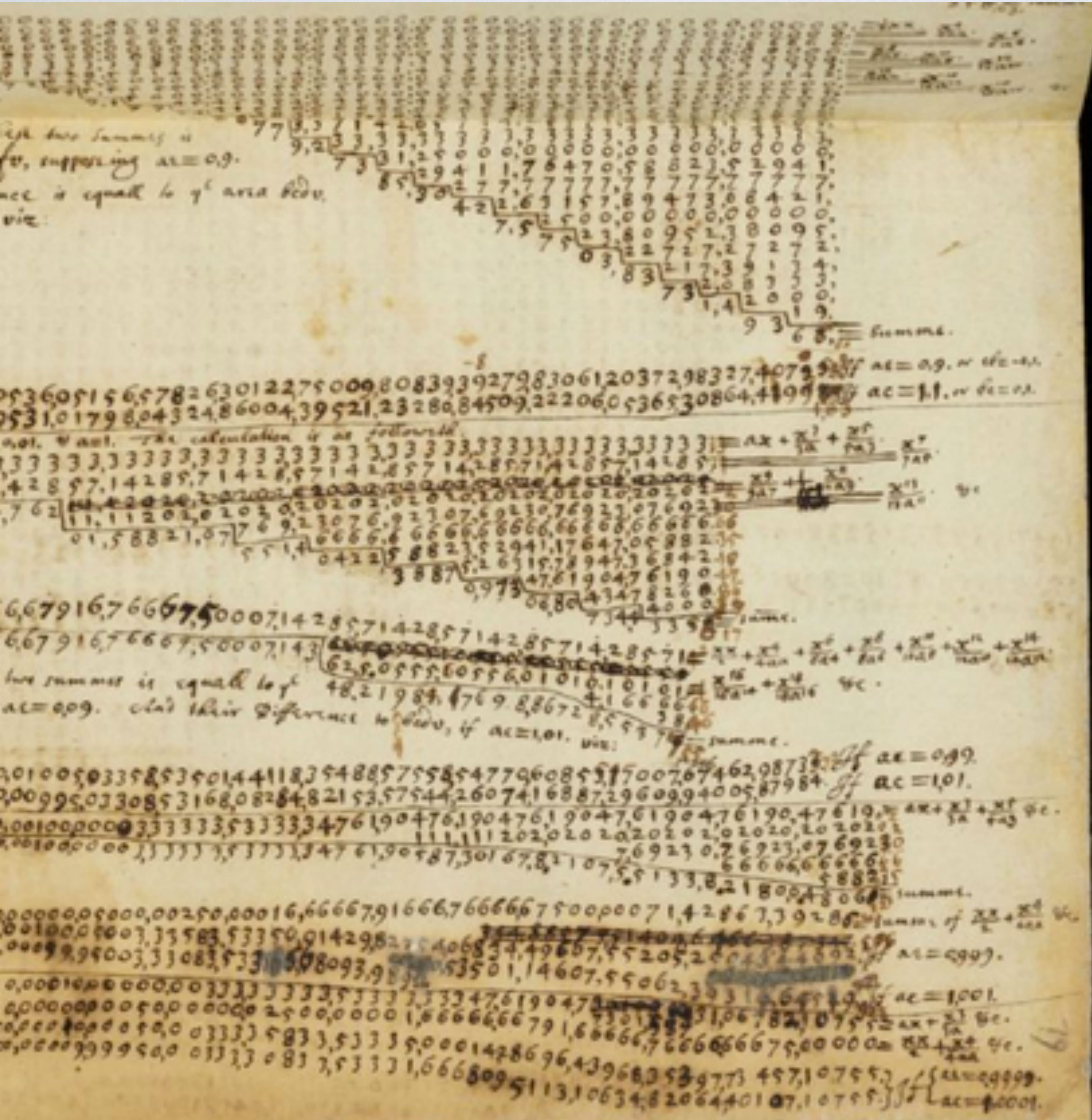
PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA.

Autore *JS.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

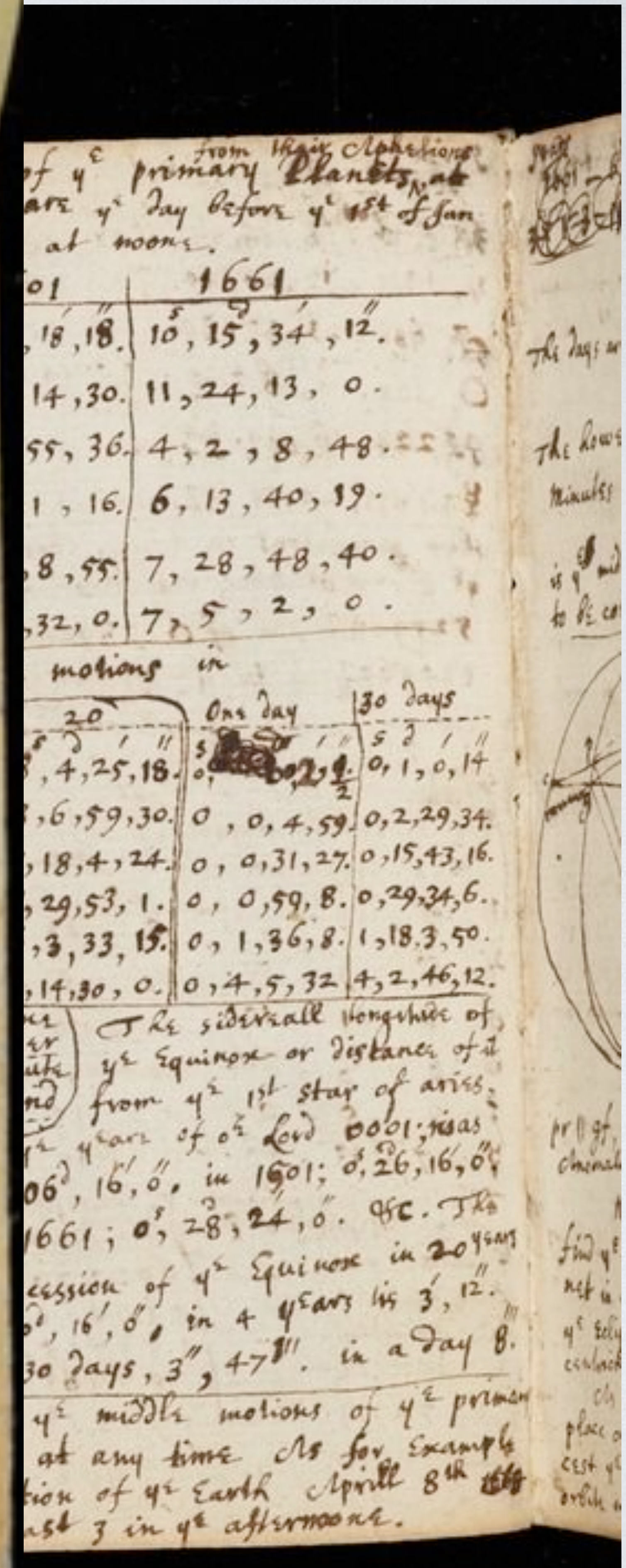
IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.



…these two summes is … suppersing ac = 0,9.
…nce is equall to yᵉ area bcdv. viz
summe.
ac = 0,9. … ac = 1,1.
…0,01. … The calculation is as followeth.
summe.
…two summes is equall to yᵉ … ac = 0,09. And their Difference to bcdv, if ac = 1,01. viz:
summe.
ac = 0,99.
ac = 1,01.
summe.
summe of …
ac = 0,999.
ac = 1,001.



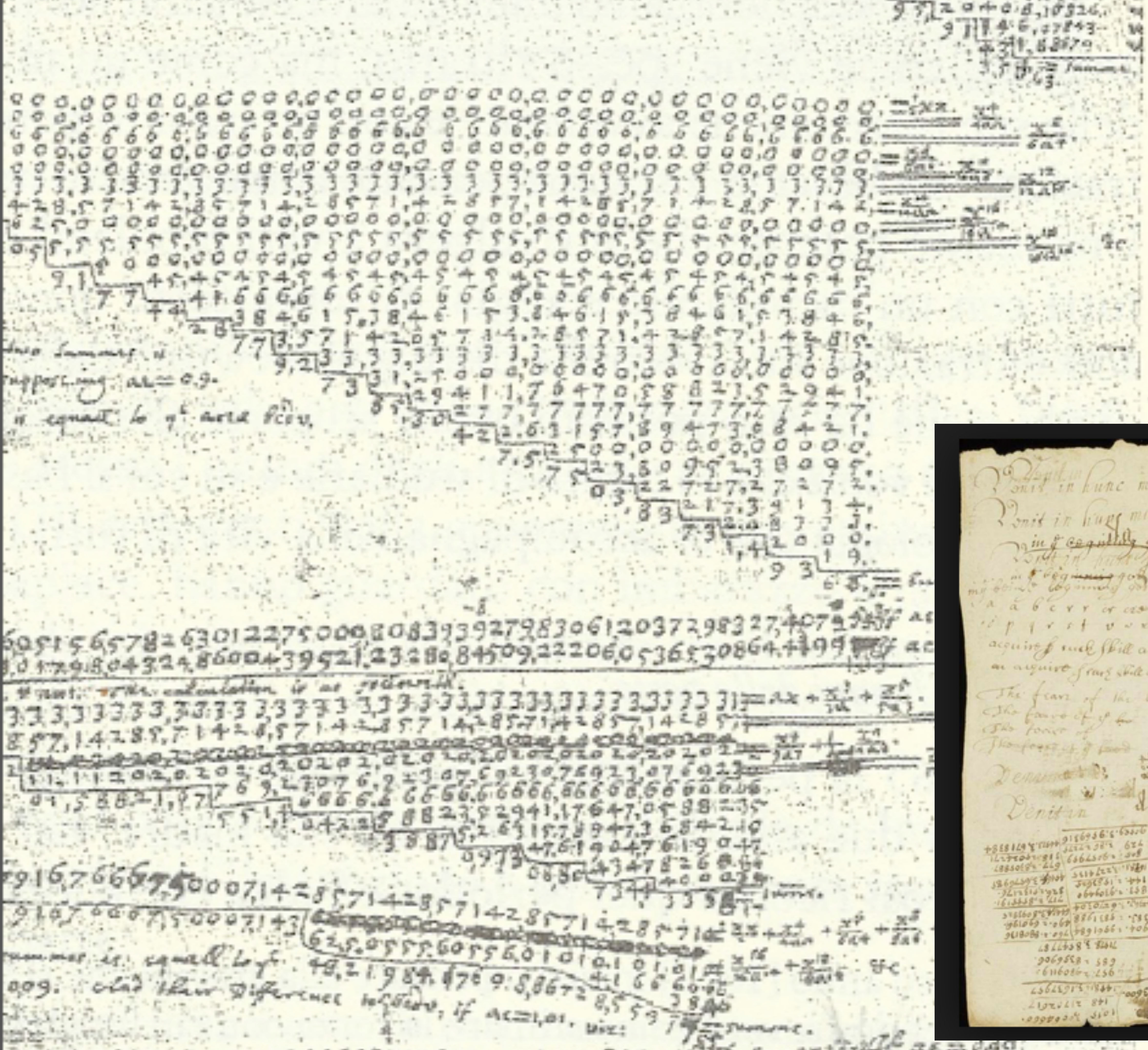
…f yᵉ primary Planets from their Aphelions at …are yᵉ day before yᵉ 1st of Jan at noone.

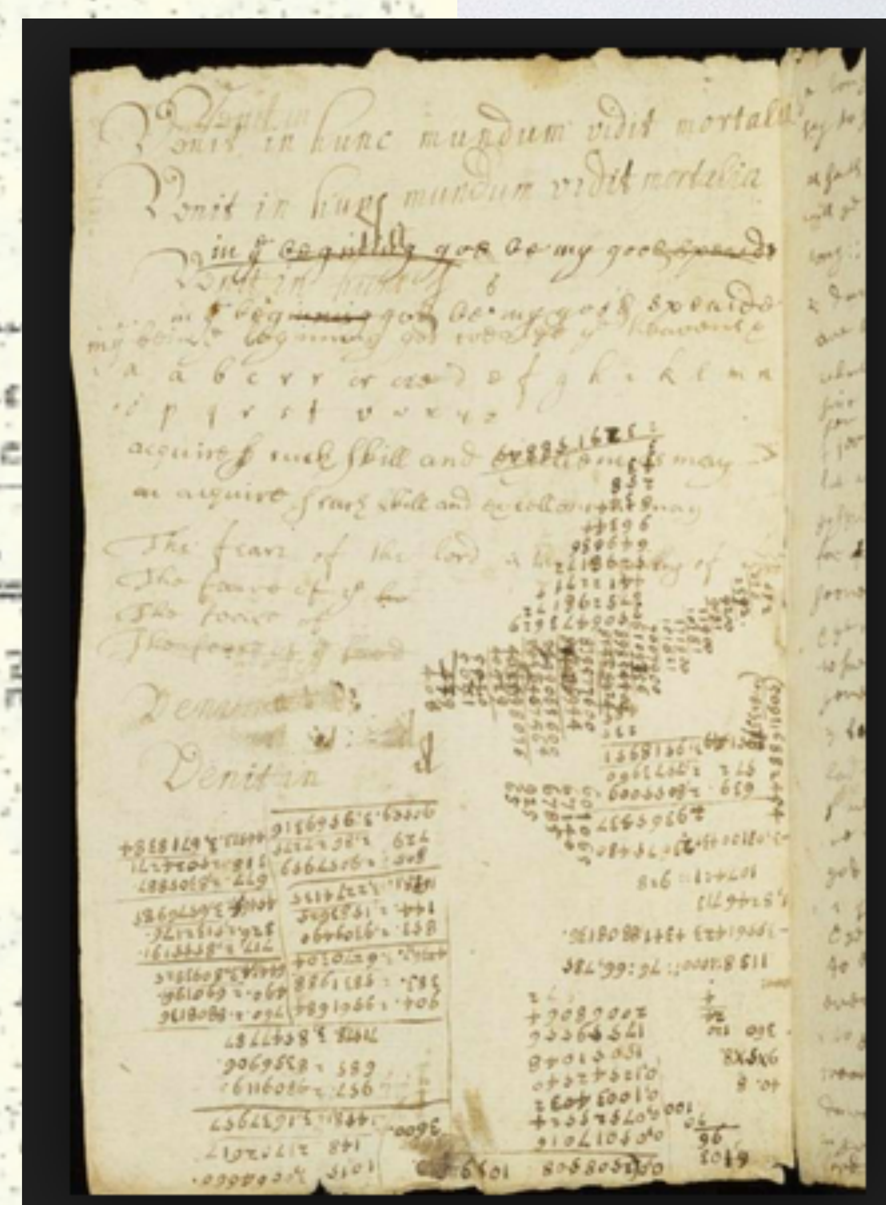
…01	1661
…18,18.	10,15,34,12.
14,30.	11,24,13,0.
55,36.	4,2,8,48.
1,16.	6,13,40,19.
8,55.	7,28,48,40.
32,0.	7,5,2,0.

…motions in

20	One day	30 days
…4,25,18.	[illegible]	0,1,0,14
6,59,30.	0,0,4,59.	0,2,29,34.
18,4,24.	0,0,31,27.	0,15,43,16.
29,53,1.	0,0,59,8.	0,29,34,6.
3,33,15.	0,1,36,8.	1,18.3,50.
14,30,0.	0,4,5,32	4,2,46,12.

The siderall longitude of yᵉ Equinox or distance of it from yᵉ 1st star of aries … yᵉ years of or Lord 0001; was … 06, 16, 0, in 1601; 0, 26, 16, 0 … 1661; 0, 28, 24, 0. &c. The …cession of yᵉ Equinox in 20 years … 16, 0, in 4 years is 3, 12. … 30 days, 3, 47. in a day 8.
… yᵉ middle motions of yᵉ primary … at any time As for example …tion of yᵉ Earth Aprill 8th … …st 3 in yᵉ afternoone.




Venit in hunc mundum vidit mortalia

..to the present days….

Physics Letters B 772 (2017) 232–238

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Physics Letters B

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High-precision calculation of the 4-loop contribution to the electron g-2 in QED

CrossMark

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ARTICLE INFO

Article history:
Received 17 May 2017
Accepted 18 June 2017
Available online 22 June 2017
Editor: G.F. Giudice

Keywords:
Quantum electrodynamics
Anomalous magnetic moment
Feynman diagram
Master integral
High-precision calculation
Analytical fit

ABSTRACT

I have evaluated up to 1100 digits of precision the contribution of the 891 4-loop Feynman diagrams contributing to the electron g-2 in QED. The total mass-independent 4-loop contribution is

$$a_e = -1.912245764926445574152647167439830054060873390658725345\ldots\left(\frac{\alpha}{\pi}\right)^4.$$

I have fit a semi-analytical expression to the numerical value. The expression contains harmonic polylogarithms of argument $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$, one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits.

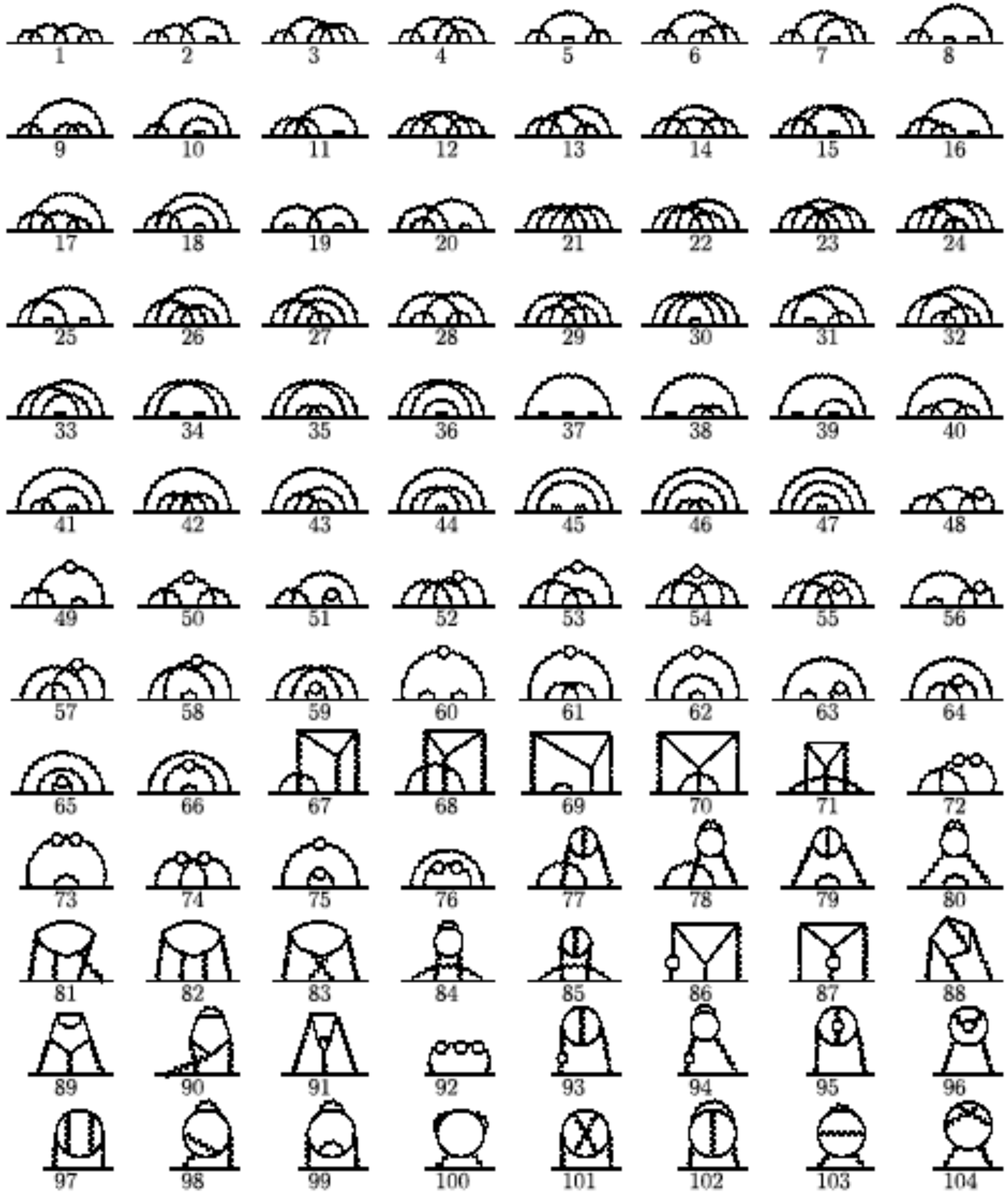


Fig. 1. The 4-loop self-mass diagrams.

$$a_e^{(4)} = T_0 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + \sqrt{3}\,(V_{4a} + V_{6a}) + V_{6b} + V_{7b} + W_{6b} + W_{7b} + \sqrt{3}\,(E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U\,.$$

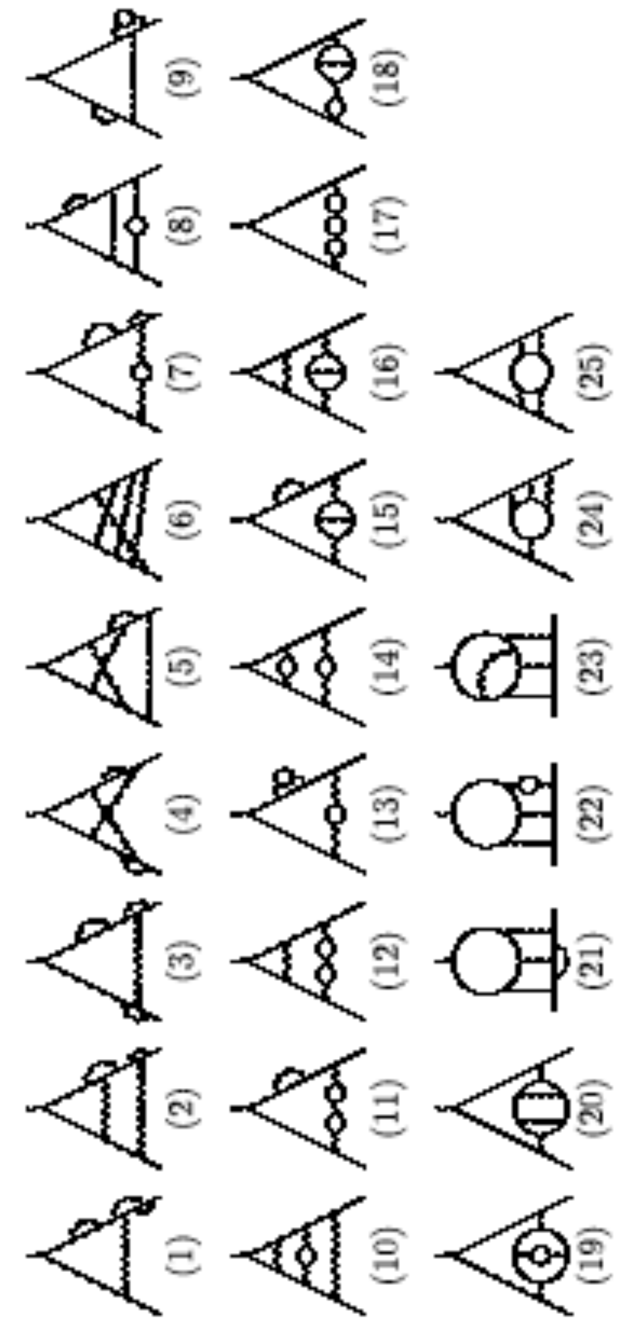


Fig. 2. Examples of vertex diagrams belonging to the 25 gauge-invariant sets. The number indicates the gauge-invariant set to which the diagram belongs. In the case of the sets 1–16, 24, 25, the other diagrams of each set can be obtained by permuting separately the vertices on the left and right side of the main electron line, and considering also the mirror images of the diagrams; in the sets containing diagrams with vacuum polarization insertions, one must also move the vacuum polarization insertion to each internal photon line. In the sets containing light-light diagrams, one must also consider the permutations of the vertices of the electron loop.

Table 1
First 1100 digits of $a_e^{(4)}$.

−1.9122457649264455741526471674398300540608733906587253451713298480060
3844398065170614276089270000363158375584153314732700563785149128545391
9028043270502738223043455789570455627293099412966997602777822115784720
3390641519081665270979708674381150121551479722743221642734319279759586
0740500578373849607018743283140248380251922494607422985589304635061404
9225266343109442400023563568812806206454940132249775943004292888367617
4889923691518087808698970526357853375377696411702453619601349757449436
1268486175162606832387186747303831505962741878015305514879400536977798
3694642786843269184311758895811597435669504330483490736134265864995311
6387811743475385423488364085584441882237217456706871041823307430517443
0557394596117155085896114899526126606124699407311840392747234002346496
9531735482584817998224097373710773657404645135211230912425281111372153
0215445372101481112115984897088422327987972048420144512282845151658523
6561786594592600991733031721302865467212345340500349104700728924487200
6160442613254490690004319151982300474881814943110384953782994062967586
7875385249781946989793132162197975750676701142904897962085050785592...

Table 2
Contribution to $a_e^{(4)}$ of the 25 gauge-invariant sets of Fig. 2.

1	−1.971075616835818943645699655337264406980
2	−0.142487379799872157235945291684857370994
3	−0.621921063535072522104091223479317643540
4	1.086698394475818687601961404690600972373
5	−1.040542410012582012539438620994249955094
6	0.512462047967986870479954030909194465565
7	0.690448347591261501528101600354802517732
8	−0.056336090170533315910959439910250595939
9	0.409217028479188586590553833614638435425
10	0.374357934811899949081953855414943578759
11	−0.091305840068696773426479566945788826481
12	0.017853686549808578110691748056565649168
13	−0.034179376078562729210191880996726218580
14	0.006504148381814640990365761897425802288
15	−0.572471862194781916152750849945181037311
16	0.151989599685819639625280516106513042070
17	0.000876865858889990697913748939713726165
18	0.015325282902013380844497471345160318673
19	0.011130913987517388830956500920570148123
20	0.049513202559526235110472234651204851710
21	−1.138822876459974505563154431181111707424
22	0.598842072031421820464649513201747727836
23	0.822284485811034346719894048799598422606
24	−0.872657392077131517978401982381415610384
25	−0.117949868787420797062780493486346339829

one finds

$$\alpha^{-1}(a_e) = 137.035\,999\,1596(27)(18)(331)\,,$$

A renowned paper

65 years ago…

Freeman J. Dyson

PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

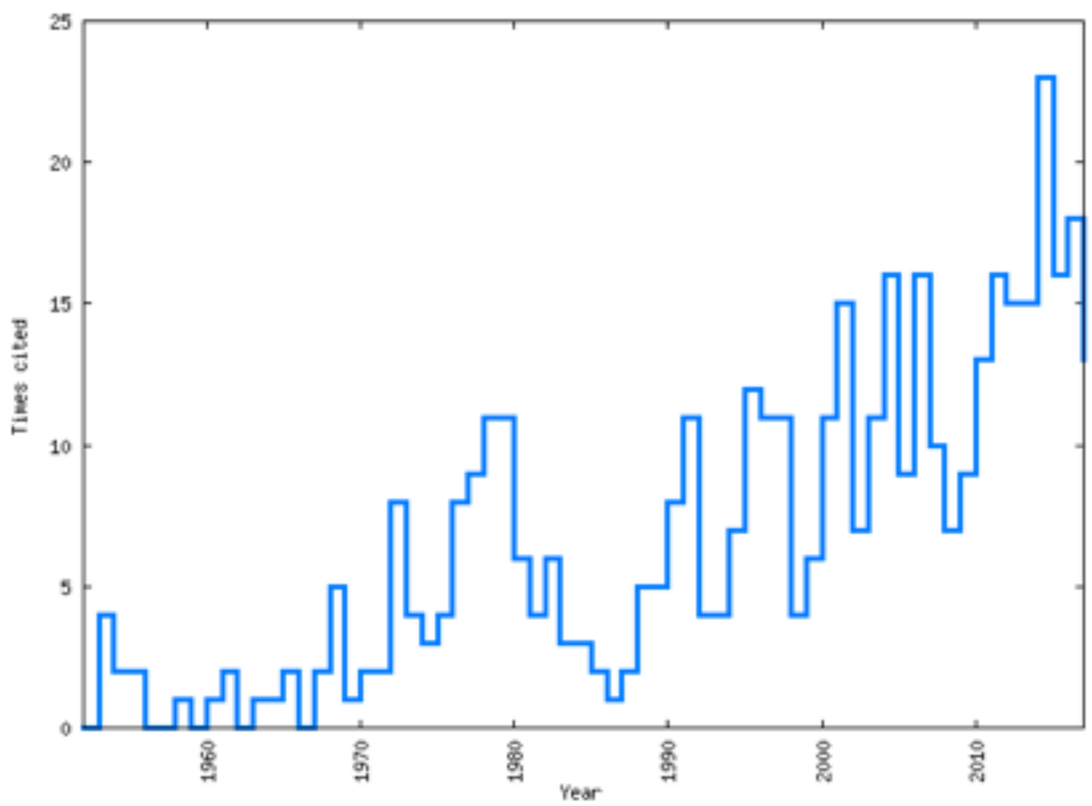
Divergence of Perturbation Theory in Quantum Electrodynamics

F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

Cronologia delle citazioni:



this (2 pages) paper has received
15 citations in 2017
(as of today)
22 citations in 2016
. . .

Informazioni | Riferimenti (7) | Citazioni (441) | File | Grafico

Divergence of perturbation theory in quantum electrodynamics

F.J. Dyson (Cornell U., LNS)

1952 - 2 pages

Phys.Rev. 85 (1952) 631-632
Also in *Le Guillou, J.C. (ed.), Zinn-Justin, J. (ed.): Large-order behaviour of perturbation theory* 32-33
DOI: 10.1103/PhysRev.85.631

Abstract (APS)
An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

Keyword(s): INSPIRE: quantum electrodynamics | perturbation theory | renormalization | asymptotic expansion

Record added 1991-05-11, last modified 2017-02-07

The indicated limit of error in this latter value is the standard deviation derived from seven measurements.

The fact that the difference ${}_{83}Bi^{209}-{}_{82}Pb^{208}$ is 5 mMU larger than unity indicates a sharp increase in the slope of the packing fraction curve. This agrees with the expectation since Bi^{209} has one proton more than the magic number 82. The addition of this single proton adds, in this case, only 3 Mev to the binding energy of the nucleus. This result is in reasonable agreement with the difference of 1.004 mass units derived from the disintegration data of Harvey.[5]

[5] J. A. Harvey, Phys. Rev. **81**, 353 (1951).

PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

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ALL existing methods of handling problems in quantum electrodynamics give results in the form of power-series in e^2. The individual coefficients in these series are finite after mass and charge renormalization. The technique of renormalization can at present be applied only to the separate coefficients, and not to the series as a whole. If the series converges, its sum is a calculable physical quantity. But if the series diverges, we have no method of calculating or even of defining the quantity which is supposed to be represented by the series.

Several authors have remarked[1] that the series after renormalization will be divergent in a trivial way, if the series represents a scattering amplitude of a free particle, in circumstances where the particle has a possibility of being captured into a permanently bound system. In this situation a perturbation expansion of the scattering amplitude will diverge, even in nonrelativistic quantum mechanics,[2] and in the relativistic theory the series will diverge for the same reason. It is to be expected that such trivial divergences will not impose any fundamental limitations on the use of the renormalization method. In fact, a new method of carrying through the renormalization program has been developed,[3] a method which is applicable to problems involving bound systems and in which divergences of this elementary nature cannot occur. In the new method the series expansion arises from a formal integration of the equations of motion over a finite interval of time, and in an elementary nonrelativistic theory such a perturbation expansion would necessarily be convergent. For this reason it was claimed as probable[4] that the power series arising from the application of the new method in quantum electrodynamics would always converge. If the claim had been accompanied by a proof of convergence, then the theoretical framework of quantum electrodynamics could have been considered closed, being within its limits a complete and consistent theory.

The purpose of this note is to present a simple argument which indicates that the power-series expansions obtained by integrating the equations of motion in quantum electrodynamics will be divergent after renormalization. The divergence is of a basic character, different from the trivial divergences mentioned above, and is present equally in the results obtained from the new and the older methods of calculation. The argument here presented is lacking in mathematical rigor and in physical precision. It is intended only to be suggestive, to serve as a basis for further discussions. To me it seems convincing enough to merit publication in its present incomplete form; also I am glad to have this opportunity to withdraw the erroneous argument previously put forward[5] to support the claim that the power series should converge.

The argument for divergence is as follows. According to Feynman,[6] quantum electrodynamics is equivalent to a theory of the motion of charges acting on each other by a direct action at a distance, the interaction between two like charges being given by the formula

$$e^2\delta_+(s_{12}^2), \tag{1}$$

where e is the electron charge. The action-at-a-distance formulation is precisely equivalent to the usual formulation of the theory, in circumstances where all emitted radiation is ultimately absorbed. We shall suppose that

[1] B. Ferretti, Nuovo cimento **8**, 108 (1951); K. Nishijima, Prog. Theor. Phys. **6**, 37 (1951).
[2] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).
[3] F. J. Dyson, Proc. Roy. Soc. (London) **A207**, 395 (1951). Phys. Rev. **83**, 608, 1207 (1951).
[4] Phys. Rev. **83**, 608 (1951), Section XII.
[5] See reference 4. The error in the argument lay in using the concept "the number of times that an interaction operates" in an intuitive and imprecise way.
[6] R. P. Feynman, Phys. Rev. **76**, 769 (1949), Eq. (4); Phys. Rev. **80**, 440 (1950), Appendix B.

conditions are such as to justify the use of the Feynman formulation of the theory. Then let

$$F(e^2)=a_0+a_2e^2+a_4e^4+\cdots \tag{2}$$

be a physical quantity which is calculated as a formal power series in e^2 by integrating the equations of motion of the theory over a finite or an infinite time. Suppose, if possible, that the series (2) converges for some positive value of e^2; this implies that $F(e^2)$ is an analytic function of e at $e=0$. Then for sufficiently small values of e, $F(-e^2)$ will also be a well-behaved analytic function with a convergent power-series expansion.

But for $F(-e^2)$ we can also make a physical interpretation. Namely, $F(-e^2)$ is the value that would be obtained for F in a fictitious world where the interaction between like charges is $[-e^2\delta_+(s_{12}^2)]$ instead of (1). In the fictitious world, like charges attract each other. The potential between static charges, in the classical limit of large distances and large numbers of elementary charges, will be just the classical Coulomb potential with the sign reversed. But it is clear that in the fictitious world the vacuum state as ordinarily defined is not the state of lowest energy. By creating a large number N of electron-positron pairs, bringing the electrons together in one region of space and the positrons in another separate region, it is easy to construct a "pathological" state in which the negative potential energy of the Coulomb forces is much greater than the total rest energy and kinetic energy of the particles. This can be done without using particularly small regions or high charge densities, so that the validity of the classical Coulomb potential is not in doubt. Suppose that in the fictitious world the state of a system is known at a certain time to be an ordinary physical state with only a few particles present. There is a high potential barrier separating the physical state from the pathological states of equal energy; to overcome the barrier it is necessary to supply the rest-energy for the creation of many particles. Nevertheless, because of the quantum-mechanical tunnel effect, there will always be a finite probability that in any finite time-interval the system will find itself in a pathological state. Thus every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization. In these circumstances it is impossible that the integration of the equations of motion of the theory over any finite or infinite time interval, starting from a given state of the fictitious world, should lead to well-defined analytic functions. Therefore $F(-e^2)$ cannot be analytic and the series (2) cannot be convergent.

The divergence of the series in the real world is associated with virtual processes in which large numbers of particles are involved. Therefore the divergence will only become noticeable when terms of very high order in the expansion (2) are considered. A crude quantitative estimate indicates that the terms of (2) will decrease to a minimum and then increase again without limit, the index of the minimum term being roughly of the order of magnitude 137. This estimate assumes the system to be such that the trivial kind of divergence discussed earlier does not occur. The nontrivial and unavoidable divergence will not prevent practical calculations being made with the series (2), to an accuracy far beyond anything at present required or contemplated. Only if similar arguments should be found to be applicable to meson theory, the divergence might impose a severe limitation on the possible accuracy of practical calculations in that field.[7]

If the conclusion of the foregoing argument is accepted, then there are two alternative possibilities for the future development of quantum electrodynamics. Alternative A: There may be discovered a new method of carrying through the renormalization program, not making use of power series expansions. In this case every physical quantity $F(e^2)$ will be well-defined and calculable, and the series (2) will be an asymptotic expansion for it in the limit of small e. Since $F(e^2)$ is not analytic at $e=0$, the asymptotic expansion will not be sufficient to determine the function uniquely. The additional information necessary to determine $F(e^2)$ will be obtained from the existing formalism, using no new physical hypotheses but only some improved mathematical methods. Alternative B: All the information that can in principle be obtained from the formalism of quantum electrodynamics is contained in the coefficients a_0, a_2, a_4, $\cdots$ of series such as (2). In this case the quantity $F(e^2)$ is neither physically well-defined nor mathematically calculable, except in so far as the asymptotic expansion (2) gives some workable approximation to it. In order to define $F(e^2)$ precisely, not merely new mathematical methods but a new physical theory is needed.

I wish to call attention to the attractive features of alternative B in the present state of physics. If B were true, it would imply that quantum electrodynamics is in its mathematical nature not a closed theory, but only a half-theory giving insufficient information for the exact prediction of events. Experimentally we know that the world contains one group of phenomena which is accurately in agreement with the results of quantum electrodynamics, and another group of phenomena which is not understood at all. We need to develop new physical ideas to understand the second group, and still we cannot abandon the theory which successfully accounts for the first. If quantum electrodynamics were a closed theory, this would be a difficult dilemma. But if the theory itself leaves room for new ideas, no such dilemma arises. In conclusion, I wish to thank Professors Pauli, Bethe, Pais, and Oppenheimer for valuable discussions of these problems.

[7] C. A. Hurst in a private communication informs me that he has discovered by direct calculation the fact that the S-matrix diverges in the way here described, in the case of a simple scalar meson theory, assuming that certain terms which are not yet calculated do not decisively change the behavior of the series.

Silvan S. Schweber in
"QED and the men who made it: Dyson, Feynman, Scwinger and Tomonaga",
Princeton University Press, 1994.

The paper on the divergence of perturbation theory in QED marked the end of Dyson's involvement with QED. Although during 1952/53 Dyson was deeply involved in the Cornell project analyzing meson-nucleon scattering using field-theoretic methods (Dyson 1954), Dyson confesses "my heart was not in this Tamm-Dancoff work. There were no grand hopes."[202] Dyson had been deeply hurt by the reception of his extended hard work in QED. The community had not appreciated what he had done and "being a practical person, [he] didn't feel like going on all by [himself] into the wilderness and wait for the world to catch up." The difference between Feynman, Schwinger, and Dyson was that Feynman and Schwinger had "stopped at the crest of the wave." Dyson had been more courageous, only to find "the river disappearing into sand."[203]

Thereafter he never invested the same amount of energy and commitment into any fundamental physics program. "I wasn't so much disappointed that these papers were not noticed," Dyson claims. "I thought to myself: well really it was silly of me to have worked so hard on something which turned out not to be important."[204]

Quantum electrodynamics occupies a unique position in contemporary physics. It is the only part of our science which has been completely reduced to a set of precise equations. It is the only field in which we can choose a hypothetical experiment and predict the result to five places of decimals confident that the theory takes into account all the factors that are involved. Quantum electrodynamics gives us a complete description of what an electron does; therefore in a certain sense it gives us an understanding of what an electron is. It is only in quantum electrodynamics that our knowledge is so exact that we can feel we have some grasp of the nature of an elementary particle. (Dyson 1953a)

I always felt it was a miracle that electrons actually behaved the way the theory said. To me it was always an amazing experimental fact that this perturbation series was somehow real, and everything the perturbation series said turned out to be right. I never felt that we really had understood the theory in the philosophical sense—where by understood I mean having a well defined and consistent mathematical scheme. [Nonetheless] I always felt that it was obviously true, true even with a big T. Truth to me, means agreeing with the experiments. . . . For a theory to be true it has to describe accurately what really happens in the experiments.[207]

207 Silvan S Schweber FJD interview in "QED and the men who made it"

For Dyson the function of theories is to account for experimental phenomena. Should a theory fail to do so, it will be replaced. Moreover, "The nature of a future theory is not a profitable subject for theoretical speculation. [A] future theory will be built, first of all upon the results of future experiments" (Dyson 1949b, p. 1755).

P.A.M. Dirac at the Tahlasse conference

DOES RENORMALIZATION MAKE SENSE?

P. A. M. Dirac
Florida State University, Tallahassee, Fl. 32306

The physics of elementary … steady progress but there is no re… basis for it. One has a well-defi… quantum mechanics, built up from … tions of Heisenberg and Schrödi… attempts to make it conform to spe… always gets infinities appearin… except in trivial cases. Such e… sense.

People have developed a technique for handling the infinities in certain theories. For these theories the infinities can all be collected into certain parameters representing physical constants, which are then renormalized to their experimental values, and so the infinities get discarded. The resulting equations are well-defined and can be used to calculate results that can be compared with experiment. The agreement is often very good, and many physicists are well-satisfied with this situation.

However there is no logical justification for it. One is discarding certain infinitely large terms simply because one does not want them in the equations. It is quite correct to discard terms that are small, but to neglect large terms is not allowed. The whole perturbation procedure used for solving the equations breaks down. In these circumstances one does not have a mathematical theory, but just a set of working rules.

Some physicists may be happy … working rules leading to results … observation. They may think that … physics. But it is not enough. One wants to understand how Nature works. There is strong reason to believe that Nature works according to mathematical laws. All the substantial progress of science supports this view. In elementary particle physics we do not have these mathematical laws, only working rules. One should not be satisfied with them, but should continue to search for laws based on sound mathematics.

In defense of working rules one could say that the whole history of atomic physical theory has been built up in terms of them, with continual improvements being made. It began with the Balmer formula, which was at first simply a rule with no justification. Then Bohr obtained a justification for it, in terms of an atomic model based on various assumptions of the nature of working rules.

With Heisenberg and Schrödinger one had a true mathematical theory. But it turned out to be inadequate to include relativistic effects. One again has to bring in working rules. It means that the fundamental laws of physics have not yet been found.

In the case of electrodynamics one can get a logical mathematical theory by introducing a cut-off into the interaction between the charged particles and the electromagnetic field for the high frequencies of the field. The infinities then become finite, and can be made small by suitably choosing the point of cut-off. This is possible owing to the smallness of the coupling constant 1/137. Of course if we do this the theory ceases to be accurately relativistic, but remains approximately so. We can then use a perturbation method for solving the equations. We have restored the logic at the expense of the relativity.

The resulting theory gives correctly the first order effects, but not the higher orders. But the ordinary theory with infinite renormalization does give the higher order effects for QED in agreement with observation. So the working rules are superior to the cut-off theory. Also the cut-off is ugly and is not to be recommended.

Some further changes are needed in the foundations of atomic theory, perhaps just as drastic as the change from Bohr orbits to Heisenberg's quantum mechanics. The present situation is in some ways analogous to the Bohr-orbit era. We then had Bohr orbits, giving good results ...ms in which only one ...ole. Theoretical ...n the problem of how ...ger succes. ...e quite ineffective. ...n methods, which was ...apid development of ...dequate to solve the ...hrough was finally ... from an entirely ... mathematics. ...ifficulties will get ...pment involving new ...resentations of the ... The irreducible representations are all known, but among the non-irreducible ones there is a wide field for further investigation.

The resulting theory gives correctly the first order effects, but not the higher orders. But the ordinary theory with infinite renormalization does give the higher order effects for QED in agreement with observation. So the working rules are superior to the cut-off theory. Also the cut-off is ugly and is not to be recommended.

Some further changes are needed in the foundations of atomic theory, perhaps just as drastic as the change from Bohr orbits to Heisenberg's quantum mechanics. The present situation is in some ways analogous to the Bohr-orbit era. We then had Bohr orbits, giving good results

Alpha

New Determination of the Fine Structure Constant and Test of the Quantum Electrodynamics

Rym Bouchendira,[1] Pierre Cladé,[1] Saïda Guellati-Khélifa,[2] François Nez,[1] and François Biraben[1]
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(Received 15 December 2010; published 24 February 2011)

We report a new measurement of the ratio h/m_{Rb} between the Planck constant and the mass of ^{87}Rb atom. A new value of the fine structure constant is deduced, $\alpha^{-1} = 137.035\,999\,037\,(91)$ with a relative uncertainty of 6.6×10^{-10}. Using this determination, we obtain a theoretical value of the electron anomaly $a_e = 0.001\,159\,652\,181\,13(84)$, which is in agreement with the experimental measurement of Gabrielse [$a_e = 0.001\,159\,652\,180\,73(28)$]. The comparison of these values provides the most stringent test of the QED. Moreover, the precision is large enough to verify for the first time the muonic and hadronic contributions to this anomaly.

DOI: 10.11

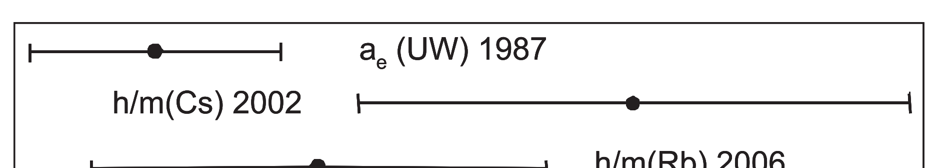
The fine structure c... of the electromagnetic interaction. This dimensionless quantity is defined as: $\alpha = e^2/4\pi\epsilon_0\hbar c$, where ϵ_0 is the permittivity of vacuum, c the speed of light, e the electron charge and $\hbar$ the reduced Planck constant ($\hbar = h/2\pi$). It appears in the expressions of the ionization energy of hydrogen atom, of the fine and hyperfine structures of atomic energy levels, and it is the parameter of the quantum electrodynamics (QED) calculations. Its measurement in different domains of physics is a test of the consistency of the theory. The most accurate value is deduced from the combination of the measurement of the electron anomaly a_e with a very difficult QED calculation. The last result, by Gabrielse at Harvard University, gives a value of α with a relative uncertainty of 3.7×10^{-10} [1,2]. This impressive result depends completely upon QED calculations. Thus, when in 2007 Aoyama *et al.* detected an error, the α value shifted by 4.7×10^{-9} [2–4]. Consequently, to check these calculations, another determination of α is required. Up to now all values of α that depend upon QED much less were

... surement of Gabrielse by comparison with the value obtained by Dehmelt at the University of Washington [8] and the recent correction found in the calculation of the electron anomaly [2]. The discussion on this agreement will be presented at the end of this Letter.

The fine structure constant is deduced from the measurement of h/m_{Rb} thanks to the relation

$$\alpha^2 = \frac{2R_\infty}{c}\frac{m_{\mathrm{Rb}}}{m_e}\frac{h}{m_{\mathrm{Rb}}}, \tag{2}$$

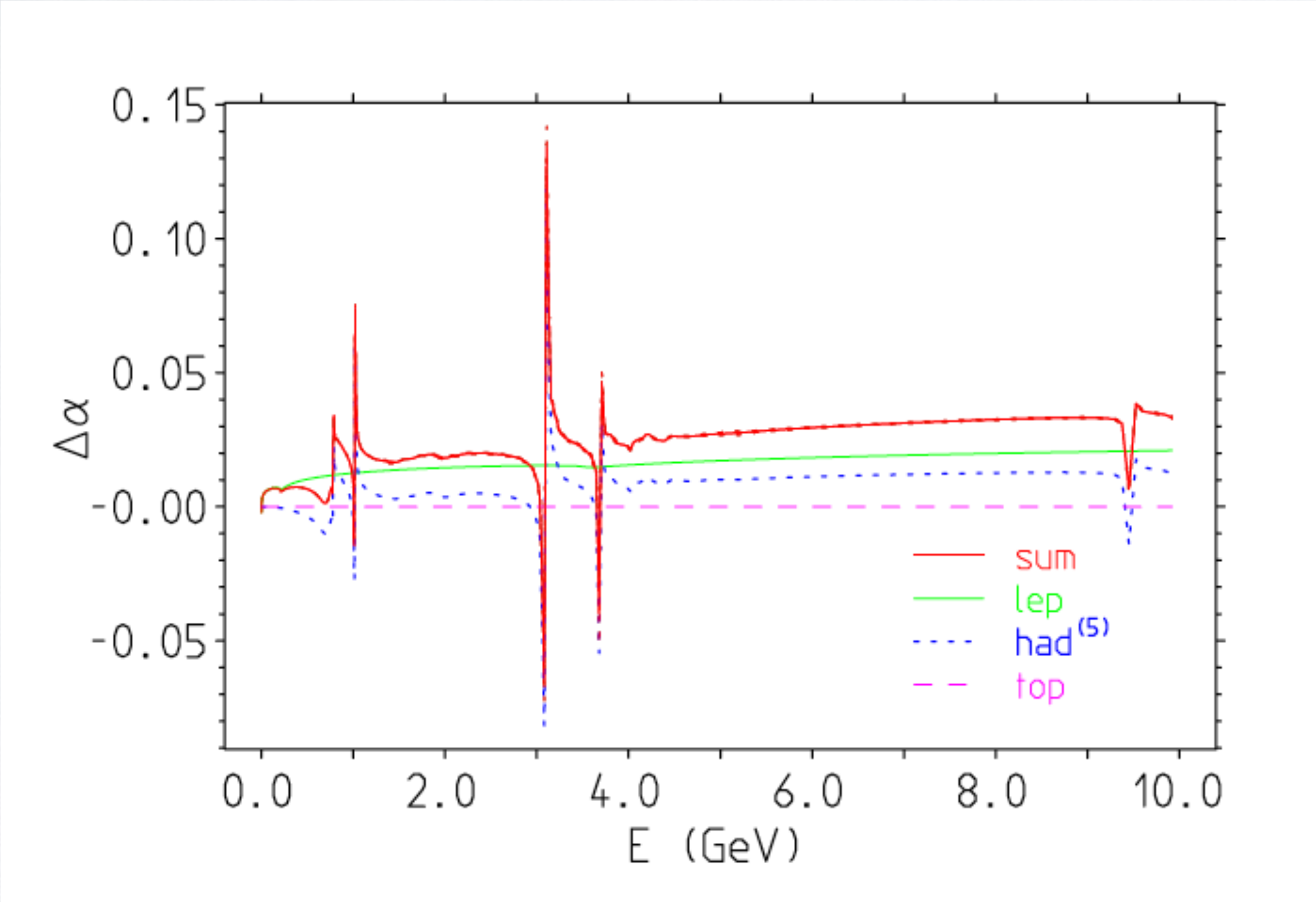
where m_e is the electron mass. In Eq. (2), the Rydberg constant R_∞ and the mass ratio m_{Rb}/m_e are known with an accuracy of 7×10^{-12} [5,9,10] and 4.4×10^{-10} [11,12], respectively: the limiting factor is the ratio h/m_{Rb}. In our

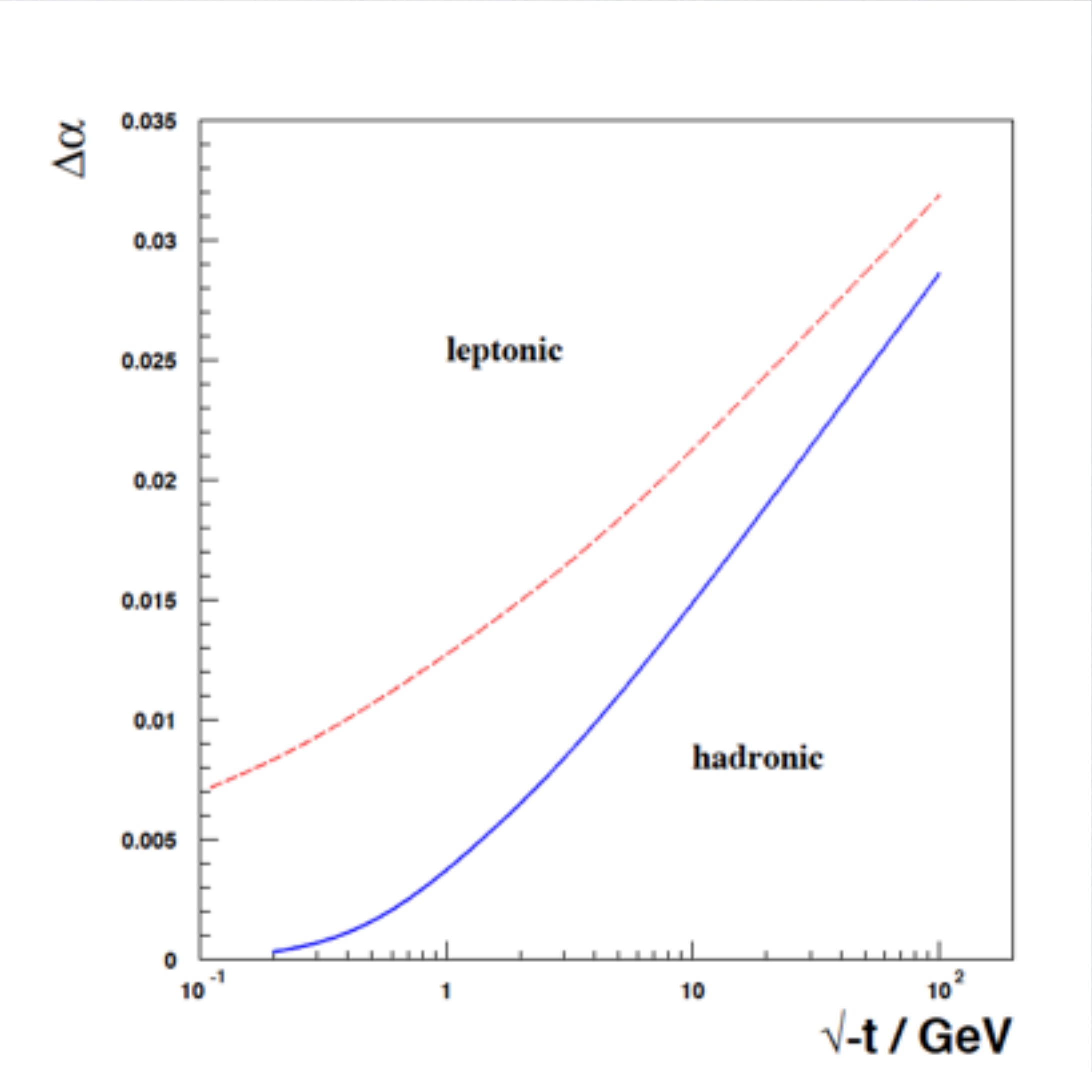


Alpha is not a constant !

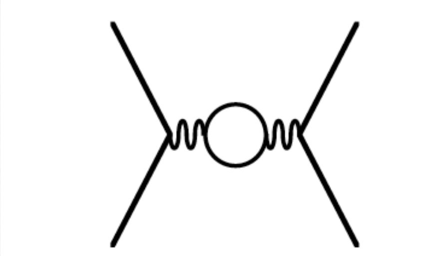
Radiative corrections start to (take) play a prominent role when the field becomes assessed and mature
and when, eventually, the collection of experimental data becomes abundant and accurate on the same time
(as a solid base for further developments)

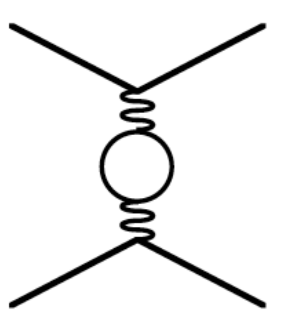
Running of α_{em}





time-like





space-like

$$\Delta\alpha^{(5)}_{had}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi}\mathrm{Re}\int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{s(s-M_Z^2-i\varepsilon)}$$

Physics Letters B 746 (2015) 325–329

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

A new approach to evaluate the leading hadronic corrections to the muon g-2

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2. Theoretical framework

The leading-order hadronic contribution to t[...] given by the well-known formula [4,15]

$$a_\mu^{\rm HLO} = \frac{\alpha}{\pi^2} \int_0^\infty \frac{ds}{s} K(s) \, {\rm Im}\Pi_{\rm had}(s + i\epsilon),$$

where $\Pi_{\rm had}(s)$ is the hadronic part of the photon vacuum polarization, $\epsilon > 0$,

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(s/m_\mu^2)} \tag{2}$$

is a positive kernel function, and m_μ is the muon mass. As the total cross section for hadron production in low-energy e^+e^- annihilations is related to the imaginary part of $\Pi_{\rm had}(s)$ via the optical theorem, the dispersion integral in Eq. (1) is computed integrating experimental time-like ($s > 0$) data up to a certain value of s [2,18,19]. The high-energy tail of the integral is calculated using perturbative QCD [20].

Alternatively, if we exchange the x and s integrations in Eq. (1) we obtain [21]

$$a_\mu^{\rm HLO} = \frac{\alpha}{\pi} \int_0^1 dx \, (x-1) \, \overline{\Pi}_{\rm had}[t(x)] \, , \tag{3}$$

where $\overline{\Pi}_{\rm had}(t) = \Pi_{\rm had}(t) - \Pi_{\rm had}(0)$ and

Eur. Phys. J. C (2017) 77:139
DOI 10.1140/epjc/s10052-017-4633-z

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Experimental Physics

Measuring the leading hadronic contribution to the muon g-2 via μe scattering

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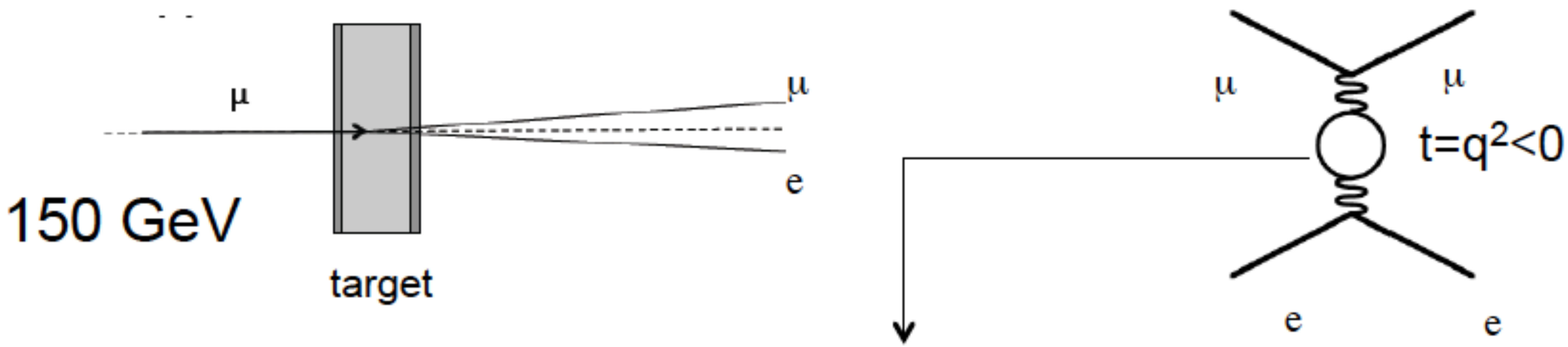
Received: 17 October 2016 / Accepted: 17 January 2017 / Published online: 1 March 2017

Experimental approach:

Use of a 150 GeV μ beam on Be target at CERN (elastic scattering $\mu e \rightarrow \mu e$)



$$\left|\frac{\alpha(t)}{\alpha_0}\right|^2 = \left|\frac{1}{1-\Delta\alpha(t)}\right|^2$$

The talks in this Workshop by Umberto (Marconi)
“Measurements of alpha and proposal for amu^HLO space-like”

as well as by Marina (Marinkovic), Pierpaolo (Mastrolia) and Fulvio (Piccinini)

Our proposal therefore has to deal with some experimental issues….

Experimental Issues

Using Bhabha at small angle (to emphasize t-channel contribution) to extract $\Delta\alpha$:

$$\left(\frac{\alpha(t)}{\alpha(0)}\right)^2 \sim \frac{d\sigma_{ee\to ee}(t)}{d\sigma^0_{MC}(t)}$$

Where $d\sigma^0_{MC}$ is the MC prediction for Bhabha process with $\alpha(t) = \alpha(0)$ and there are corrections due to RC…

$$\Delta\alpha_{had}(t) = 1 - \left(\frac{\alpha(t)}{\alpha(0)}\right)^{-1} - \Delta\alpha_{lept}(t)$$

since $\Delta\alpha_{lep}(t)$ is theoretically well known

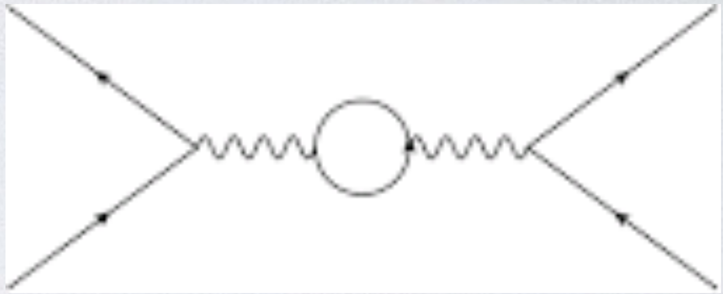
Which experimental accuracy we are aiming at ?
$\delta\Delta\alpha_{had} \simeq$ 1/2 fractional accuracy on $d\sigma(t)/d\sigma^0_{MC}(t)$.

If we assume to measure $\delta\Delta\alpha_{had}$ at 0.5% at the peak of the integrand ($\Delta\alpha_{had}$ ~10^{-3} at x=0.92) fractional accuracy on

$d\sigma(t)/d\sigma^0_{MC}(t) \sim 10^{-5}$

Very challenging measurement (one order of magnitude improvement respect to date) due to the systematics

Vacuum Polarization makes α_{em} running assuming a well defined "effective" value at any scale



vacuum polarization and the "effective charge" are defined by:

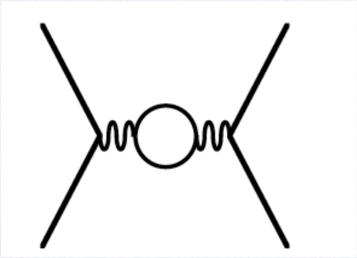
$$e^2 \to e^2(q^2) = \frac{e^2}{1 + (\Pi(q^2) - \Pi(0))} \qquad \alpha(q^2) = \frac{\alpha(0)}{1 - \Delta\alpha}; \quad \Delta\alpha = -\Re e\left(\Pi(q^2) - \Pi(0)\right)$$

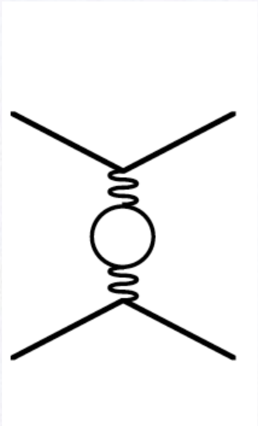
$\Delta\alpha$ takes contributions from leptonic and hadronic and gauge bosons elementary states

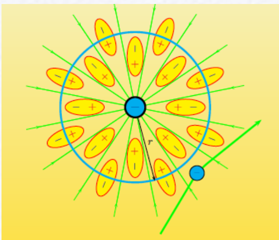
Among these the non-perturbative $\Delta\alpha_{had}$

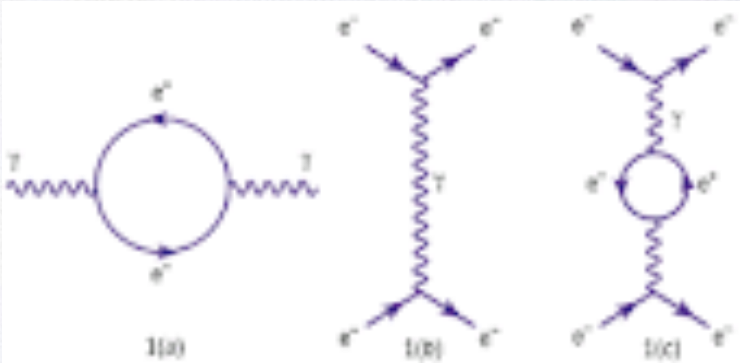
$$\Delta\alpha = \Delta\alpha_{leptonic} + \Delta\alpha_{gb} + \Delta\alpha_{had} + \Delta\alpha_{top}$$

α

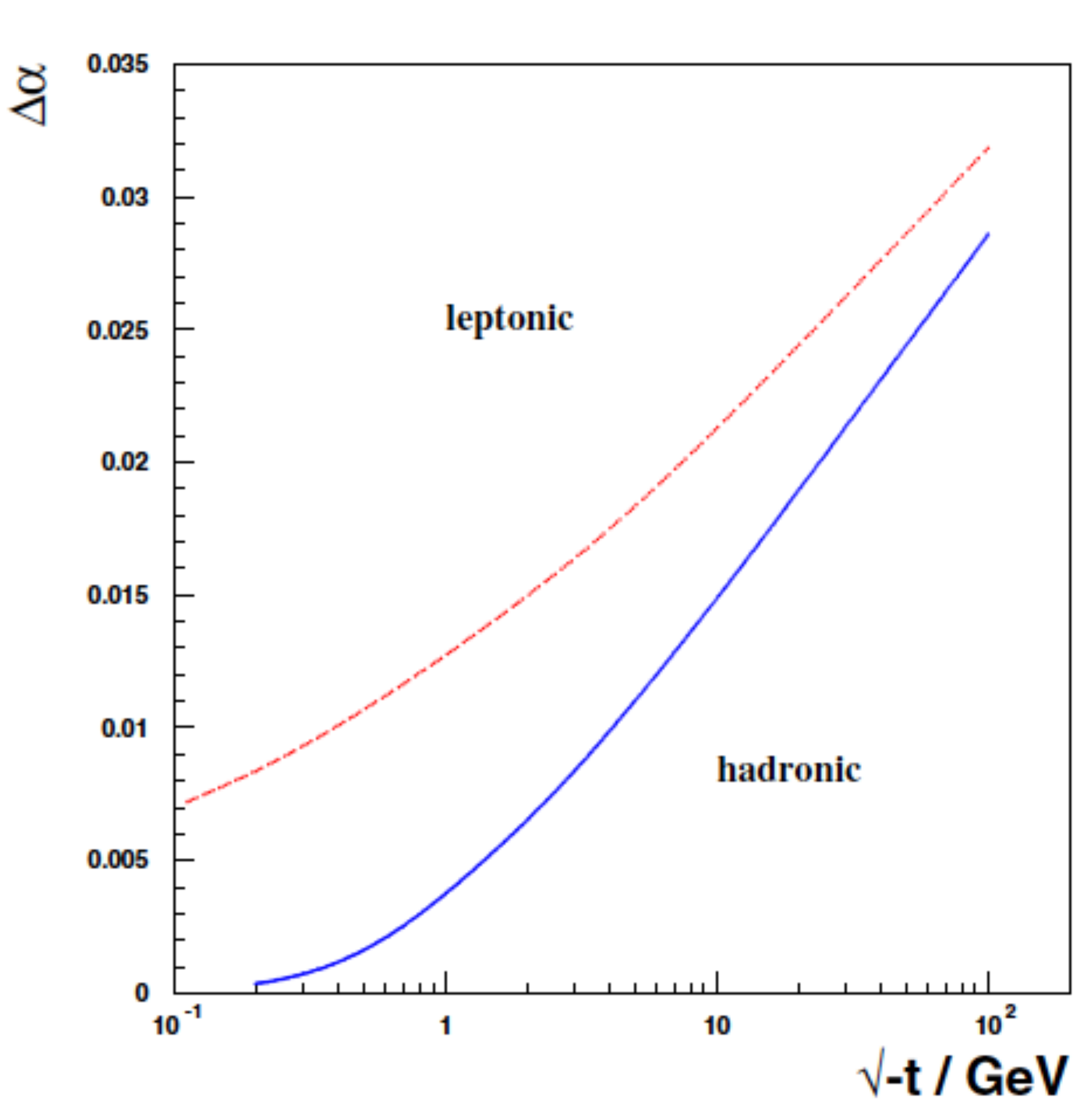








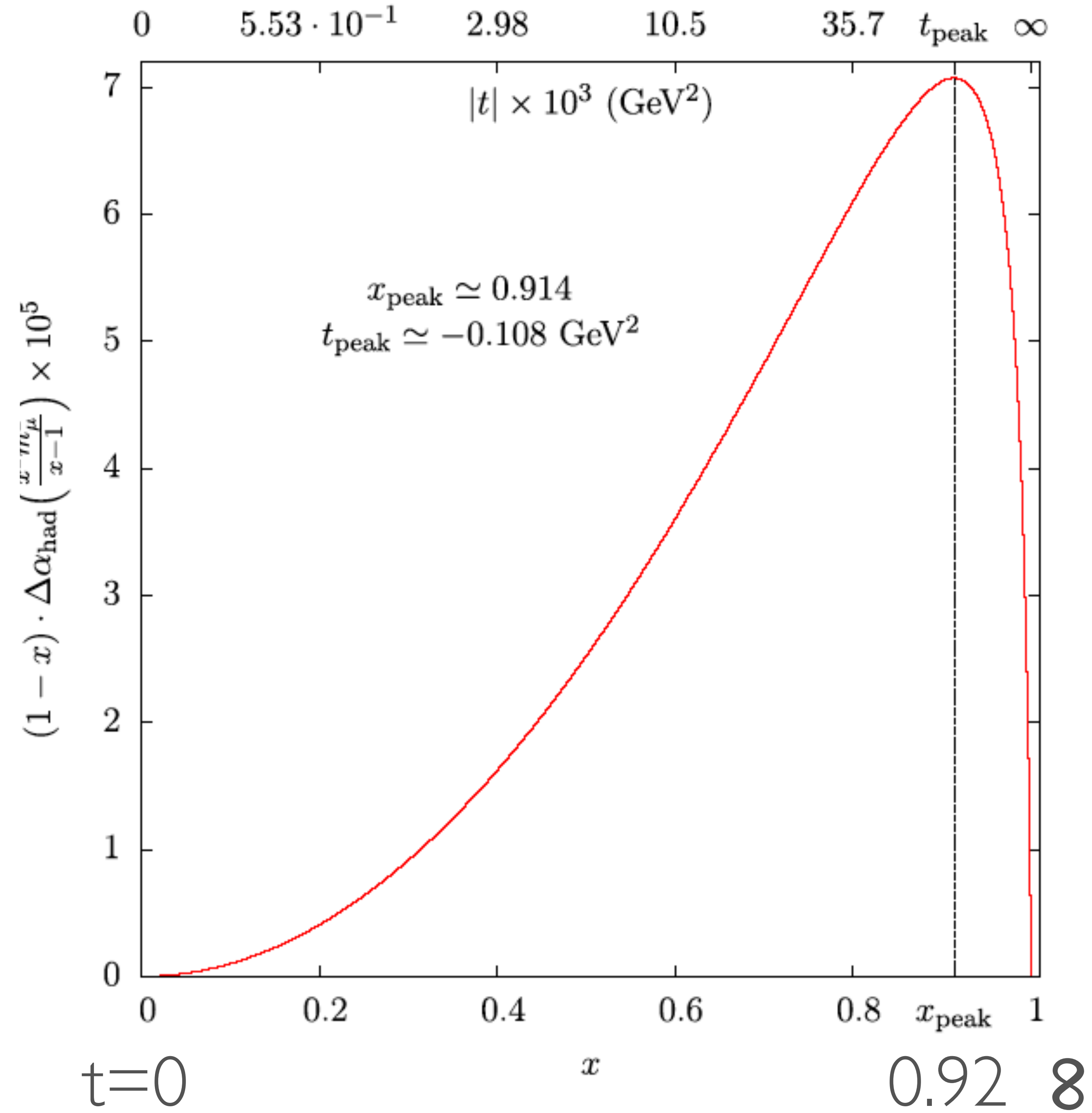
functional form of the kernel



$\Delta\alpha$ is dominated at low t by the leptonic contribution

A. Arbuzov, D.Haidt, C.Matteuzzi,M.Paganoni, L.T. Eur. Phys. J. C 34 (2004) 267

C.M. Carloni Calame et al. / Physics Letters B 746 (2015) 325–329



large t-values are depressed by x-1 denominator

The integrand is peaked at ~x=0.92
t=-0.11 GeV^2 (~330 MeV) for which
$\Delta\alpha_{had}(0.92)$~ 10^{-3}

The running of alpha in the space-like region
some years ago

arXiv:hep-ph/0402211v1 19 Feb 2004

The running of the electromagnetic coupling α in small-angle Bhabha scattering

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Abstract

A method to determine the running of α from a measurement of small-angle Bhabha scattering is proposed and worked out. The method is suited to high statistics experiments at e^+e^- colliders, which are equipped with luminometers in the appropriate angular region. A new simulation code predicting small-angle Bhabha scattering is also presented.

A. Arbuzov, D. Haidt, C. Matteuzzi, M. Paganoni and L.T.
A. Eur. Phys. J. C 34 (2004) 267

The method to measure the running of α exploits the fact that the cross section for the process $e^+e^- \to e^+e^-$ can be conveniently decomposed into three factors :

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma^0}{\mathrm{d}t}\left(\frac{\alpha(t)}{\alpha(0)}\right)^2 (1+\Delta r(t)) \tag{3}$$

each one of them known with an accuracy of at least 0.1%

1st factor

$$\frac{\mathrm{d}\sigma^0}{\mathrm{d}t} = \frac{\mathrm{d}\sigma^B}{\mathrm{d}t}\left(\frac{\alpha(0)}{\alpha(t)}\right)^2 .$$

Born cross section contains all the soft and virtual corrections

Bhabha is a pure QED processes
quarks enter only in loops

$$\frac{\mathrm{d}\sigma^B}{\mathrm{d}t} = \frac{\pi\alpha_0^2}{2s^2}\mathrm{Re}\{B_t + B_s + B_i\},$$

$$\begin{aligned}
B_t &= \left(\frac{s}{t}\right)^2 \left\{\frac{5+2c+c^2}{(1-\Pi(t))^2} + \xi\frac{2(g_v^2+g_a^2)(5+2c+c^2)}{(1-\Pi(t))}\right. \\
&+ \left.\xi^2\left(4(g_v^2+g_a^2)^2 + (1+c)^2(g_v^4+g_a^4+6g_v^2g_a^2)\right)\right\} \\
B_s &= \frac{2(1+c^2)}{|1-\Pi(s)|^2} + 2\chi\frac{(1-c)^2(g_v^2-g_a^2)+(1+c)^2(g_v^2+g_a^2)}{1-\Pi(s)} \\
&+ \chi^2[(1-c)^2(g_v^2-g_a^2)^2 + (1+c)^2(g_v^4+g_a^4+6g_v^2g_a^2)] \\
B_i &= 2\frac{s}{t}(1+c)^2\left\{\frac{1}{(1-\Pi(t))(1-\Pi(s))}\right. \\
&+ (g_v^2+g_a^2)\left(\frac{\xi}{1-\Pi(s)} + \frac{\chi}{1-\Pi(t)}\right) \\
&+ \left.(g_v^4+6g_v^2g_a^2+g_a^4)\xi\chi\right\}
\end{aligned}$$

2nd factor

$$\left(\frac{\alpha(t)}{\alpha(0)}\right)^2$$

Vacuum polarization effects

contains the running of alpha

3rd factor

$$(1 + \Delta r(t))$$

contains all the real and virtual effects not incorporated in the running of alpha

$$\alpha(q^2) = \frac{\alpha(0)}{1 - \Delta\alpha(q^2)},$$

$\alpha(0)$ is the Sommerfeld fine structure constant measured with a precision of $O(10^{-9})$

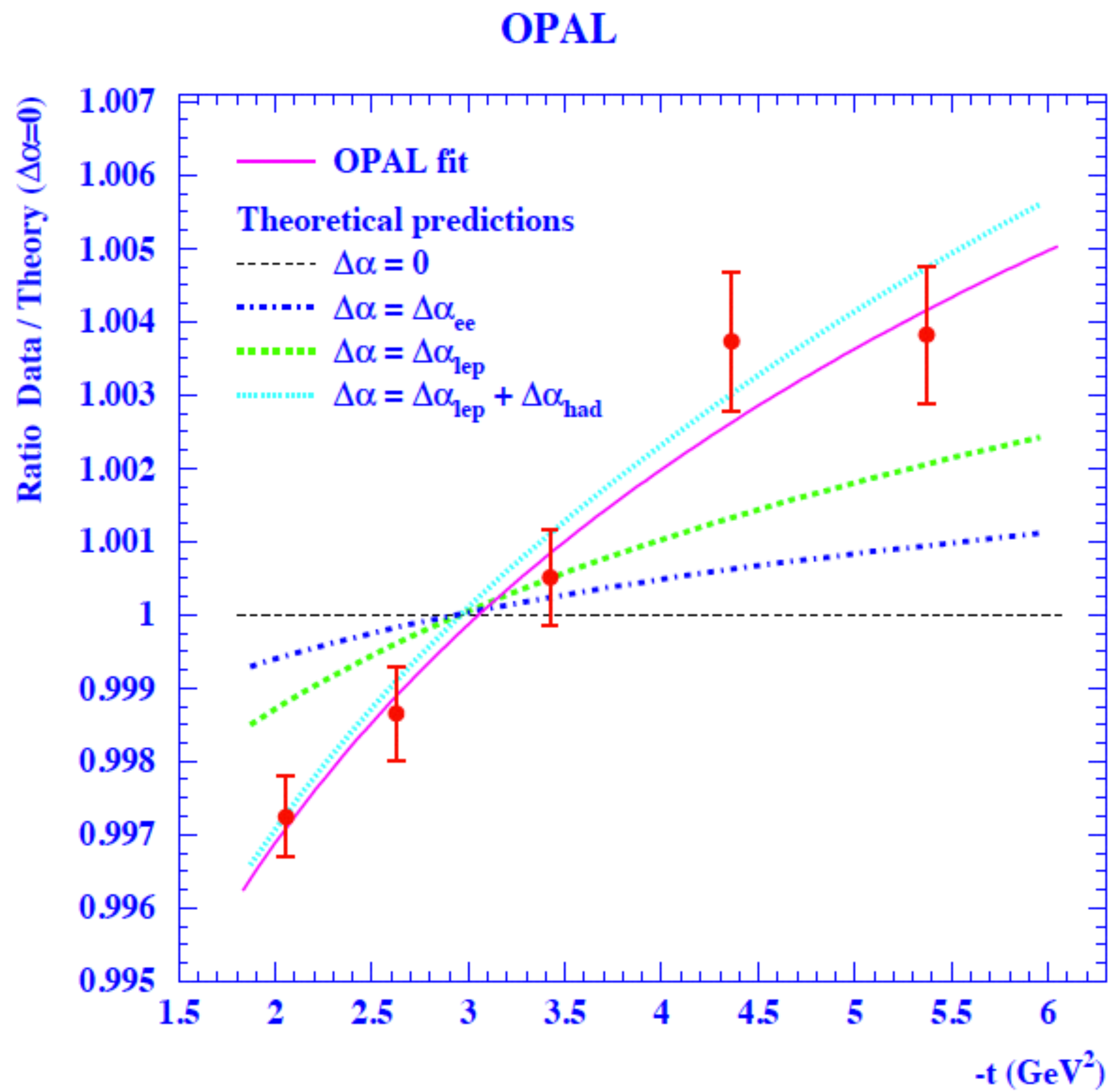
$\Delta\alpha(q^2)$ from loop contributions to the photon propagator

EUROPEAN ORGANISATION FOR NUCLEAR RESEARCH

CERN-PH-EP/2005-014
21 February 2005
Revised 28 June 2005

Measurement of the running of the QED coupling in small-angle Bhabha scattering at LEP

OPAL Collaboration


OPAL
Ratio Data / Theory (Δα=0)
1.007
1.006
1.005
1.004
1.003
1.002
1.001
1
0.999
0.998
0.997
0.996
0.995
OPAL fit
Theoretical predictions
Δα = 0
Δα = Δα_ee
Δα = Δα_lep
Δα = Δα_lep + Δα_had
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
-t (GeV²)

Abstract

Using the OPAL detector at LEP, the running of the effective QED coupling $\alpha(t)$ is measured for space-like momentum transfer from the angular distribution of small-angle Bhabha scattering. In an almost ideal QED framework, with very favourable experimental conditions, we obtain:

$$\Delta\alpha(-6.07\,\mathrm{GeV}^2) - \Delta\alpha(-1.81\,\mathrm{GeV}^2) = (440 \pm 58 \pm 43 \pm 30) \times 10^{-5}\,,$$

where the first error is statistical, the second is the experimental systematic and the third is the theoretical uncertainty. This agrees with current evaluations of $\alpha(t)$. The null hypothesis that α remains constant within the above interval of $-t$ is excluded with a significance above $5\,\sigma$. Similarly, our results are inconsistent at the level of $3\,\sigma$ with the hypothesis that only leptonic loops contribute to the running. This is currently the most significant direct measurement where the running $\alpha(t)$ is probed differentially within the measured t range.

The method used follows the above parametrization/factorization of the Bhabha cross-section

$$\frac{d\sigma}{dt} = \frac{d\sigma^{(0)}}{dt}\left(\frac{\alpha(t)}{\alpha_0}\right)^2 (1+\epsilon)\,(1+\delta_\gamma) + \delta_Z$$

$$\frac{d\sigma^{(0)}}{dt} = \frac{4\pi\alpha_0^2}{t^2}$$

We determined the effective slope of the Bhabha momentum transfer distribution which is simply related to the average derivative of $\Delta\alpha$ as a function of $\ln t$ in the range $2\ \mathrm{GeV}^2 \leq -t \leq 6\ \mathrm{GeV}^2$. The observed t-spectrum is in good agreement with Standard Model predictions. We find:

$$\Delta\alpha(-6.07\,\mathrm{GeV}^2) - \Delta\alpha(-1.81\,\mathrm{GeV}^2) = (440 \pm 58 \pm 43 \pm 30) \times 10^{-5},$$

where the first error is statistical, the second is the experimental systematic and the third is the theoretical uncertainty.

This measurement is one of only a very few experimental tests of the running of $\alpha(t)$ in the space-like region, where $\Delta\alpha$ has a smooth behaviour. We obtain the strongest direct evidence for the running of the QED coupling ever achieved differentially in a single experiment, with a significance above $5\,\sigma$. Moreover we report clear experimental evidence for the hadronic contribution to the running in the space-like region, with a significance of $3\,\sigma$.

Such an approach was possible
with a per-mille accuracy
of the Bhabha cross-section
1992-1997….

all started in 1992 with a preprint:

Small angles Bhabha scattering: Two loop approximation

Victor S. Fadin (Novosibirsk, IYF) , E.A. Kuraev (Dubna, JINR) , L.N. Lipatov (St. Petersburg, INP) , N.P. Merenkov (Kharkov, KIPT) , L. Trentadue (CERN)

Dec 1992 - 20 pages

JINR-E2-92-577

then the general program :

Generalized eikonal representation of the small angle e+ e- scattering amplitude at high-energy

Victor S. Fadin, E.A. Kuraev, L. Trentadue (Dubna, JINR) , L.N. Lipatov (St. Petersburg, INP) , N.P. Merenkov (Kharkov, KIPT)

1993

Phys.Atom.Nucl. 56 (1993) 1537-1540
Yad.Fiz. 56N11 (1993) 145-150

After almost three years at a CERN workshop
("Reports of the Working Group on Precision Calculations for the Z Resonance" CERN 95-03, March 1995.):

Small Angle Bhabha Scattering for LEP

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We present the results of our calculations to a one, two, and three loop approximation of the $e^+e^- \to e^+e^-$ Bhabha scattering cross-section at small angles. All terms contributing to the radiatively corrected cross-section, within an accuracy of $\delta\sigma/\sigma = 0.1\%$, are explicitely evaluated and presented in an analytic form. $O(\alpha)$ and $O(\alpha^2)$ contributions are kept up to next-to-leading logarithmic accuracy, and $O(\alpha^3)$ terms are taken into account to the leading logarithmic approximation. We define an experimentally measurable cross-section by integrating the calculated distributions over a given range of final-state energies and angles. The cross-sections for exclusive channels as well as for the totally integrated distributions are also given.

and a few months later:

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NUCLEAR PHYSICS B

Nuclear Physics B 485 (1997) 457–499

Small-angle electron–positron scattering with a per mille accuracy ★

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Received 22 December 1995; revised 25 June 1996; accepted 6 September 1996

The goal of the analytical result was aiming at a precision

$$|\frac{\delta\sigma}{\sigma}| < 0.001$$

since the accuracy reached at the time was still inadequate.
According to the evaluations the theoretical estimates were still incomplete, moreover, are in disagreement with each other up to 0.5%, far from the required theoretical and experimental accuracy

$$\theta_1 < \theta_- = \widehat{\vec{p}_1\vec{p}_{1'}} \equiv \theta < \theta_3 \quad , \qquad \theta_2 < \theta_+ = \widehat{\vec{p}_2\vec{p}_{2'}} < \theta_4 \quad , \qquad 0.01 \lesssim \theta_i \lesssim 0.1 \text{ rad} \,, \qquad (2)$$

where $\vec{p}_1$, $\vec{p}_{1'}$, ($\vec{p}_2$, $\vec{p}_{2'}$) represent the momenta of the initial and of the scattered electron (positron) in the center-of-mass frame.

At small angles the main contribution comes from one photon exchanged in the t-channel

(due to the eikonal approximation logarithmic terms from multiple-photon exchange diagrams do cancel)

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4} \quad .$$

Let us now estimate the correction of order θ^2 to this contribution. If

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}(1 + c_1\theta^2) \quad ,$$

then, after integration over θ^2 in the angular range as Eq. (2), one obtains:

$$\int_{\theta^2_{\min}}^{\theta^2_{\max}} \frac{d\sigma}{d\theta^2}\, d\theta^2 \sim \theta^{-2}_{\min}(1 + c_1\theta^2_{\min} \ln \frac{\theta^2_{\max}}{\theta^2_{\min}}).$$

Also terms of the type: $\frac{m^2}{Q^2}$ $m = m_e, m_\mu$

if $Q^2 \simeq 1 GeV^2$ may be omitted

2. Born cross section and one-loop virtual and soft corrections

The Born cross section for Bhabha scattering within the Standard Model is well known [8]:

$$\frac{d\sigma^B}{d\Omega} = \frac{\alpha^2}{8s}\{4B_1 + (1-c)^2 B_2 + (1+c)^2 B_3\}, \tag{5}$$

where

$$B_1 = \left(\frac{s}{t}\right)^2 \left|1 + (g_v^2 - g_a^2)\xi\right|^2, \qquad B_2 = \left|1 + (g_v^2 - g_a^2)\chi\right|^2,$$

$$B_3 = \frac{1}{2}\left|1 + \frac{s}{t} + (g_v + g_a)^2\left(\frac{s}{t}\xi + \chi\right)\right|^2 + \frac{1}{2}\left|1 + \frac{s}{t} + (g_v - g_a)^2\left(\frac{s}{t}\xi + \chi\right)\right|^2,$$

$$\chi = \frac{\Lambda s}{s - m_z^2 + iM_Z\Gamma_Z}, \qquad \xi = \frac{\Lambda t}{t - M_Z^2},$$

$$\Lambda = \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} = (\sin 2\theta_w)^{-2}, \qquad g_a = -\frac{1}{2}, \qquad g_v = -\frac{1}{2}(1 - 4\sin^2\theta_w),$$

$$s = (p_1 + p_2)^2 = 4\varepsilon^2, \qquad t = -Q^2 = (p_1 - q_1)^2 = -\frac{1}{2}s(1-c),$$

$$c = \cos\theta, \qquad \theta = \widehat{\mathbf{p}_1\mathbf{q}_1}.$$

Here θ_w is the Weinberg angle. In the small-angle limit ($c = 1 - \theta^2/2 + \theta^4/24 + \ldots$), expanding formula (5) leads to

the weak interaction contribution

$$\frac{d\sigma^B}{\theta d\theta} = \frac{8\pi\alpha^2}{\varepsilon^2\theta^4}\left(1 - \frac{\theta^2}{2} + \frac{9}{40}\theta^4 + \delta_{\text{weak}}\right), \tag{6}$$

where $\varepsilon = \sqrt{s}/2$ is the electron or positron initial energy and the weak correction term δ_{weak}, connected with diagrams with Z^0-boson exchange, is given by the expression

$$\delta_{\text{weak}} = 2g_v^2\xi - \frac{\theta^2}{4}(g_v^2 + g_a^2)\text{Re}\,\chi + \frac{\theta^4}{32}(g_v^4 + g_a^4 + 6g_v^2g_a^2)|\chi|^2. \tag{7}$$

One can see from Eq. (7) that the contribution c_1^w of the weak correction δ_{weak} into the coefficient c_1 introduced in Eq. (3)

$$c_1^w \lesssim 2g_v^2 + \frac{(g_v^2 + g_a^2)}{4}\frac{M_Z}{\Gamma_Z} + \theta_{\max}^2\frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{32}\frac{M_Z^2}{\Gamma_Z^2} \simeq 1. \tag{8}$$

virtual+soft photon contribution

$$\frac{d\sigma^{(1)}_{\text{QED}}}{dc} = \frac{d\sigma^{B}_{\text{QED}}}{dc} \left(1 + \delta_{\text{virt}} + \delta_{\text{soft}}\right), \tag{9}$$

where $d\sigma^{B}_{\text{QED}}$ is the Born cross section in the pure QED case (it is equal to $d\sigma^{B}$ with $g_a = g_v = 0$) and

$$\begin{aligned}
\delta_{\text{virt}} + \delta_{\text{soft}} = 2\frac{\alpha}{\pi}\Bigg[& 2\left(1 - \ln\frac{4\varepsilon^2}{m^2} + 2\ln\left(\cot\frac{\theta}{2}\right)\right)\ln\frac{\varepsilon}{\Delta\varepsilon} + \int\limits_{\cos^2(\theta/2)}^{\sin^2(\theta/2)} \frac{dx}{x}\ln(1-x) \\
& - \frac{23}{9} + \frac{11}{6}\ln\frac{4\varepsilon^2}{m^2}\Bigg] + \frac{\alpha}{\pi}\frac{1}{(3+c^2)^2}\Bigg[\frac{\pi^2}{3}(2c^4 - 3c^3 - 15c) \\
& + 2(2c^4 - 3c^3 + 9c^2 + 3c + 21)\ln^2\left(\sin\frac{\theta}{2}\right) \\
& - 4(c^4 + c^2 - 2c)\ln^2\cos\frac{\theta}{2} - 4(c^3 + 4c^2 + 5c + 6)\ln^2\left(\tan\frac{\theta}{2}\right) \\
& + \frac{2}{3}(11c^3 + 33c^2 + 21c + 111)\ln\left(\sin\frac{\theta}{2}\right) \\
& + 2(c^3 - 3c^2 + 7c - 5)\ln\left(\cos\frac{\theta}{2}\right) \\
& + 2(c^3 + 3c^2 + 3c + 9)\delta_t - 2(c^3 + 3c)(1 - c)\delta_s\Bigg].
\end{aligned}$$

where

The value δ_t (δ_s) is defined by contributions to the photon vacuum polarization function $\Pi(t)$ ($\Pi(s)$) as follows:

$$\Pi(t) = \frac{\alpha}{\pi}\left(\delta_t + \frac{1}{3}L - \frac{5}{9}\right) + \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 L, \qquad (10)$$

where

$$L = \ln\frac{Q^2}{m^2}, \qquad Q^2 = -t = 2\varepsilon^2(1-c), \qquad (11)$$

and we took into account the leading part of the two-loop contribution in the polarization operator. In the Standard Model, δ_t contains contributions of muons, tau-leptons, W-bosons and hadrons:

$$\delta_t = \delta_t^{\mu} + \delta_t^{\tau} + \delta_t^{W} + \delta_t^{H}, \qquad \delta_s = \delta_t(Q^2 \to -s), \qquad (12)$$

the first three contributions are theoretically calculable and can be given as

$$\begin{aligned}
\delta_t^{\mu} &= \frac{1}{3}\ln\frac{Q^2}{m_\mu^2} - \frac{5}{9}, \\
\delta_t^{\tau} &= \frac{1}{2}v_\tau\left(1 - \frac{1}{3}v_\tau^2\right)\ln\frac{v_\tau+1}{v_\tau-1} + \frac{1}{3}v_\tau^2 - \frac{8}{9}, \qquad v_\tau = \sqrt{1+\frac{4m_\tau^2}{Q^2}}, \\
\delta_t^{W} &= \frac{1}{4}v_W(v_W^2-4)\ln\frac{v_W+1}{v_W-1} - \frac{1}{2}v_W^2 + \frac{11}{6}, \qquad v_W = \sqrt{1+\frac{4M_W^2}{Q^2}}.
\end{aligned} \qquad (13)$$

The contribution of hadrons cannot be calculated theoretically; instead, it can be given as integration of the experimentally measurable cross section:

$$\delta_t^H = \frac{Q^2}{4\pi\alpha^2} \int_{4m_\pi^2}^{+\infty} \frac{\sigma^{e^+e^- \to h}(x)}{x+Q^2} \, dx. \tag{14}$$

For numerical calculations we will use for $\Pi(t)$ the results of Eidelman Jegerlehner

In the small scattering angle limit we can present (9) in the following form:

$$\begin{aligned}
\frac{d\sigma_{\mathrm{QED}}^{(1)}}{dc} &= \frac{d\sigma_{\mathrm{QED}}^{B}}{dc} (1-\Pi(t))^{-2}(1+\delta), \\
\delta &= 2\frac{\alpha}{\pi}\left[2(1-L)\ln\frac{1}{\Delta} + \frac{3}{2}L - 2\right] + \frac{\alpha}{\pi}\theta^2 \Delta_\theta + \frac{\alpha}{\pi}\theta^2 \ln\Delta, \\
\Delta_\theta &= \frac{3}{16}l^2 + \frac{7}{12}l - \frac{19}{18} + \frac{1}{4}(\delta_t - \delta_s), \\
\Delta &= \frac{\Delta\varepsilon}{\varepsilon}, \qquad l = \ln\frac{Q^2}{s} \simeq \ln\frac{\theta^2}{4}.
\end{aligned} \tag{15}$$

This representation gives us a possibility to verify explicitly that the terms of relative order θ^2 in the radiative corrections are small. Taking into account that the large contribution proportional to $\ln\Delta$ disappears when we add the cross section for the hard emission, we can verify again that such terms can be neglected.

Therefore we will omit in higher orders the annihilation diagrams and multiple-photon exchange diagrams in the scattering channel. The second simplification is justified by the generalized eikonal representation for small-angle scattering amplitudes. In particular, for the case of elastic processes we have [11]

$$A(s,t) = A_0(s,t)F_1^2(t)(1-\Pi(t))^{-1}\,\mathrm{e}^{i\varphi(t)}\left[1+\mathcal{O}\left(\frac{\alpha}{\pi}\frac{Q^2}{s}\right)\right], \quad s \gg Q^2 \gg m^2, \tag{16}$$

where $A_0(s,t)$ is the Born amplitude, $F_1(t)$ is the Dirac form factor and $\varphi(t) = -\alpha\ln(Q^2/\lambda^2)$ is the Coulomb phase, λ is the *photon mass* auxiliary parameter. The eikonal representation is violated at a three-loop level, but, fortunately, the corresponding contribution to the Bhabha cross section is small enough ($\sim\alpha^5$) and can be neglected for our purposes. We may consider the eikonal representation as correct within the required accuracy. [3]

Let us now introduce the dimensionless quantity $\Sigma = Q_1^2\sigma_{\exp}/(4\pi\alpha^2)$, with $Q_1^2 = \varepsilon^2\theta_1^2$, where $\sigma_{\exp}$ is the Bhabha-process cross section integrated over the typical experimental energy and angular ranges:[4]

$$\Sigma = \frac{Q_1^2}{4\pi\alpha^2}\int \mathrm{d}x_1 \int \mathrm{d}x_2 \Theta(x_1x_2 - x_c)\int \mathrm{d}^2 \boldsymbol{q}_1^{\perp}\Theta_1^c \times \int \mathrm{d}^2\boldsymbol{q}_2^{\perp}\Theta_2^c \frac{\mathrm{d}\sigma^{e^+e^-\to e^+(\boldsymbol{q}_2^{\perp},x_2)\,e^-(\boldsymbol{q}_1^{\perp},x_1)+X}}{\mathrm{d}x_1\mathrm{d}^2\boldsymbol{q}_1^{\perp}\mathrm{d}x_2\mathrm{d}^2\boldsymbol{q}_2^{\perp}}, \qquad (17)$$

where $x_{1,2}$, $\boldsymbol{q}_{1,2}^{\perp}$ are the energy fractions and the transverse components of the momenta of the electron and positron in the final state, sx_c is the experimental cut-off on their invariant mass squared and the functions Θ_i^c do take into account the angular cuts (2):

$$\Theta_1^c = \Theta\left(\theta_3 - \frac{|\boldsymbol{q}_1^{\perp}|}{x_1\varepsilon}\right)\Theta\left(\frac{|\boldsymbol{q}_1^{\perp}|}{x_1\varepsilon} - \theta_1\right), \quad \Theta_2^c = \Theta\left(\theta_4 - \frac{|\boldsymbol{q}_2^{\perp}|}{x_2\varepsilon}\right)\Theta(\frac{|\boldsymbol{q}_2^{\perp}|}{x_2\varepsilon} - \theta_2). \qquad (18)$$

Σ as the sum of various contributions:

$$\Sigma = \Sigma_0 + \Sigma^{\gamma} + \Sigma^{2\gamma} + \Sigma^{e^+e^-} + \Sigma^{3\gamma} + \Sigma^{e^+e^-\gamma}$$

$$= \Sigma_{00}(1 + \delta_0 + \delta^{\gamma} + \delta^{2\gamma} + \delta^{e^+e^-} + \delta^{3\gamma} + \delta^{e^+e^-\gamma}),$$

3. Single hard-photon emission

In order to calculate the contribution to Σ due to the hard-photon emission we start from the corresponding differential cross section written in terms of energy fractions $x_{1,2}$ and transverse components $q^{\perp}_{1,2}$ of the final particle momenta [13]:

$$\frac{d\sigma_B^{e^+e^-\to e^+e^-\gamma}}{dx_1 d^2 q_1^{\perp} dx_2 d^2 q_2^{\perp}} = \frac{2\alpha^3}{\pi^2}\left\{\frac{R(x_1; q_1^{\perp}, q^{\perp}{}_2)\delta(1-x_2)}{(q_2^{\perp})^4(1-\Pi(-(q_2^{\perp})^2))^2} + \frac{R(x_2; q_2^{\perp}, q_1^{\perp})\,\delta(1-x_1)}{(q_1^{\perp})^4(1-\Pi(-(q_1^{\perp})^2))^2}\right\}(1+\mathcal{O}(\theta^2)), \qquad (24)$$

where

$$R(x; q_1^{\perp}, q_2^{\perp}) = \frac{1+x^2}{1-x}\left[\frac{(q_2^{\perp})^2(1-x)^2}{d_1 d_2} - \frac{2m^2(1-x)^2 x}{1+x^2}\frac{(d_1-d_2)^2}{d_1^2 d_2^2}\right], \qquad (25)$$

$$d_1 = m^2(1-x)^2 + (q_1^{\perp} - q_2^{\perp})^2,$$

$$d_2 = m^2(1-x)^2 + (q_1^{\perp} - x q_2^{\perp})^2,$$

$$\Sigma^H = \frac{\alpha}{\pi} \int\limits_{x_c}^{1-\Delta} \mathrm{d}x \frac{1+x^2}{1-x} \; F(x, D_1, D_3; D_2, D_4), \tag{26}$$

with

$$F = \int\limits_{D_1}^{D_3} \mathrm{d}z_1 \int\limits_{D_2}^{D_4} \frac{\mathrm{d}z_2}{z_2} (1 - \Pi(-z_2 Q_1^2))^{-2} \left\{ \frac{1-x}{z_1 - x z_2} \left(a_1^{-\frac{1}{2}} - x a_2^{-\frac{1}{2}} \right) \right.$$
$$\left. - \frac{4x\sigma^2}{1+x^2} \left[a_1^{-\frac{3}{2}} + x^2 a_2^{-\frac{3}{2}} \right] \right\}, \tag{27}$$

where

$$a_1 = (z_1 - z_2)^2 + 4 z_2 \sigma^2, \qquad a_2 = (z_1 - x^2 z_2)^2 + 4 x^2 z_2 \sigma^2,$$
$$\sigma^2 = \frac{m^2}{Q_1^2} (1-x)^2, \tag{28}$$

$$\Sigma^H = \frac{\alpha}{\pi} \int\limits_{x_c}^{1-\Delta} dx \frac{1+x^2}{1-x} \int\limits_{1}^{\rho^2} \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2}$$
$$\times \left\{ [1 + \Theta(x^2\rho^2 - z)] \, (L-1) + k(x,z) \right\},$$
$$k(x,z) = \frac{(1-x)^2}{1+x^2} [1 + \Theta(x^2\rho^2 - z)] + L_1$$
$$+\Theta(x^2\rho^2 - z) L_2 + \Theta(z - x^2\rho^2) L_3 \,, \tag{30}$$

where $L = \ln(zQ_1^2/m^2)$ and

$$L_1 = \ln\left|\frac{x^2(z-1)(\rho^2 - z)}{(x-z)(x\rho^2 - z)}\right|, \qquad L_2 = \ln\left|\frac{(z-x^2)(x^2\rho^2 - z)}{x^2(x-z)(x\rho^2 - z)}\right|,$$
$$L_3 = \ln\left|\frac{(z-x^2)(x\rho^2 - z)}{(x-z)(x^2\rho^2 - z)}\right|. \tag{31}$$

It is seen from Eq. (30) that Σ^H contains the auxiliary parameter Δ. This parameter disappears, as it should, in the sum $\Sigma^\gamma = \Sigma^H + \Sigma^{V+S}$, where Σ^{V+S} is the contribution of virtual and soft real photons which can be obtained using Eq. (15):

$$\Sigma^{\gamma} = \frac{\alpha}{\pi} \int_{1}^{\rho^2} \frac{\mathrm{d}z}{z^2} \int_{x_c}^{1} \mathrm{d}x (1 - \Pi(-zQ_1^2))^{-2} \Big\{ (L-1) P(x) \times [1 + \Theta(x^2\rho^2 - z)] + \frac{1+x^2}{1-x} k(x,z) - \delta(1-x) \Big\}, \tag{32}$$

where

$$P(x) = \left(\frac{1+x^2}{1-x} \right)_+ = \lim_{\Delta \to 0} \left\{ \frac{1+x^2}{1-x} \theta(1-x-\Delta) + \left(\frac{3}{2} + 2 \ln \Delta \right) \delta(1-x) \right\} \tag{33}$$

4. Radiative corrections to $\mathcal{O}(\alpha^2)$

A systematic treatment of all $\mathcal{O}(\alpha^2)$ contributions is absent up to now. This is mainly due to the extreme complexity of the analysis (more than 100 Feynman diagrams are to be taken into account considering elastic and inelastic processes). Nevertheless in the case of small scattering angles we may restrict ourselves by considering only diagrams of the scattering type. It is enough to make some rough estimates of other contributions. Contributions of pure annihilation-type diagrams, describing some $\mathcal{O}(\alpha^2)$ RC, have so-called double-logarithmical enhancement [25] but, fortunately, it is proportional to the fourth power of the small scattering angle:

$$(\Sigma^{\gamma\gamma})_{\text{annih}} \sim (\Sigma^{e^+e^-})_{\text{annih}} \sim \theta^4 \left(\frac{\alpha}{\pi}\right)^2 \mathcal{L}^4. \tag{34}$$

The contribution of interference of the scattering–type and the annihilation-type amplitudes can be estimated as

$$(\Sigma^{\gamma\gamma})_{\text{interf}} \sim (\Sigma^{e^+e^-})_{\text{interf}} \sim \theta^2 \left(\frac{\alpha}{\pi}\right)^2 \ln^4\left(\frac{Q^2}{s}\right). \tag{35}$$

We consider first virtual two-loop corrections $d\sigma_W^{(2)}$ to the elastic scattering cross section. Using the representation (16) and the loop expansion for the Dirac form factor F_1,

$$F_1 = 1 + \frac{\alpha}{\pi} F_1^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 F_1^{(2)}, \tag{36}$$

one obtains

$$\frac{\mathrm{d}\sigma_W^{(2)}}{\mathrm{d}c} = \frac{\mathrm{d}\sigma_0}{\mathrm{d}c} \left(\frac{\alpha}{\pi}\right)^2 (1 - \Pi(t))^{-2} \left[6(F_1^{(1)})^2 + 4F_1^{(2)} \right]. \tag{37}$$

$$F_1^{(1)} = (L-1)\ln\frac{\lambda}{m} + \frac{3}{4}L - \frac{1}{4}L^2 - 1 + \frac{1}{2}\zeta_2. \tag{38}$$

The two-loop correction can be obtained from the results of Ref. [14]. Let us present it in the form

$$F_1^{(2)} = F_1^{\gamma\gamma} + F_1^{e^+e^-}, \tag{39}$$

where the contribution $F_1^{e^+e^-}$ is related to the vacuum polarization by e^+e^- pairs:

$$F_1^{e^+e^-} = -\frac{1}{36}L^3 + \frac{19}{72}L^2 - \left(\frac{265}{216} + \frac{1}{6}\zeta_2\right)L + \mathcal{O}(1), \tag{40}$$

$$F_1^{\gamma\gamma} = \frac{1}{32}L^4 - \frac{3}{16}L^3 + \left(\frac{17}{32} - \frac{1}{8}\zeta_2\right)L^2 + \left(-\frac{21}{32} - \frac{3}{8}\zeta_2 + \frac{3}{2}\zeta_3\right)L$$
$$+\frac{1}{2}(L-1)^2\ln^2\frac{m}{\lambda} + (L-1)\left[-\frac{1}{4}L^2 + \frac{3}{4}L - 1 + \frac{1}{2}\zeta_2\right]\ln\frac{\lambda}{m} + \mathcal{O}(1),$$

$$\zeta_2 = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta_3 = \sum_1^\infty \frac{1}{n^3} \approx 1.202\,. \tag{41}$$

The *photon mass* λ entering Eqs. (38)–(41) is canceled in the expression $d\sigma^{(2)}/dc$ for the sum of the virtual and soft-photon corrections of the second order $d\sigma_W^{(2)}/dc$ (see Eq. (37)), $d\sigma_{SS}^{(2)}/dc$ and $d\sigma_{SV}^{(2)}/dc$.

The cross section $\mathrm{d}\sigma_{SS}^{(2)}/\mathrm{d}c$ for the emission of two soft photons, each of energy smaller than $\Delta\varepsilon = \varepsilon\Delta$, is $(\Delta \ll 1)$:

$$\mathrm{d}\sigma_{SS}^{(2)} = \mathrm{d}\sigma_0 \left(\frac{\alpha}{\pi}\right)^2 (1 - \Pi(t))^{-2}\, 8 \left[(L-1)\ln\frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\zeta_2\right]^2, \tag{42}$$

and the virtual correction $\mathrm{d}\sigma_{SV}^{(2)}/\mathrm{d}c$ to the cross section of the single soft-photon emission is

$$\mathrm{d}\sigma_{SV}^{(2)} = \mathrm{d}\sigma_0 \left(\frac{\alpha}{\pi}\right)^2 (1 - \Pi(t))^{-2} 16 F_1^{(1)} \left[(L-1)\ln\frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\zeta_2\right]. \tag{43}$$

The contribution to Σ of this sum, excepting the part coming from $F_1^{e^+e^-}$ connected with the vacuum polarization, contains no more than a second power of L. It has the following form:

$$\Sigma_{S+V}^{\gamma\gamma} = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{SS} = \left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2} (1 - \Pi(-zQ_1^2))^{-2} R_{S+V}^{\gamma\gamma}\,. \tag{44}$$

Thus for the contribution of the virtual and soft $e^+ e^-$ pairs to Σ we have

$$\Sigma_{S+V}^{e^+e^-} = \left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2} (1 - \Pi(-zQ_1^2))^{-2} R_{S+V}^{e^+e^-}\,, \tag{47}$$

$$\begin{aligned} R_{S+V}^{e^+e^-} &= R_S^{e^+e^-} + 4F_1^{e^+e^-} \\ &= L^2\left(\frac{2}{3}\ln\Delta + \frac{1}{2}\right) + L\left(-\frac{17}{6} + \frac{4}{3}\ln^2\Delta - \frac{20}{9}\ln\Delta - \frac{4}{3}\zeta_2\right) + \mathcal{O}(1). \end{aligned}$$

5. Virtual and soft corrections to the hard-photon emission

By evaluating the corrections arising from the emission of virtual and real soft photons which accompany a single hard photon we will consider two cases. The first case corresponds to the emission of the photons by the same fermion. The second one occurs when the hard photon is emitted by another fermion:

$$\mathrm{d}\sigma|_{H(S+V)} = \mathrm{d}\sigma^{H(S+V)} + \mathrm{d}\sigma_{H(S+V)} + \mathrm{d}\sigma^{H}_{(S+V)} + \mathrm{d}\sigma^{(S+V)}_{H}. \quad (48)$$

In the case when both fermions emit, one finds that

$$\Sigma^{H}_{(S+V)} + \Sigma^{(S+V)}_{H} = 2\Sigma^{H}\left(\frac{\alpha}{\pi}\right)\left[(L-1)\ln\Delta + \frac{3}{4}L - 1\right], \quad (49)$$

$$\mathrm{d}\sigma^{H(S+V)} = \frac{\alpha^4 \mathrm{d}x\mathrm{d}^2\boldsymbol{q}_1^{\perp}\mathrm{d}^2\boldsymbol{q}_2^{\perp}}{4x(1-x)(\boldsymbol{q}_2^{\perp})^4\pi^3}\left[\left(B_{11}(s_1,t_1) + x^2 B_{11}(t_1,s_1)\right)h + T\right], \quad (51)$$

$$h = T_{11}(s_1,t_1) + x^2 T_{11}(t_1,s_1) + x\left(T_{12}(s_1,t_1) + T_{12}(t_1,s_1)\right),$$

$$\rho = 2\left(L - \ln\frac{(\boldsymbol{q}_2^{\perp})^2}{-u_1} - 1\right)(2\ln\Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2},$$

The final result (see Appendix C for details) has the form

$$
\begin{aligned}
\Sigma^{H(S+V)} &= \Sigma_{H(S+V)} \\
&= \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2} \int_{x_c}^{1-\Delta} \frac{\mathrm{d}x(1+x^2)}{1-x} L\left\{\left(2\ln\Delta - \ln x + \frac{3}{2}\right)\right. \\
&\quad \times[(L-1)(1+\Theta) + k(x,z)] + \frac{1}{2}\ln^2 x + (1+\Theta)[-2+\ln x - 2\ln\Delta] \\
&\quad +(1-\Theta)\left[\frac{1}{2}L\ln x + 2\ln\Delta\ln x - \ln x\ln(1-x)\right. \\
&\quad \left.\left. -\ln^2 x - \mathrm{Li}_2(1-x) - \frac{x(1-x)+4x\ln x}{2(1+x^2)}\right] - \frac{(1-x)^2}{2(1+x^2)}\right\}, \qquad (52)
\end{aligned}
$$

$$
\mathrm{Li}_2(x) \equiv -\int_0^x \frac{\mathrm{d}t}{t}\ln(1-t),
$$

where $k(x,z)$ is given in Eq. (30) and $\Theta \equiv \Theta(x^2\rho^2 - z)$.

6. Double hard-photon bremsstrahlung

We now consider the contribution given by the process of emission of two hard photons. We will distinguish two cases: (a) the double simultaneous bremsstrahlung in opposite directions along electron and positron momenta, and (b) the double bremsstrahlung in the same direction along electron or positron momentum. The differential cross section in the first case can be obtained by using the factorization property of cross sections within the impact parameter representation [16]. It takes the following form [13] (see Appendix A):

$$\frac{\mathrm{d}\sigma^{e^+e^-\to(e^+\gamma)(e^-\gamma)}}{\mathrm{d}x_1\mathrm{d}^2\boldsymbol{q}_1^\perp\mathrm{d}x_2\mathrm{d}^2\boldsymbol{q}_2^\perp} = \frac{\alpha^4}{\pi^3}\int\frac{\mathrm{d}^2\boldsymbol{k}^\perp}{\pi(\boldsymbol{k}^\perp)^4}\left(1-\Pi(-(\boldsymbol{k}^\perp)^2)\right)^{-2} \times R(x_1;\boldsymbol{q}_1^\perp,\boldsymbol{k}^\perp)R(x_2;\boldsymbol{q}_2^\perp,-\boldsymbol{k}^\perp), \tag{53}$$

$$\Sigma_H^H = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\int_0^\infty\frac{\mathrm{d}z}{z^2}\left(1-\Pi(-zQ_1^2)\right)^{-2} \times\int_{x_c}^{1-\Delta}\mathrm{d}x_1\int_{x_c/x_1}^{1-\Delta}\mathrm{d}x_2\frac{1+x_1^2}{1-x_1}\frac{1+x_2^2}{1-x_2}\Phi(x_1,z)\Phi(x_2,z), \tag{54}$$

$$\begin{aligned}\Phi(x,z) = {} & (L-1)[\Theta(z-1)\Theta(\rho^2-z)+\Theta(z-x^2)\Theta(\rho^2x^2-z)] \\ & +L_3[-\Theta(x^2-z)+\Theta(z-x^2\rho^2)] \\ & +\left(L_2+\frac{(1-x)^2}{1+x^2}\right)\Theta(z-x^2)\Theta(x^2\rho^2-z) \\ & +\left(L_1+\frac{(1-x)^2}{1+x^2}\right)\Theta(z-1)\Theta(\rho^2-z) \\ & +(\Theta(1-z)-\Theta(z-\rho^2))\ln\left|\frac{(z-x)(\rho^2-z)}{(x\rho^2-z)(z-1)}\right|.\end{aligned} \tag{55}$$

Let us now turn to the double hard-photon emission in the same direction and the hard $e^+ e^-$ pair production. Here we use the method developed by one of us [17,18]. We will distinguish the collinear and semi-collinear kinematics of final particles. In the first case all produced particles move in the cones within the polar angles $\theta_i < \theta_0 \ll 1$ centered along the charged-particle momenta (final or initial). In the semi-collinear region only one produced particle moves inside those cones, while the other moves outside them. For the totally inclusive cross section, such a distinction no longer has physical meaning and the dependence on the auxiliary parameter θ_0 disappears. We underline that in this way all double and single-logarithmical contributions may be extracted rigorously. The contribution of the region when both the photons move outside the small cones does not contain any large logarithm L. The systematic omission of those contributions in

The contribution of both collinear and semi-collinear regions (we consider for definiteness the emission of both hard photons along the electron, since the contribution of the emission along the positron is the same) has the form (see Appendix B for details)

$$\Sigma^{HH} = \Sigma_{HH} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2}(1 - \Pi(-zQ_1^2))^{-2}$$

$$\times \int_{x_c}^{1-2\Delta} \mathrm{d}x \int_{\Delta}^{1-x-\Delta} \mathrm{d}x_1 \frac{I^{HH} L}{x_1(1-x-x_1)(1-x_1)^2},$$

$$I^{HH} = A\,\Theta(x^2\rho^2 - z) + B + C\,\Theta((1-x_1)^2\rho^2 - z), \qquad (56)$$

$$A = \gamma\beta\left(\frac{L}{2} + \ln\frac{(\rho^2 x^2 - z)^2}{x^2(\rho^2 x(1-x_1) - z)^2}\right)$$
$$+(x^2 + (1-x_1)^4)\ln\frac{(1-x_1)^2(1-x-x_1)}{x x_1} + \gamma_A,$$
$$B = \gamma\beta\left(\frac{L}{2} + \ln\left|\frac{x^2(z-1)(\rho^2 - z)(z-x^2)(z-(1-x_1)^2)^2(\rho^2 x(1-x_1)-z)^2}{(\rho^2 x^2 - z)(z-(1-x_1))^2(\rho^2(1-x_1)^2 - z)^2(z - x(1-x_1))^2}\right|\right)$$
$$+(x^2 + (1-x_1)^4)\ln\frac{(1-x_1)^2 x_1}{x(1-x-x_1)} + \delta_B,$$
$$C = \gamma\beta\left(L + 2\ln\left|\frac{x(\rho^2(1-x_1)^2 - z)^2}{(1-x_1)^2(\rho^2 x(1-x_1) - z)(\rho^2(1-x_1) - z)}\right|\right)$$
$$-2(1-x_1)\beta - 2x(1-x_1)\gamma,$$

The total expression $\Sigma^{2\gamma}$, which describes the contribution to (20) from the two-photon (real and virtual) emission processes is determined by expressions (43), (47), (49), (51) , (53) and (55). Furthermore it does not depend on the auxiliary parameter Δ and has the form

$$\Sigma^{2\gamma} = \Sigma^{\gamma\gamma}_{S+V} + 2\Sigma^{H(V+S)} + 2\Sigma^{H}_{S+V} + \Sigma^{H}_{H} + 2\Sigma^{HH}$$
$$= \Sigma^{\gamma\gamma} + \Sigma^{\gamma}_{\gamma} + \left(\frac{\alpha}{\pi}\right)^2 \mathcal{L}(\phi^{\gamma\gamma} + \phi^{\gamma}_{\gamma}), \qquad \mathcal{L} = \ln\frac{\varepsilon^2\theta_1^2}{m^2}. \tag{58}$$

The leading contributions $\Sigma^{\gamma\gamma}$, Σ_{γ}^{γ} have the following forms (see Appendix D):

$$\Sigma^{\gamma\gamma} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2} L^2 (1 - \Pi(-Q_1^2 z))^{-2}$$
$$\times \int_{x_c}^{1} \mathrm{d}x \left\{ \frac{1}{2} P^{(2)}(x) [\Theta(x^2\rho^2 - z) + 1] \right.$$
$$\left. + \int_x^1 \frac{\mathrm{d}t}{t} P(t) P(\frac{x}{t}) \Theta(t^2\rho^2 - z) \right\}, \qquad (59)$$

$$P^{(2)}(x) = \int_x^1 \frac{\mathrm{d}t}{t} P(t) P(\frac{x}{t})$$
$$= \lim_{\Delta \to 0} \left\{ \left[\left(2\ln\Delta + \frac{3}{2}\right)^2 - 4\zeta_2 \right] \delta(1 - x) \right.$$
$$+2\left[\frac{1 + x^2}{1 - x}\left(2\ln(1 - x) - \ln x + \frac{3}{2}\right) + \frac{1}{2}(1 + x)\ln x - 1 + x\right]$$
$$\left. \times \Theta(1 - x - \Delta) \right\}, \qquad (60)$$

$$\Sigma_{\gamma}^{\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} \frac{\mathrm{d}z}{z^2} L^2 (1 - \Pi(-Q_1^2 z))^{-2} \int_{x_c}^{1} \mathrm{d}x_1 \int_{x_c/x_1}^{1} \mathrm{d}x_2 P(x_1) P(x_2)$$
$$\times \left[\Theta(z - 1)\Theta(\rho^2 - z) + \Theta(z - x_1^2)\Theta(x_1^2\rho^2 - z)\right]$$
$$\times \left[\Theta(z - 1)\Theta(\rho^2 - z) + \Theta(z - x_2^2)\Theta(x_2^2\rho^2 - z)\right]. \qquad (61)$$

We see that the leading contributions to $\Sigma^{2\gamma}$ may be expressed in terms of kernels for the evolution equation for structure functions.

7. Pair production

The method, developed by one of us [17,18], of calculating the real hard pair production cross section within logarithmic accuracy (see the discussion in Section 6) consists in separating the contributions of the collinear and semi-collinear kinematical regions. In the first one (CK) we suggest that both electron and positron from the created pair go in the narrow cone around the direction of one charged particle [the projectile (scattered) electron $\boldsymbol{p}_1$ ($\boldsymbol{q}_1$) or the projectile (scattered) positron $\boldsymbol{p}_2$ ($\boldsymbol{q}_2$)]:

$$\widehat{\boldsymbol{p}_+\boldsymbol{p}_-} \sim \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} \sim \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} < \theta_0 \ll 1, \qquad \varepsilon\theta_0/m \gg 1, \qquad \boldsymbol{p}_i = \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2 . \quad (62)$$

The contribution of the CK contains terms of order $(\alpha L/\pi)^2$, $(\alpha/\pi)^2 L\ln(\theta_0/\theta)$ and $(\alpha/\pi)^2 L$, where $\theta = \widehat{\boldsymbol{p}_-\boldsymbol{q}_1}$ is the scattering angle. In the semi-collinear region only one of conditions (62) on the angles is fulfilled:

$$\widehat{\boldsymbol{p}_+\boldsymbol{p}_-} < \theta_0, \quad \widehat{\boldsymbol{p}_\pm\boldsymbol{p}_i} > \theta_0 ; \quad \text{or} \quad \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} < \theta_0, \quad \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} > \theta_0 ; \quad (63)$$
$$\text{or} \quad \widehat{\boldsymbol{p}_-\boldsymbol{p}_i} > \theta_0, \quad \widehat{\boldsymbol{p}_+\boldsymbol{p}_i} < \theta_0 .$$

The contribution of the SCK contains terms of the form

$$\left(\frac{\alpha}{\pi}\right)^2 L\ln\frac{\theta_0}{\theta}, \qquad \left(\frac{\alpha}{\pi}\right)^2 L. \quad (64)$$

The auxiliary parameter θ_0 drops out in the total sum of the CK and SCK contributions.

All possible mechanisms for pair creation (singlet and non-singlet) and the identity of the particles in the final state are taken into account [22]. In the case of small-angle Bhabha scattering only a part of the total 36 tree-type Feynman diagrams are relevant, i.e. the scattering diagrams. [5]

Consider first the collinear kinematics. There are four different CK regions, when the created pair goes in the direction of the incident (scattered) electron or positron. We will consider only two of them, corresponding to the initial and the final electron directions. For the case of pair emission parallel to the initial electron, it is useful to decompose the particle momenta into longitudinal and transverse components:

$$\begin{aligned} &p_+ = x_1 p_1 + p_+^\perp, \qquad p_- = x_2 p_1 + p_-^\perp, \qquad q_1 = x p_1 + q_1^\perp, \\ &x = 1 - x_1 - x_2, \qquad q_2 \approx p_2, \qquad p_+^\perp + p_-^\perp + q_1^\perp = 0, \end{aligned} \tag{66}$$

where $p_i^\perp$ are the two-dimensional momenta of the final particles, which are transverse with respect to the initial electron beam direction. It is convenient to introduce dimensionless quantities for the relevant kinematical invariants:

$$\begin{aligned} &z_i = \left(\frac{\varepsilon\theta_i}{m}\right)^2, \qquad 0 < z_i < \left(\frac{\varepsilon\theta_0}{m}\right)^2 \gg 1, \\ &A = \frac{(p_+ + p_-)^2}{m^2} = (x_1 x_2)^{-1}\left[(1-x)^2 + x_1^2 x_2^2 (z_1 + z_2 - 2\sqrt{z_1 z_2}\cos\phi)\right], \\ &A_1 = \frac{2 p_1 p_-}{m^2} = x_2^{-1}\left[1 + x_2^2 + x_2^2 z_2\right], \qquad A_2 = \frac{2 p_1 p_+}{m^2} = x_1^{-1}\left[1 + x_1^2 + x_1^2 z_1\right], \\ &C = \frac{(p_1 - p_-)^2}{m^2} = 2 - A_1, \qquad D = \frac{(p_1 - q_1)^2}{m^2} - 1 = A - A_1 - A_2, \end{aligned} \tag{67}$$

where ϕ is the azimuthal angle between the $(\boldsymbol{p}_1 \boldsymbol{p}_+^\perp)$ and $(\boldsymbol{p}_1 \boldsymbol{p}_-^\perp)$ planes.

Keeping only the terms from the sum over spin states of the square of the absolute value of the matrix element, which give non-zero contributions to the cross section in the limit $\theta_0 \to 0$, we find that only 8 from the total 36 Feynman diagrams are essential [22].

The result has the factorized form

$$\sum_{\text{spins}} |M|^2\Big|_{p_+,p_-\|p_1} = \sum_{\text{spins}} |M_0|^2\, 2^7\pi^2\alpha^2 \frac{I}{m^4}, \tag{68}$$

where one of the multipliers corresponds to the matrix element in the Born approximation (without pair production):

$$\sum_{\text{spins}} |M_0|^2 = 2^7\pi^2\alpha^2 \left(\frac{s^4+t^4+u^4}{s^2t^2}\right), \tag{69}$$

$$s = 2p_1p_2, \qquad t = -Q^2x, \qquad u = -s-t,$$

$$\begin{aligned} I &= (1-x_2)^{-2}\left(\frac{A(1-x_2)+Dx_2}{DC}\right)^2 + (1-x)^{-2}\left(\frac{C(1-x)-Dx_2}{AD}\right)^2 \\ &+ \frac{1}{2xAD}\left[\frac{2(1-x_2)^2-(1-x)^2}{1-x} + \frac{x_1x-x_2}{1-x_2} + 3(x_2-x)\right] \\ &+ \frac{1}{2xCD}\left[\frac{(1-x_2)^2-2(1-x)^2}{1-x_2} + \frac{x-x_1x_2}{1-x} + 3(x_2-x)\right] \\ &+ \frac{x_2(x^2+x_2^2)}{2x(1-x_2)(1-x)AC} + \frac{3x}{D^2} + \frac{2C}{AD^2} + \frac{2A}{CD^2} + \frac{2(1-x_2)}{xA^2D} \\ &- \frac{4C}{xA^2D^2} - \frac{4A}{D^2C^2} + \frac{1}{DC^2}\left[\frac{(x_1-x)(1+x_2)}{x(1-x_2)} - 2\frac{1-x}{x}\right]. \end{aligned} \tag{70}$$

We rewrite the phase volume of the final particles as

$$\begin{aligned} \mathrm{d}\Gamma &= \frac{\mathrm{d}^3q_1\mathrm{d}^3q_2}{(2\pi)^6 2q_1^0 2q_2^0}(2\pi)^4\delta^{(4)}(p_1x+p_2-q_1-q_2) \\ &\times m^4 2^{-8}\pi^{-4}x_1x_2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2\frac{\mathrm{d}\phi}{2\pi}. \end{aligned} \tag{71}$$

$$\mathrm{d}\sigma_{\text{coll}} = \frac{\alpha^4 \mathrm{d}x}{\pi Q_1^2} \int_1^{\rho^2} \frac{\mathrm{d}z}{z^2} L \Big\{ R_0(x) \left(L + 2 \ln \frac{\lambda^2}{z} \right) (1 + \Theta)$$

$$+ 4 R_0(x) \ln x + 2\Theta f(x) + 2 f_1(x) \Big\}, \qquad \lambda = \frac{\theta_0}{\theta_{\min}}, \tag{72}$$

$$\Theta \equiv \Theta(x^2\rho^2 - z) = \begin{cases} 1 & x^2\rho^2 > z, \\ 0 & x^2\rho^2 \leqslant z, \end{cases}$$

$$R_0(x) = \frac{2}{3}\frac{1+x^2}{1-x} + \frac{(1-x)}{3x}(4 + 7x + 4x^2) + 2(1+x)\ln x,$$

$$f(x) = -\frac{107}{9} + \frac{136}{9}x - \frac{2}{3}x^2 - \frac{4}{3x} - \frac{20}{9(1-x)} + \frac{2}{3}\Big[-4x^2 - 5x + 1$$

$$+ \frac{4}{x(1-x)}\Big] \ln(1-x) + \frac{1}{3}\left[8x^2 + 5x - 7 - \frac{13}{1-x}\right] \ln x - \frac{2}{1-x} \ln^2 x$$

$$+ 4(1+x) \ln x \ln(1-x) - \frac{2(3x^2-1)}{1-x} \mathrm{Li}_2(1-x),$$

$$f_1(x) = -x \,\mathrm{Re}\, f(1/x) = -\frac{116}{9} + \frac{127}{9}x + \frac{4}{3}x^2 + \frac{2}{3x} - \frac{20}{9(1-x)} + \frac{2}{3}\Big[-4x^2$$

$$-5x + 1 + \frac{4}{x(1-x)}\Big] \ln(1-x) + \frac{1}{3}\left[8x^2 - 10x - 10 + \frac{5}{1-x}\right] \ln x$$

$$-(1+x)\ln^2 x + 4(1+x)\ln x \ln(1-x) - \frac{2(x^2-3)}{1-x} \mathrm{Li}_2(1-x),$$

$$Q_1 = \varepsilon\theta_{\min}, \qquad L = \ln \frac{z Q_1^2}{m^2}.$$

Consider now semi-collinear kinematical regions. We will restrict ourselves again to the case in which the created pair goes close to the electron momentum (initial or final). A similar treatment applies in the CM system in the case in which the pair follows the positron momentum. There are three different semi-collinear regions, which contribute to the cross section within the required accuracy. The first region includes the events for which the created pair has very small invariant mass:

$$4m^2 \ll (p_+ + p_-)^2 \ll |q^2|,$$

and the pair escapes the narrow cones (defined by θ_0) in both the incident and scattered electron momentum directions. We will refer to this SCK region as $\boldsymbol{p}_+ \parallel \boldsymbol{p}_-$. The reason is the smallness (in comparison with s) of the square of the four-momentum of the virtual photon converting to the pair [22].

The second SCK region includes the events for which the invariant mass of the created positron and the scattered electron is small, $4m^2 \ll (p_+ + q_1)^2 \ll |q^2|$, with the restriction that the positron should escape the narrow cone in the initial electron momentum direction. We refer to it as $\boldsymbol{p}_+ \parallel \boldsymbol{q}_1$ [22].

The third SCK region includes the events in which the created electron goes inside the narrow cone in the initial electron momentum direction, but the created positron does not. We refer to it as $\boldsymbol{p}_- \parallel \boldsymbol{p}_1$ [22].

The differential cross section takes the following form:

$$\mathrm{d}\sigma = \frac{\alpha^4}{8\pi^4 s^2} \frac{|M|^2}{q^4} \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x}{x_1 x_2 x} \mathrm{d}^2\boldsymbol{p}_+^\perp \mathrm{d}^2\boldsymbol{p}_-^\perp \mathrm{d}^2\boldsymbol{q}_1^\perp \mathrm{d}^2\boldsymbol{q}_2^\perp \delta(1 - x_1 - x_2 - x)$$
$$\times \delta^{(2)}(\boldsymbol{p}_+^\perp + \boldsymbol{p}_-^\perp + \boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp)\,, \qquad (73)$$

where x_1 (x_2), x and $\boldsymbol{p}_+^\perp$ ($\boldsymbol{p}_-^\perp$), $\boldsymbol{q}_1^\perp$ are the energy fractions and the perpendicular momenta of the created positron (electron) and the scattered electron (positron) respectively; $s = (p_1 + p_2)^2$ and $q^2 = -Q^2 = (p_2 - q_2)^2 = -\varepsilon^2\theta^2$ are the center-of-mass energy squared and the momentum transferred squared; the matrix element squared $|M|^2$ takes different forms according to the different SCK regions.

For the differential cross section in the $\boldsymbol{p}_+ \parallel \boldsymbol{p}_-$ region we have (see, for details, Ref. [21])

$$\mathrm{d}\sigma_{\boldsymbol{p}_+\parallel\boldsymbol{p}_-} = \frac{\alpha^4}{\pi} L \,\mathrm{d}x\,\mathrm{d}x_2 \frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2} \frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp)^2} \times \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\boldsymbol{q}_1^\perp + x\boldsymbol{q}_2^\perp)^2} \left[(1-x_1)^2 + (1-x_2)^2 - \frac{4xx_1x_2}{(1-x)^2} \right], \qquad (74)$$

where ϕ is the angle between the two-dimensional vectors $\boldsymbol{q}_1^\perp$ and $\boldsymbol{q}_2^\perp$, $\boldsymbol{q}_{1,2}^\perp$ are the transverse momentum components of the final electrons, $x_{1,2}$ are their energy fractions ($x = 1 - x_1 - x_2$). At this stage it is necessary to use the restrictions on the two-dimensional momenta $\boldsymbol{q}_1^\perp$ and $\boldsymbol{q}_2^\perp$. They appear when the contribution of the CK region (which here represents the narrow cones with opening angle θ_0 in the momentum directions of both incident and scattered electrons) is excluded:

$$\left| \frac{\boldsymbol{p}_+^\perp}{\varepsilon_+} \right| > \theta_0, \qquad |\boldsymbol{r}^\perp| = \left| \frac{\boldsymbol{p}_+^\perp}{\varepsilon_+} - \frac{\boldsymbol{q}_1^\perp}{\varepsilon_2} \right| > \theta_0, \qquad (75)$$

where ε_+ and ε_2 are the energies of the created positron and the scattered electron respectively. In order to exclude $\boldsymbol{p}_+^\perp$ from the above equation we use the conservation of the perpendicular momentum, in this case:

$$\boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp + \frac{1-x}{x_1} \boldsymbol{p}_+^\perp = 0.$$

In the semi-collinear region $\boldsymbol{p}_+ \parallel \boldsymbol{q}_1$ we obtain

$$\mathrm{d}\sigma_{\boldsymbol{p}_+\parallel \boldsymbol{q}_1} = \frac{\alpha^4}{\pi} L\,\mathrm{d}x\,\mathrm{d}x_2 \frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2} \frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp)^2} \times \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\boldsymbol{q}_1^\perp + x\boldsymbol{q}_2^\perp)^2} \frac{x^2}{(1-x_2)^2} \left[(1-x)^2 + (1-x_1)^2 - \frac{4xx_1x_2}{(1-x_2)^2} \right], \tag{76}$$

Finally for the $\boldsymbol{p}_- \parallel \boldsymbol{p}_1$ semi-collinear region we get

$$\mathrm{d}\sigma_{\boldsymbol{p}_-\parallel \boldsymbol{p}_1} = \frac{\alpha}{4\pi} L\,\mathrm{d}x\,\mathrm{d}x_2 \frac{\mathrm{d}(\boldsymbol{q}_2^\perp)^2}{(\boldsymbol{q}_2^\perp)^2} \frac{\mathrm{d}(\boldsymbol{q}_1^\perp)^2}{(\boldsymbol{q}_1^\perp)^2} \times \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\boldsymbol{q}_1^\perp + \boldsymbol{q}_2^\perp)^2} \left[\frac{(1-x)^2 + (1-x_1)^2}{(1-x_2)^2} - \frac{4xx_1x_2}{(1-x_2)^4} \right]. \tag{78}$$

In order to obtain the finite expression for the cross section we have to add $d\sigma_{p_+ \| p_-} + d\sigma_{p_+ \| q_1} + d\sigma_{p_- \| p_1}$ to the contribution of the collinear kinematics region (72) and those due to the production of virtual and soft pairs. Taking into account the leading and next-to-leading terms we can write the full hard pair contribution including also the pair emission along the positron direction, after the integration over x_2 as

$$\sigma_{\text{hard}} = 2\frac{\alpha^4}{\pi Q_1^2}\int\limits_1^{\rho^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta} dx\left\{L^2(1+\Theta)R(x) + \mathcal{L}[\Theta F_1(x) + F_2(x)]\right\} \qquad (80)$$

$$F_1(x) = d(x) + C_1(x), \qquad F_2(x) = d(x) + C_2(x),$$

$$d(x) = \frac{1}{1-x}\left(\frac{8}{3}\ln(1-x) - \frac{20}{9}\right),$$

$$C_1(x) = -\frac{113}{9} + \frac{142}{9}x - \frac{2}{3}x^2 - \frac{4}{3x} - \frac{4}{3}(1+x)\ln(1-x)$$
$$+\frac{2}{3}\frac{1+x^2}{1-x}\left[\ln\frac{(x^2\rho^2-z)^2}{(x\rho^2-z)^2} - 3\mathrm{Li}_2(1-x)\right] + \Big(8x^2+3x-9$$
$$-\frac{7}{1-x}\Big)\ln x + \frac{2(5x^2-6)}{1-x}\ln^2 x + \beta(x)\ln\frac{(x^2\rho^2-z)^2}{\rho^4},$$

$$C_2(x) = -\frac{122}{9} + \frac{133}{9}x + \frac{4}{3}x^2 + \frac{2}{3x} - \frac{4}{3}(1+x)\ln(1-x)$$
$$+\frac{2}{3}\frac{1+x^2}{1-x}\left[\ln\left|\frac{(z-x^2)(\rho^2-z)(z-1)}{(x^2\rho^2-z)(z-x)^2}\right| + 3\mathrm{Li}_2(1-x)\right]$$
$$+\frac{1}{3}\left(-8x^2 - 32x - 20 + \frac{13}{1-x} + \frac{8}{x}\right)\ln x + 3(1+x)\ln^2 x$$
$$+\beta(x)\ln\left|\frac{(z-x^2)(\rho^2-z)(z-1)}{x^2\rho^2-z}\right|, \qquad \beta = 2R(x) - \frac{2}{3}\frac{1}{1}$$

$$R(x) = \frac{1}{3}\frac{1+x^2}{1-x} + \frac{1-x}{6x}(4+7x+4x^2) + (1+x)\ln x. \qquad (81)$$

The contribution to the cross section of the small-angle Bhabha scattering connected with the real soft (with energy lower than $\Delta\varepsilon$) and virtual pair production can be defined [22] by the formula

$$\sigma_{\text{soft+virt}} = \frac{4\alpha^4}{\pi Q_1^2} \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2} \left\{ L^2 \left(\frac{2}{3} \ln \Delta + \frac{1}{2} \right) + \mathcal{L} \left(-\frac{17}{6} + \frac{4}{3} \ln^2 \Delta - \frac{20}{9} \ln \Delta - \frac{4}{3} \zeta_2 \right) \right\}. \tag{82}$$

Using Eqs. (80) and (82) it is easy to verify that the auxiliary parameter Δ is canceled in the sum $\sigma_{\text{pair}} = \sigma_{\text{hard}} + \sigma_{\text{soft+virt}}$. We can, therefore, write the total contribution σ_{pair} as

$$\begin{aligned} \sigma_{\text{pair}} = \frac{2\alpha^4}{\pi Q_1^2} \int\limits_1^{\rho^2} \frac{\mathrm{d}z}{z^2} \Bigg\{ & L^2 \Big(1 + \frac{4}{3} \ln(1 - x_c) - \frac{2}{3} \int\limits_{x_c}^{1} \frac{\mathrm{d}x}{1 - x} \bar{\Theta} \Big) + \mathcal{L} \Bigg[-\frac{17}{3} \\ & - \frac{8}{3}\zeta_2 - \frac{40}{9} \ln(1 - x_c) + \frac{8}{3} \ln^2(1 - x_c) + \int\limits_{x_c}^{1} \frac{\mathrm{d}x}{1 - x} \bar{\Theta} \cdot \left(\frac{20}{9} - \frac{8}{3} \ln(1 - x) \right) \Bigg] \\ & + \int\limits_{x_c}^{1} \mathrm{d}x \left[L^2 (1 + \Theta) \bar{R}(x) + \mathcal{L}(\Theta C_1(x) + C_2(x)) \right] \Bigg\}, \end{aligned}$$

$$\bar{R}(x) = R(x) - \frac{2}{3(1 - x)}, \qquad \bar{\Theta} = 1 - \Theta. \tag{83}$$

The right-hand side of Eq. (83) gives the contribution to the small-angle Bhabha scattering cross section for pair production. It is finite and can be used for numerical estimations. The leading term can be described by the electron structure function $D_e^{\bar{e}}(x)$ [20].

8. Terms of $\mathcal{O}(\alpha\mathcal{L})^3$

We may, therefore, limit ourselves to consider the emission by the initial electron and positron. Three photons (virtual and real) contribution to Σ have the form

$$\Sigma^{3\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\mathcal{L}\right)^3 \int\limits_1^{\rho^2} \frac{dz}{z^2} \int\limits_{x_c}^{1} dx_1 \int\limits_{x_c}^{1} dx_2 \Theta(x_1 x_2 - x_c) \left[\frac{1}{6}\delta(1-x_2)\, P^{(3)}(x_1) \right.$$
$$\left. \times \Theta(x_1^2\rho^2 - z) + \frac{1}{2x_1^2} P^{(2)}(x_1) P(x_2) \Theta_1\Theta_2 \right] (1 + \mathcal{O}(x_c^3)), \quad (85)$$

where $P(x)$ and $P^{(2)}(x)$ are given by Eqs. (33) and (59) correspondingly,

$$\Theta_1\Theta_2 = \Theta\left(z - \frac{x_2^2}{x_1^2}\right)\Theta\left(\rho^2\frac{x_2^2}{x_1^2} - z\right),$$
$$P^{(3)}(x) = \delta(1-x)\,\Delta_t + \Theta(1-x-\Delta)\Theta_t,$$
$$\Delta_t = 48\left[\frac{1}{3}\zeta_3 - \frac{1}{2}\zeta_2\left(\ln\Delta + \frac{3}{4}\right) + \frac{1}{6}\left(\ln\Delta + \frac{3}{4}\right)^3\right],$$
$$\Theta_t = 48\left\{\frac{1}{2}\frac{1+x^2}{1-x}\left[\frac{9}{32} - \frac{1}{2}\zeta_2 + \frac{3}{4}\ln(1-x) - \frac{3}{8}\ln x + \frac{1}{2}\ln^2(1-x) \right.\right.$$
$$\left. + \frac{1}{12}\ln^2 x - \frac{1}{2}\ln x \ln(1-x)\right] + \frac{1}{8}(1+x)\ln x \ln(1-x)$$
$$- \frac{1}{4}(1-x)\ln(1-x) + \frac{1}{32}(5-3x)\ln x - \frac{1}{16}(1-x) - \frac{1}{32}(1+x)\ln^2 x$$
$$\left. + \frac{1}{8}(1+x)\mathrm{Li}_2(1-x)\right\}. \quad (86)$$

The contribution to Σ of the process of pair production accompanied by photon emission when both, pair and photons, may be real and virtual has the form (with respect to Ref. [20] we include also the non-singlet mechanism of pair production)

$$\Sigma^{e^+e^-\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\mathcal{L}\right)^3 \int\limits_1^{\rho^2} \mathrm{d}z\ z^{-2} \int\limits_{x_c}^{1} \mathrm{d}x_1 \int\limits_{x_c}^{1} \mathrm{d}x_2 \Theta(x_1 x_2 - x_c)$$
$$\times \left\{ \frac{1}{3}\left[R^P(x_1) - \frac{1}{3}R^s(x_1)\right]\delta(1-x_2)\Theta(x_1^2\rho^2 - z) + \frac{1}{2x_1^2} P(x_2) R(x_1) \Theta_1 \Theta_2 \right\},$$

where

$$R(x) = R^s(x) + \frac{2}{3}P(x), \qquad R^s(x) = \frac{1-x}{3x}(4 + 7x + 4x^2) + 2(1+x)\ln x,$$

$$R^P(x) = R^s(x)\left(\frac{3}{2} + 2\ln(1-x)\right) + (1+x)(-\ln^2 x + 4\mathrm{Li}_2(1-x)$$
$$+\frac{1}{3}(-9 - 3x + 8x^2)\ln x + \frac{2}{3}\left(-\frac{3}{x} - 8 + 8x + 3x^2\right) + \frac{2}{3}P^{(2)}(x). \qquad (87)$$

The total expression for Σ in Eq. (20) is the sum of the contributions in Eqs. (21), (32), (56), (60), (66) and (68). The quantity Σ depends on the parameters x_c, ρ and Q_1^2.

9. Estimates of neglected terms and numerical results

The uncertainty of our calculations is defined by neglected terms. Let us list them.

(a) Terms of the first order RC coming from annihilation-type diagrams (15):

$$\frac{\alpha}{\pi}\theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{\mathrm{d}\theta}{\theta^2} \Delta_\theta \leqslant 0.10 \times 10^{-4}. \tag{88}$$

(b) Similar terms in the second order do not exceed (see Section 4)

$$\left(\frac{\alpha}{\pi}\right)^2 \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{\mathrm{d}\theta}{\theta^2}\, l^4 \leqslant 0.23 \times 10^{-4},$$

$$\left(\frac{\alpha}{\pi}\right)^2 (\theta_2^4 - \theta_1^4)\mathcal{L}^4 \leqslant 0.5 \times 10^{-5}. \tag{89}$$

(c) We neglect terms which violate the eikonal approximation:

$$\frac{\alpha}{\pi}\frac{Q^2}{s} \leqslant 0.3 \times 10^{-6}. \tag{90}$$

(d) We omit term of the second order which are not enhanced by large logarithms:

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.5 \times 10^{-5}. \tag{91}$$

(e) Creation of heavy pairs ($\mu\mu$, $\tau\tau$, $\pi\pi$, ...) gives in sum at least one order of magnitude smaller than the corresponding contribution due to light particle production [24]:

$$\Sigma_{\pi\pi} + \Sigma_{\mu\mu} + \Sigma_{\tau\tau} \leqslant 0.1\,\Sigma^{e^+e^-} \leqslant 0.5 \times 10^{-4}. \tag{92}$$

(f) Higher-order corrections, including soft and collinear multi-photon contributions, can be neglected since they only give contributions of the type $(\alpha L/\pi)^4 \leqslant 0.2 \times 10^{-5}$ or less.

(g) The terms in the third order associated with the emission off the final particles: [7]

Let us define Σ_0^0 to be equal to $\Sigma_0|_{\Pi=0}$ (see Eq. (21)), which corresponds to the Born cross section obtained by switching off the vacuum polarization contribution $\Pi(t)$. For the experimentally observable cross section we obtain

$$\sigma = \frac{4\pi\alpha^2}{Q_1^2}\Sigma_0^0(1 + \delta_0 + \delta^\gamma + \delta^{2\gamma} + \delta^{e^+e^-} + \delta^{3\gamma} + \delta^{e^+e^-\gamma}), \tag{94}$$

where

$$\Sigma_0^0 = \Sigma_0|_{\Pi=0} = 1 - \rho^{-2} + \Sigma_W + \Sigma_\theta|_{\Pi=0} \tag{95}$$

and

$$\delta_0 = \frac{\Sigma_0 - \Sigma_0^0}{\Sigma_0^0}, \qquad \delta^\gamma = \frac{\Sigma^\gamma}{\Sigma_0^0}, \qquad \delta^{2\gamma} = \frac{\Sigma^{2\gamma}}{\Sigma_0^0}, \ldots \tag{96}$$

The numerical results are presented in Table 1.

Table 1
The values of δ^i in per cent for $\sqrt{s} = 91.161$ GeV, $\theta_1 = 1.61°$, $\theta_2 = 2.8°$, $\sin^2\theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV

x_c	δ_0	δ^γ	$\delta^{2\gamma}_{\text{leading}}$	$\delta^{2\gamma}_{\text{non-leading}}$	$\delta^{e^+e^-}$	$\delta^{e^+e^-\gamma}$	$\delta^{3\gamma}$	$\sum\delta^i$
0.1	4.120	−8.918	0.657	0.162	−0.016	−0.017	−0.019	−4.031±0.006
0.2	4.120	−9.226	0.636	0.156	−0.027	−0.011	−0.016	−4.368±0.006
0.3	4.120	−9.626	0.615	0.148	−0.033	−0.008	−0.013	−4.797±0.006
0.4	4.120	−10.147	0.586	0.139	−0.039	−0.005	−0.010	−5.356±0.006
0.5	4.120	−10.850	0.539	0.129	−0.044	−0.003	−0.006	−6.115±0.006
0.6	4.120	−11.866	0.437	0.132	−0.049	−0.002	−0.001	−7.229±0.006
0.7	4.120	−13.770	0.379	0.130	−0.057	−0.001	0.005	−9.194±0.006
0.8	4.120	−17.423	0.608	0.089	−0.069	0.001	0.013	−12.661±0.006
0.9	4.120	−25.269	1.952	−0.085	−0.085	0.005	0.017	−19.379±0.006

$$x_c\left(\frac{\alpha\mathcal{L}}{\pi}\right)^3 \leqslant 0.3\times 10^{-4} \quad (\text{for } x_c = 0.5). \tag{93}$$

Regarding all the uncertainties (a)–(g) and (82) as independent ones we conclude the total theoretical uncertainty of our results to be ±0.006%.

Each of these contributions to σ has a sign that can change because of the interplay between real and virtual corrections. The cross section corresponding to the Born diagrams for producing a real particle is always positive, whereas the sign of the radiative corrections depends on the order of perturbation theory. For the virtual corrections at odd orders it is negative, and at even orders it is positive. When the aperture of the counters is small the compensation between real and virtual corrections is not complete. In the limiting case of small aperture ($\rho \to 1,\ x_c \to 1$) the virtual contributions dominate.

Some examples of subsequent work

Physics Letters B 760 (2016) 816–822

Contents lists available at ScienceDirect

Physics Letters B

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Two-loop Bhabha scattering at high energy beyond leading power approximation

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ARTICLE INFO

Article history:
Received 2 July 2016
Accepted 31 July 2016
Available online 3 August 2016
Editor: A. Ringwald

Keywords:
QED
Perturbation theory
Bhabha scattering

ABSTRACT

We evaluate the two-loop $\mathcal{O}(m_e^2/s)$ contribution to the wide-angle high-energy electron–positron scattering in the double-logarithmic approximation. The origin and the general structure of the power-suppressed double logarithmic corrections are discussed in detail.

1. Introduction

High-energy electron–positron or *Bhabha* scattering [1] is among the classical applications of the perturbative quantum electrodynamics (QED). Beside its phenomenological importance as a standard candle for luminosity calibration at the electron–positron colliders, Bhabha scattering has become a testing ground for new techniques of multiloop calculations. The analysis of high-order corrections to this process often sheds new light on perturbative structure of gauge theories. In general the radiative corrections for the scattering of two massive particles are known only in the one-loop approximation. Despite significant progress over the last decade [2–7], the two-loop corrections have been computed only in the high energy limit neglecting the terms suppressed by the ratio of the electron mass m_e to the center-of-mass energy $\sqrt{s}$ [8–15].[1] The logarithmically enhanced two-loop electroweak corrections are available in this approximation as well [17–21]. At the same time the power-suppressed terms in two loops are still beyond the reach of existing computational techniques. In general the power-suppressed contributions are of great interest. At intermediate energies the power corrections in many cases are phenomenologically important. Moreover, in contrast to the leading-power contribution very little is known about the infrared structure of the power-suppressed terms. This problem has been studied already in early days of QED [22] and currently attracts much attention in various context [23–27]. However, a systematic renormalization group analysis of the high-energy behavior of on-shell amplitudes beyond the leading-power approximation is still elusive for the existing effective field theory methods.

In this paper we consider the $\mathcal{O}(m_e^2/s)$ two-loop QED corrections to the differential cross section of the high-energy large-angle Bhabha scattering. The corrections are evaluated in the double-logarithmic approximation *i.e.* retaining the terms enhanced by two powers of the large logarithm $\ln(s/m_e^2)$ per each power of the coupling constant. These terms dominate the power-suppressed contribution and in a wide energy interval are numerically comparable to the nonlogarithmic leading-power terms. The leading power-suppressed double-logarithmic corrections have been obtained in Ref. [26] to all orders in fine structure constant α for the electromagnetic form factor of electron. In this paper we elaborate the approach [26] and apply it to the electron–positron scattering amplitude in two-loop approximation. Our main result is given by Eq. (24).

The paper is organized as follows. In the next section we describe the perturbative expansion of the cross section at high energy. In Sect. 3 we discuss the origin and general structure of the double-logarithmic corrections. In Sect. 4 we describe the evaluation of the one and two-loop double-logarithmic power-suppressed corrections to Bhabha scattering. Sect. 5 is our summary and conclusion.

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[1] For a review see Ref. [16].

http://dx.doi.org/10.1016/j.physletb.2016.07.077

Two-loop correction to Bhabha scattering

Z. Bern, L. Dixon, and A. Ghinculov
Phys. Rev. D **63**, 053007 – Published 7 February 2001

ABSTRACT

We present the two-loop virtual QED corrections to $e^+e^- \to \mu^+\mu^-$ and Bhabha scattering in dimensional regularization. The results are expressed in terms of polylogarithms. The form of the infrared divergences agrees with previous expectations. These results are a crucial ingredient in the complete next-to-next-to-leading order QED corrections to these processes. A future application will be to reduce theoretical uncertainties associated with luminosity measurements at e^+e^- colliders. The calculation also tests methods that may be applied to analogous QCD processes.

Received 12 October 2000

DOI: https://doi.org/10.1103/PhysRevD.63.053007

Physics Letters B

Volume 516, Issues 1–2, 6 September 2001, Pages 33-38

Second order contributions to elastic large-angle Bhabha scattering

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Show more

https://doi.org/10.1016/S0370-2693(01)00927-3

Abstract

We derive the coefficient of the $\mathrm{O}(\alpha^2 \log(\mathrm{s}/\mathrm{m_e}^2))$ fixed order contribution to elastic large-angle Bhabha scattering. We adapt the classification of infrared divergences, that was recently developed within dimensional regularization, and apply it to the regularization scheme with a massive photon and electron.

Two-loop $N_F = 1$ QED Bhabha scattering differential cross section

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Received 4 June 2004; received in revised form 18 August 2004; accepted 13 September 2004

Available online 28 September 2004

Abstract

We calculate the two-loop virtual, UV renormalized corrections at order $\alpha^4(N_F = 1)$ in QED to the Bhabha scattering differential cross section, for arbitrary values of the squared c.m. energy s and momentum transfer t, and on-shell electrons and positrons of finite mass m. The calculation is carried out within the dimensional regularization scheme; the remaining IR divergences appear as polar singularities in $(D - 4)$. The result is presented in terms of 1- and 2-dimensional harmonic polylogarithms, of maximum weight 3.

Appendix A
from page 142 ….to page 170

A.1. *Auxiliary functions for the one-loop cross section*

$$\Pi_0^{(1l,0)}(-P^2) = \frac{5}{9} - \frac{4}{3(1-x)^2} + \frac{4}{3(1-x)} - \left[\frac{1}{3} + \frac{4}{3(1-x)^3} - \frac{2}{(1-x)^2}\right]H(0;x), \tag{A.14}$$

$$\begin{aligned}\Pi_0^{(1l,1)}(-P^2) &= \frac{14}{27} - \frac{16}{9(1-x)^2} + \frac{16}{9(1-x)} + \left[\frac{2}{3(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{6}\right]\zeta(2)\\ &- \left[\frac{5}{18} + \frac{16}{9(1-x)^3} - \frac{8}{3(1-x)^2} + \frac{1}{3(1-x)}\right]H(0;x)\\ &- \left[\frac{1}{6} + \frac{2}{3(1-x)^3} - \frac{1}{(1-x)^2}\right]H(0,0;x)\\ &+ \left[\frac{1}{3} + \frac{4}{3(1-x)^3} - \frac{2}{(1-x)^2}\right]H(-1,0;x),\end{aligned} \tag{A.15}$$

$$F_1^{(1l,-1)}(-P^2) = 1 - \left[1 - \frac{1}{(1-x)} - \frac{1}{(1+x)}\right]H(0;x), \tag{A.16}$$

$$\begin{aligned}F_1^{(1l,0)}(-P^2) &= -1 - \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(1+x)}\right]H(0;x)\\ &- \frac{1}{2}\left[1 - \frac{1}{(1-x)} - \frac{1}{(1+x)}\right][\zeta(2) - 2H(0;x) - H(0,0;x)\\ &+ 2H(-1,0;x)],\end{aligned} \tag{A.17}$$

$$F_2^{(1l,0)}(-P^2) = -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]H(0;x), \tag{A.18}$$

$$\begin{aligned}&B_1^{(1l,-1)}(-P^2,-Q^2)\\ &= \Big(-48 - \frac{8}{x^2(1-y)^2} + \frac{8}{x^2(1-y)} + \frac{32}{x(1-y)^2} - \frac{32}{x(1-y)} - \frac{16}{x}\\ &- \frac{32x}{(1-y)^2} + \frac{32x}{(1-y)} + 16x + \frac{8x^2}{(1-y)^2} - \frac{8x^2}{(1-y)} - \frac{8}{y(1+x)}\\ &- \frac{8}{y(1-x)} + \frac{8}{y} - \frac{8y}{(1+x)} - \frac{8y}{(1-x)} + 8y - \frac{96}{(1+x)(1-y)^2}\\ &+ \frac{96}{(1+x)(1-y)} + \frac{80}{(1+x)} + \frac{32}{(1-x)(1-y)^2} - \frac{32}{(1-x)(1-y)}\\ &+ \frac{16}{(1-x)} + \frac{32}{(1-y)^2} - \frac{32}{(1-y)}\Big)H(0;x),\end{aligned} \tag{A.19}$$

$$\begin{aligned}&B_1^{(1l,0)}(-P^2,-Q^2)\\ &= \zeta(2)\Big(\frac{8}{x^2(1+y)} - \frac{8}{x^2(1-y)} - \frac{32}{x(1+y)^3} + \frac{48}{x(1+y)^2} - \frac{88}{x(1+y)}\\ &+ \frac{40}{x(1-y)} + \frac{12}{x} - \frac{32x}{(1+y)^3} + \frac{48x}{(1+y)^2} - \frac{88x}{(1+y)} + \frac{40x}{(1-y)} + 20x\\ &+ \frac{8x^2}{(1+y)} - \frac{8x^2}{(1-y)} - \frac{4}{y(1-x)} - \frac{6}{y} - \frac{4y}{(1-x)} + 10y + \frac{32}{(1+x)}\\ &+ \frac{64}{(1+y)^3} - \frac{96}{(1+y)^2} + \frac{208}{(1+y)} - \frac{48}{(1-y)} - 80\Big)\end{aligned}$$

………….

$$\begin{aligned}&- \frac{64}{x(1-y)^3} - \frac{32}{3x(1-y)^2} + \frac{32}{3x(1-y)} + \frac{128x}{3(1-y)^5} - \frac{320x}{3(1-y)^4}\\ &+ \frac{64x}{(1-y)^3} + \frac{32x}{3(1-y)^2} - \frac{32x}{3(1-y)} + \frac{32x^2}{3(1-y)^5} - \frac{80x^2}{3(1-y)^4}\\ &+ \frac{16x^2}{(1-y)^3} + \frac{8x^2}{3(1-y)^2} - \frac{8x^2}{3(1-y)} + \frac{2}{3y(1-x)} - \frac{2}{3y(x+1)}\\ &+ \frac{2y}{3(1-x)} - \frac{2y}{3(x+1)} - \frac{128}{(1-x)(1-y)^5} + \frac{320}{(1-x)(1-y)^4}\\ &- \frac{192}{(1-x)(1-y)^3} - \frac{32}{(1-x)(1-y)^2} + \frac{32}{(1-x)(1-y)} + \frac{4}{3(1-x)}\\ &+ \frac{128}{3(1-y)^5(x+1)} + \frac{128}{3(1-y)^5} - \frac{320}{3(1-y)^4(x+1)} - \frac{320}{3(1-y)^4}\\ &+ \frac{64}{(1-y)^3(x+1)} + \frac{64}{(1-y)^3} + \frac{32}{3(1-y)^2(x+1)} + \frac{32}{3(1-y)^2}\\ &- \frac{32}{3(1-y)(x+1)} - \frac{32}{3(1-y)} - \frac{4}{(x+1)}\Big)H(0,0;y)H(0;x)\\ &+ \Big(-\frac{8}{3x^2(1-y)^5} + \frac{20}{3x^2(1-y)^4} - \frac{4}{x^2(1-y)^3} - \frac{2}{3x^2(1-y)^2}\\ &+ \frac{2}{3x^2(1-y)} - \frac{32}{3x(1-y)^5} + \frac{80}{3x(1-y)^4} - \frac{16}{x(1-y)^3} - \frac{8}{3x(1-y)^2}\\ &+ \frac{8}{3x(1-y)} + \frac{32x}{3(1-y)^5} - \frac{80x}{3(1-y)^4} + \frac{16x}{(1-y)^3}\\ &+ \frac{8x}{3(1-y)^2} - \frac{8x}{3(1-y)} + \frac{8x^2}{3(1-y)^5} - \frac{20x^2}{3(1-y)^4} + \frac{4x^2}{(1-y)^3}\\ &+ \frac{2x^2}{3(1-y)^2} - \frac{2x^2}{3(1-y)} - \frac{32}{(1-x)(1-y)^5} + \frac{80}{(1-x)(1-y)^4}\\ &- \frac{48}{(1-x)(1-y)^3} - \frac{8}{(1-x)(1-y)^2} + \frac{8}{(1-x)(1-y)}\\ &+ \frac{32}{3(1-y)^5(x+1)} + \frac{32}{3(1-y)^5} - \frac{80}{3(1-y)^4(x+1)}\\ &- \frac{80}{3(1-y)^4} + \frac{16}{(1-y)^3(x+1)} + \frac{16}{(1-y)^3} + \frac{8}{3(1-y)^2(x+1)}\\ &+ \frac{8}{3(1-y)^2} - \frac{8}{3(1-y)(x+1)} - \frac{8}{3(1-y)}\Big)\Big[\big(G(-y,0,0;x)\\ &- G(-1/y,0,0;x)\big) + \big(G(-y,0;x) + G(-1/y,0,x)\big)H(0;y)\\ &+ \big(G(-1/y;x) - G(-y;x)\big)\big(H(0,0;y) + 3\zeta(2)\big) + 2H(0;x)H(1,0;y)\\ &- 2H(-1,0;x)H(0;y) - 6H(-1,0;y)H(0;x)\Big].\end{aligned} \tag{A.33}$$

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NUCLEAR PHYSICS B

Nuclear Physics B 716 (2005) 280–302

Two-loop $N_F = 1$ QED Bhabha scattering: Soft emission and numerical evaluation of the differential cross-section

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Received 13 December 2004; received in revised form 24 February 2005; accepted 8 March 2005

Available online 8 April 2005

Appendix B. Leading corrections at order α^3 and order $\alpha^4 (N_F = 1)$

In this appendix, we provide the explicit expressions of the leading radiative corrections defined in Eq. (27):

$$\left.\frac{d\sigma_1^T}{d\Omega}\right|_L = \alpha^2\Big\{-\ln\left(\frac{m^2}{s}\right)\left(\frac{7}{3}\frac{t^2}{s^3}+\frac{14}{3}\frac{t}{s^2}+\frac{7}{s}+\frac{7}{3}\frac{s}{t^2}+\frac{14}{3t}\right)$$
$$-\ln^2\left(-\frac{t}{s}\right)\left(\frac{t^2}{s^3}+\frac{13}{4}\frac{t}{s^2}+\frac{25}{4s}+2\frac{s}{t^2}+\frac{19}{4t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\ln\left(-\frac{u}{s}\right)\left(2\frac{t^2}{s^3}+5\frac{t}{s^2}+\frac{19}{2s}+4\frac{s}{t^2}+\frac{8}{t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\left(\frac{7}{6}\frac{t}{s^2}+\frac{7}{2s}+\frac{7}{3}\frac{s}{t^2}+\frac{3}{t}\right)$$

$$+2\ln\left(\frac{\omega^2}{s}\right)\Big(1$$

$$\left.\frac{d\sigma_2^T}{d\Omega}\right|_L = \alpha^2\Big\{-\zeta(2)\Big(\frac{46}{9}$$
$$+\zeta(3)\Big(2\frac{t^2}{s^3}+$$
$$+\mathrm{Li}_2\left(-\frac{t}{s}\right)\ln\Big($$
$$-\mathrm{Li}_2\left(-\frac{u}{s}\right)\ln$$
$$-\mathrm{Li}_2\left(-\frac{u}{t}\right)\ln$$
$$+\mathrm{Li}_2\left(-\frac{u}{t}\right)\ln$$
$$-\mathrm{Li}_2\left(-\frac{u}{t}\right)\ln$$
$$-\mathrm{Li}_2\left(-\frac{u}{t}\right)\Big($$
$$-\mathrm{Li}_3\left(-\frac{t}{s}\right)\Big($$
$$+\mathrm{Li}_3\left(-\frac{u}{s}\right)\Big($$
$$-\mathrm{Li}_3\left(-\frac{u}{t}\right)\Big($$
$$-\ln\left(\frac{m^2}{s}\right)\zeta(2)$$
$$+\ln^3\left(\frac{m^2}{s}\right)\Big($$

$$+\ln^2\left(\frac{m^2}{s}\right)\ln\left(-\frac{t}{s}\right)\Big($$
$$-\ln^2\left(\frac{m^2}{s}\right)\ln\left(-\frac{u}{s}\right)\Big($$
$$-\ln^2\left(\frac{m^2}{s}\right)\ln\left(\frac{\omega^2}{s}\right)\Big($$
$$-\ln^2\left(\frac{m^2}{s}\right)\Big(\frac{17}{18}\frac{t^2}{s^3}+$$
$$-\ln\left(\frac{m^2}{s}\right)\ln^2\left(-\frac{t}{s}\right)\Big($$
$$+\ln\left(\frac{m^2}{s}\right)\ln\left(-\frac{t}{s}\right)\ln\Big($$
$$+\ln\left(\frac{m^2}{s}\right)\ln\left(-\frac{t}{s}\right)\ln\Big($$
$$+\ln\left(\frac{m^2}{s}\right)\ln\left(-\frac{t}{s}\right)\Big($$
$$-\ln\left(\frac{m^2}{s}\right)\ln^2\left(-\frac{u}{s}\right)\Big($$
$$-\ln\left(\frac{m^2}{s}\right)\ln\left(-\frac{u}{s}\right)\ln\Big($$
$$-\ln\left(\frac{m^2}{s}\right)\ln\left(-\frac{u}{s}\right)\Big($$

$$-\ln\left(-\frac{t}{s}\right)\ln^2\left(-\frac{u}{s}\right)\left(\frac{1}{3}\frac{t^2}{s^3}-\frac{13}{12s}-\frac{s}{t^2}-\frac{3}{2t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\ln\left(-\frac{u}{s}\right)\ln\left(\frac{\omega^2}{s}\right)\left(\frac{2}{3}\frac{t}{s^2}+\frac{2}{s}+\frac{4}{3}\frac{s}{t^2}+\frac{2}{t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\ln\left(-\frac{u}{s}\right)\left(\frac{20}{9}\frac{t^2}{s^3}+\frac{58}{9}\frac{t}{s^2}+\frac{83}{6s}+\frac{56}{9}\frac{s}{t^2}+\frac{215}{18t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\ln\left(\frac{\omega^2}{s}\right)\left(\frac{20}{9}\frac{t^2}{s^3}+\frac{46}{9}\frac{t}{s^2}+\frac{26}{3s}+\frac{32}{9}\frac{s}{t^2}+\frac{58}{9t}\right)$$
$$+\ln\left(-\frac{t}{s}\right)\left(\frac{56}{27}\frac{t^2}{s^3}+\frac{211}{54}\frac{t}{s^2}+\frac{11}{2s}+\frac{43}{27}\frac{s}{t^2}+\frac{61}{27t}\right)$$
$$+\ln\left(-\frac{u}{s}\right)\zeta(2)\left(2\frac{t^2}{s^3}+\frac{19}{3}\frac{t}{s^2}+\frac{29}{3s}+\frac{4}{3}\frac{s}{t^2}+\frac{17}{3t}\right)$$
$$-\ln^2\left(-\frac{u}{s}\right)\left(\frac{10}{9}\frac{t^2}{s^3}+\frac{61}{18}\frac{t}{s^2}+\frac{5}{s}+\frac{13}{9}\frac{s}{t^2}+\frac{67}{18t}\right)$$
$$+\ln^3\left(-\frac{u}{s}\right)\left(\frac{1}{9}\frac{t^2}{s^3}+\frac{5}{18}\frac{t}{s^2}+\frac{5}{18s}+\frac{1}{6t}\right)$$
$$-\ln\left(-\frac{u}{s}\right)\ln\left(\frac{\omega^2}{s}\right)\left(\frac{20}{9}\frac{t^2}{s^3}+\frac{40}{9}\frac{t}{s^2}+\frac{20}{3s}+\frac{20}{9}\frac{s}{t^2}+\frac{40}{9t}\right)$$
$$-\ln\left(-\frac{u}{s}\right)\left(\frac{56}{27}\frac{t^2}{s^3}+\frac{161}{54}\frac{t}{s^2}+\frac{56}{9s}+\frac{56}{27}\frac{s}{t^2}+\frac{161}{54t}\right)$$
$$-\ln\left(\frac{\omega^2}{s}\right)\left(\frac{20}{9}\frac{t^2}{s^3}+\frac{40}{9}\frac{t}{s^2}+\frac{20}{3s}+\frac{20}{9}\frac{s}{t^2}+\frac{40}{9t}\right)$$

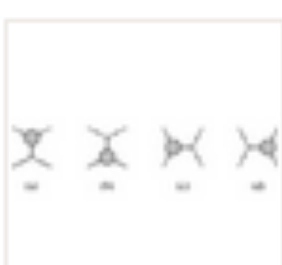
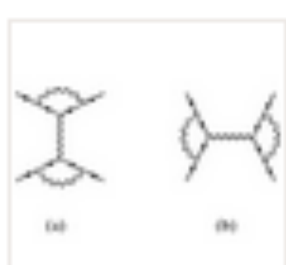
$$-\ln(2)\ln\left(-\frac{u}{s}\right)\left(\frac{40}{9}\frac{t^2}{s^3}+\frac{80}{9}\frac{t}{s^2}+\frac{40}{3s}+\frac{40}{9}\frac{s}{t^2}+\frac{80}{9t}\right)$$
$$-\ln(2)\left(\frac{40}{9}\frac{t^2}{s^3}+\frac{80}{9}\frac{t}{s^2}+\frac{40}{3s}+\frac{40}{9}\frac{s}{t^2}+\frac{80}{9t}\right)$$
$$+\frac{1007}{108}\frac{t^2}{s^3}+\frac{1007}{54}\frac{t}{s^2}+\frac{1007}{36s}+\frac{1007}{108}\frac{s}{t^2}+\frac{1007}{54t}\Big\}. \quad \text{(B.2)}$$

Two-loop Bhabha scattering in QED: Vertex and one-loop by one-loop contributions

R. Bonciani and A. Ferroglia
Phys. Rev. D **72**, 056004 – Published 22 September 2005

ABSTRACT

In the context of pure QED, we obtain analytic expressions for the contributions to the Bhabha scattering differential cross section at order α^4, which originate from the interference of two-loop photonic vertices with tree-level diagrams and from the interference of one-loop photonic diagrams amongst themselves. The ultraviolet renormalization is carried out. The IR-divergent soft-photon emission corrections are evaluated and added to the virtual cross section. The cross section obtained in this manner is valid for on-shell electrons and positrons of finite mass and for arbitrary values of the center of mass energy and momentum transfer. We provide the expansion of our results in powers of the electron mass, and we compare them with the corresponding expansion of the complete order α^4 photonic cross section, recently obtained by A.A. Penin [Phys. Rev. Lett. **95**, 010408 (2005).]. As a by-product, we obtain the contribution to the Bhabha scattering differential cross section of the interference of the two-loop photonic boxes with the tree-level diagrams, up to terms suppressed by positive powers of the electron mass. We evaluate numerically the various contributions to the cross section, paying particular attention to the comparison between exact and expanded results.



For a review:

The European Physical Journal C
April 2010, Volume 66, Issue 3–4, pp 585–686

Quest for precision in hadronic cross sections at low energy: Monte Carlo tools vs. experimental data

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Working Group on Radiative Corrections and Monte Carlo Generators for Low Energies, S. Actis, A. Arbuzov, G. Balossini, P. Beltrame, C. Bignamini, R. Bonciani, C. M. Carloni Calame, V. Cherepanov, M. Czakon, H. Czyż, A. Denig, S. Eidelman, G. V. Fedotovich, A. Ferroglia, show 42 more

Both experimental and theoretical improvements, tests, calculations, have to be worked out….. and eventually the mission will be ……


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