Muon-electron Scattering @NNLO in QED

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FCCP2017

Anacapri, September 7-9, 2017

In collaboration with:

- Passera, Peraro, Primo, Ossola, Schubert, Torres-Bobadilla

Physics and Astronomy Department
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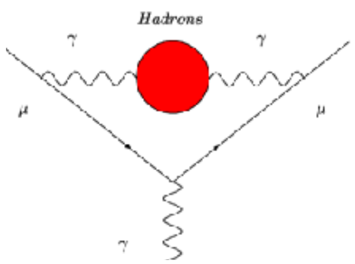
Outline

- Motivations
- Adaptive Integrand Decomposition
  - Improved reduction @ 1- and 2-loop
  - Automated two-loop corrections for generic processes
- Feynman Integrals in Dimensional Regularization
  - Integration-by-parts identities, Master Integrals & Differential Equations
  - Magnus Exponential Matrix and Canonical Forms
- Results
- Conclusions/Outlook

Motivations

New space-like proposal for HLO

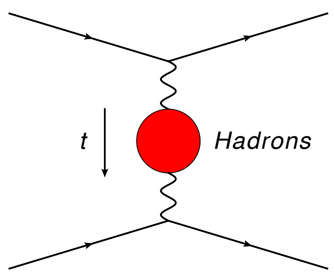
- At present, the leading hadronic contribution a_μ^{HLO} is computed via the time-like formula:



$$a_\mu^{\mathrm{HLO}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} ds\, K(s)\, \sigma^0_{\mathrm{had}}(s)$$

$$K(s) = \int_0^1 dx\, \frac{x^2\,(1-x)}{x^2 + (1-x)\,(s/m_\mu^2)}$$

- Alternatively, exchanging the x and s integrations in a_μ^{HLO}



$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx\,(1-x)\,\Delta\alpha_{\mathrm{had}}[t(x)]$$

$$t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$

which involves $\Delta\alpha_{had}(t)$, the hadronic contribution to the running of α in the space-like region. It can be extracted from scattering data!

Muon-electron scattering

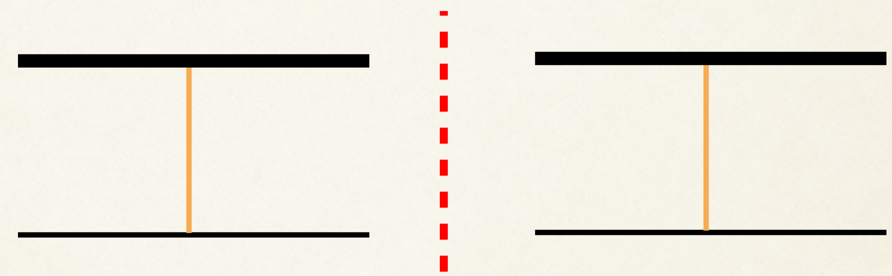
Abbiendi, Carloni Calame, Marconi, Matteuzzi, Montagna, Nicrosini, MP, Piccinini, Tenchini, Trentadue, Venanzoni
EPJC 2017 - arXiv:1609.08987

>> see Trentadue & Marconi's talk

What we *know* :: Anatomy of NLO

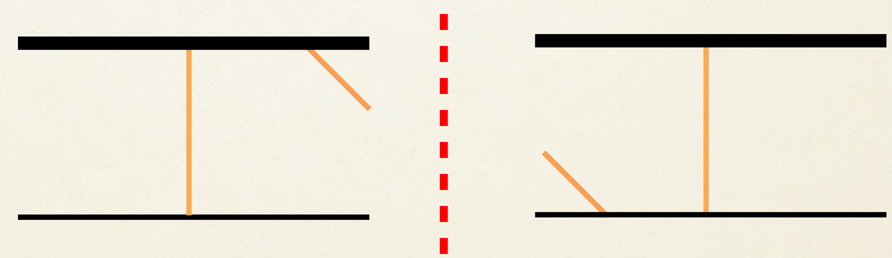
- **Born matrix-element**

  $d\sigma_{LO}$



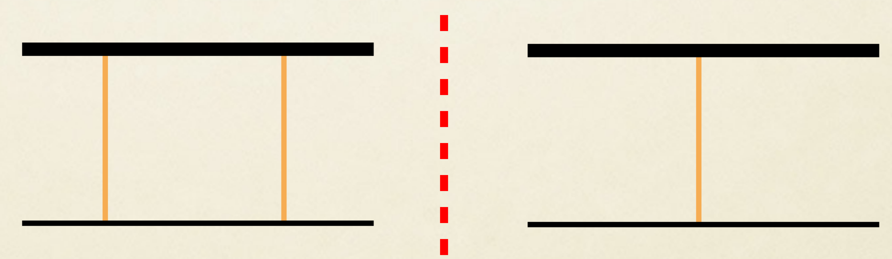
- **Real Radiation tree-level matrix elements**

  $d\sigma^{R}_{NLO}$



- **1-loop matrix elements**

  $d\sigma^{V}_{NLO}$



Known since long
vanNieuwenhuizen

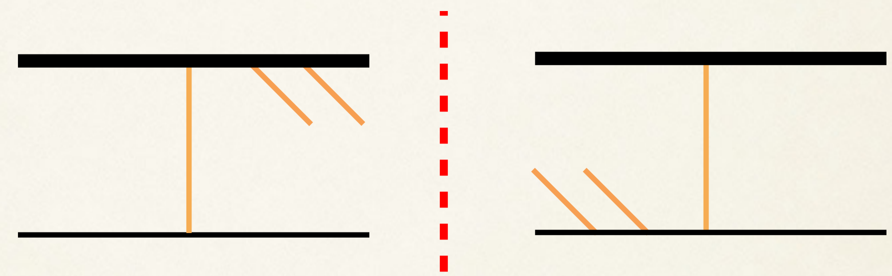
- double checked
  Fael Passera

$$\sigma_{NLO} \sim \int_{d\Phi_{m+1}} d\sigma^{R}_{NLO} + \int_{d\Phi_{m}} d\sigma^{V}_{NLO}$$

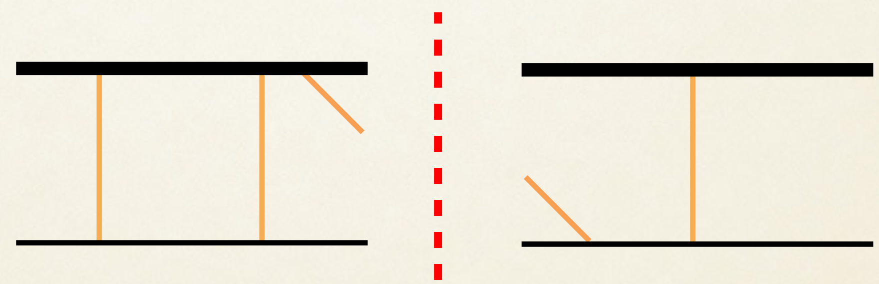
- **Subtractions and MC-integration**

What we *need* :: Anatomy of NNLO

- **Double-real Radiation**
  tree-level matrix elements
  $\mathrm{d}\hat{\sigma}^{RR}_{NNLO}$

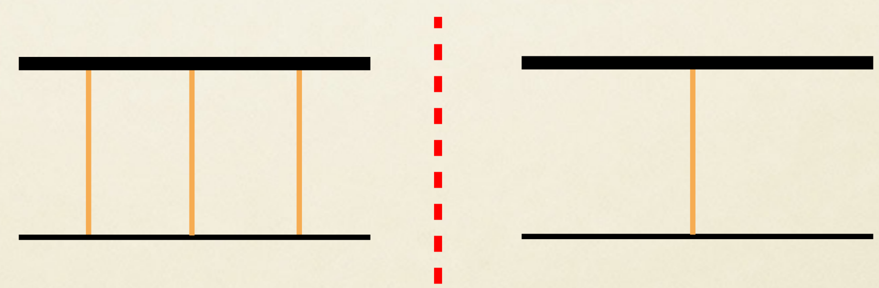


- **Single-real Radiation**
  1-loop matrix elements
  $\mathrm{d}\hat{\sigma}^{RV}_{NNLO}$



GOSAM
Recent improvements
for QED/EW corrections
Chiesa Greiner Tramontano

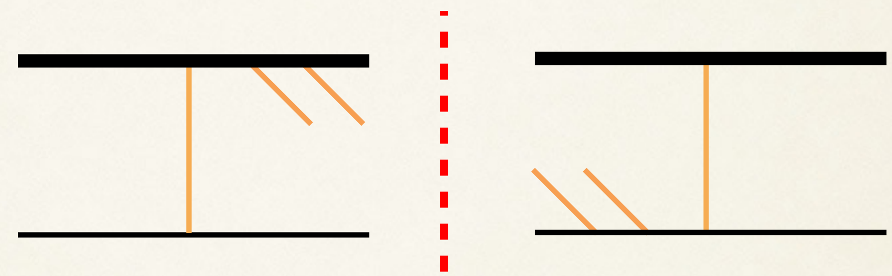
- **Virtual 2-loop matrix element**
  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$



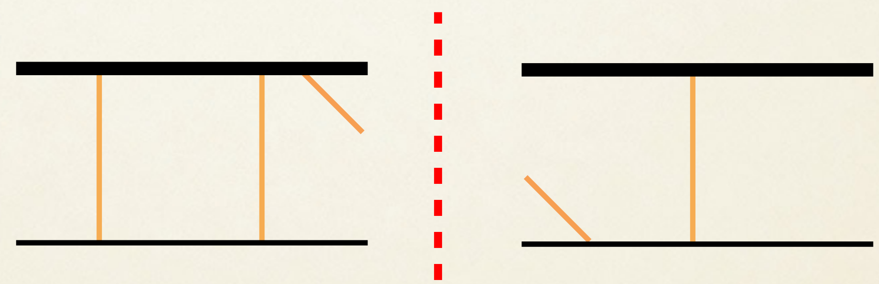
this talk!

What we *need* :: Anatomy of NNLO

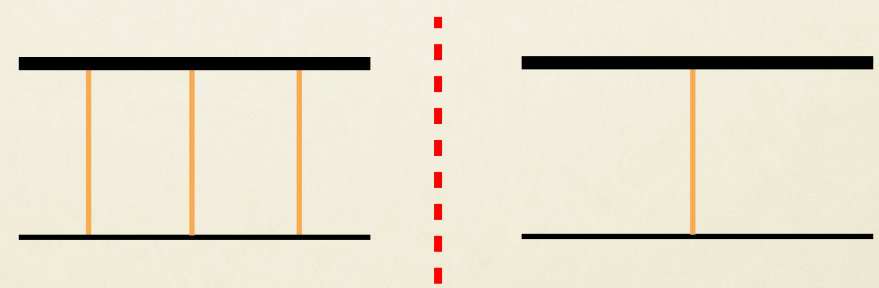
- **Double-real Radiation**
  tree-level matrix elements
  $\mathrm{d}\hat{\sigma}^{RR}_{NNLO}$



- **Single-real Radiation**
  1-loop matrix elements
  $\mathrm{d}\hat{\sigma}^{RV}_{NNLO}$



- **Virtual 2-loop matrix element**
  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$

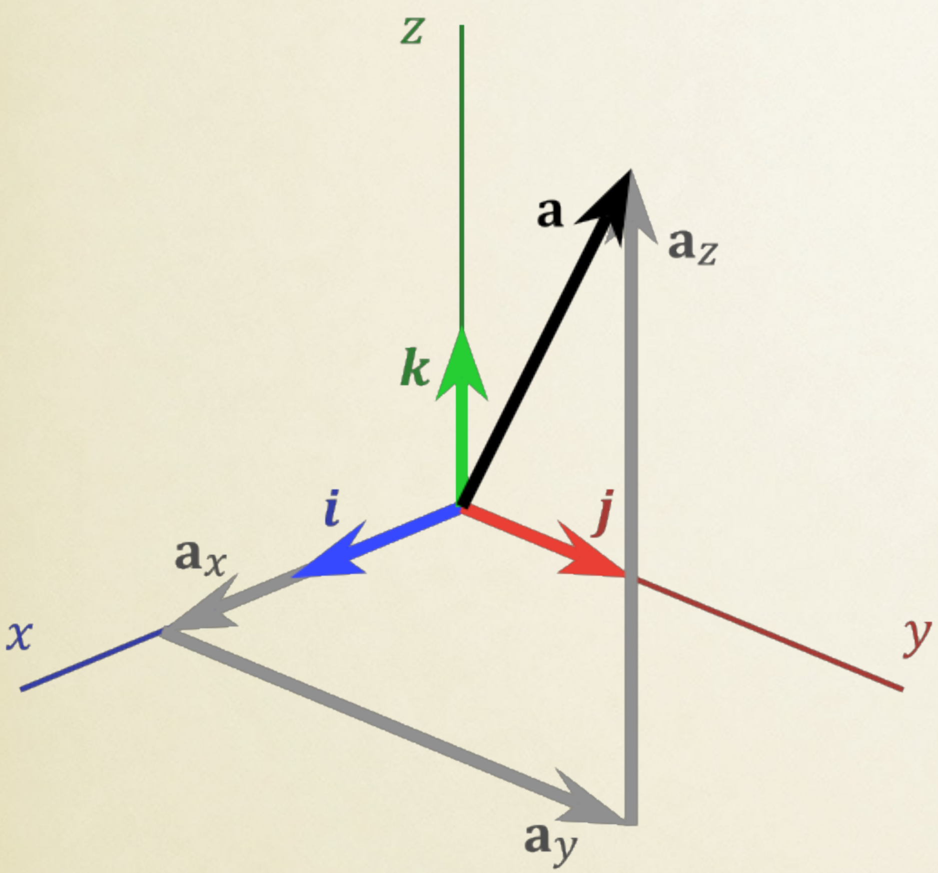


$$\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

- **Subtractions and MC-integration?**

Amplitudes Decomposition:

the algebraic way



$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

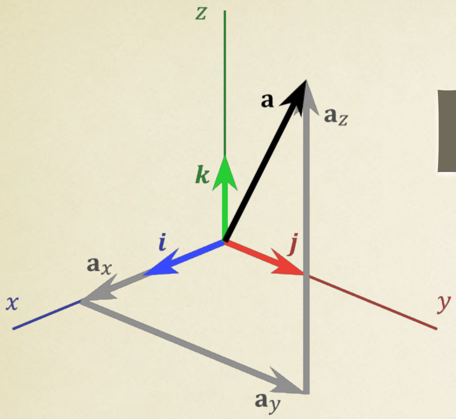
- Basis: {i j k}

- Scalar product/Projection:
  to extract the components

$a_x = \mathbf{a}.\mathbf{i}$

$a_y = \mathbf{a}.\mathbf{j}$

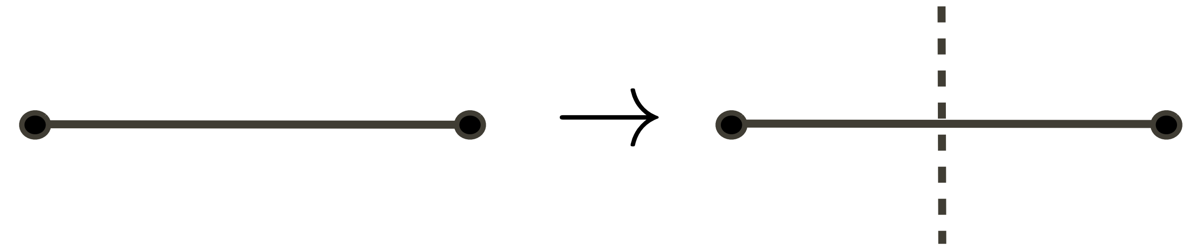
$a_z = \mathbf{a}.\mathbf{k}$



Projections :: On-Shell Cut-Conditions

vanishing denominators

$$\frac{1}{p^2 - m^2 - i0} \to \delta(p^2 - m^2)$$



Completeness Relations: cutting "1"

- **the richness of factorization**

$$i\,(\text{-}i) = 1$$

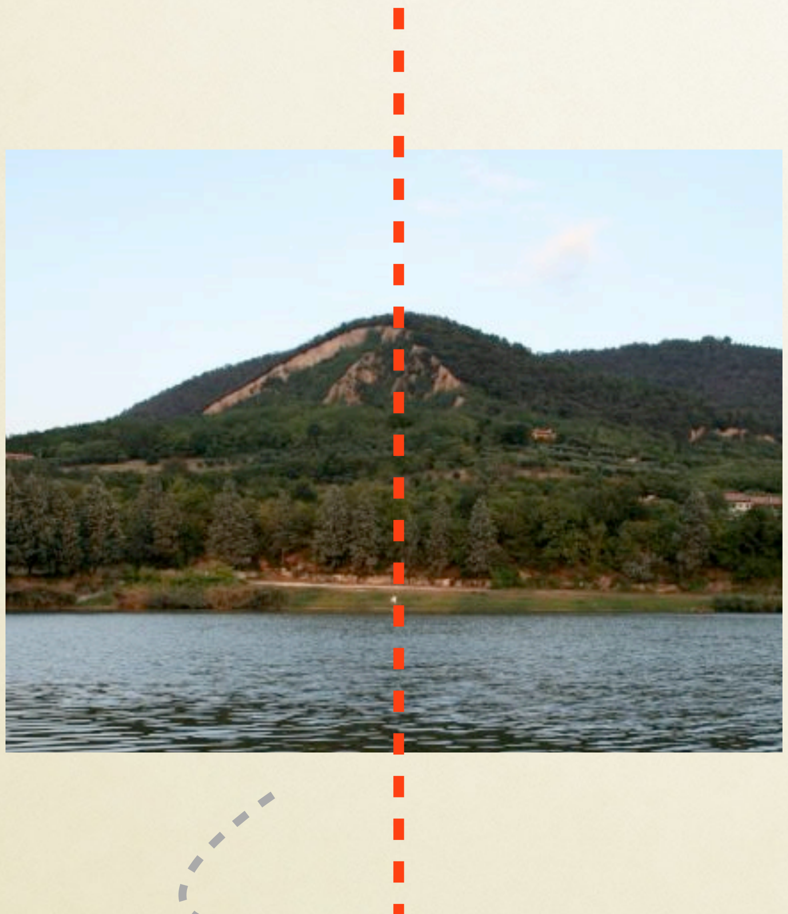
$$\sum_n |\,\psi_{\mathrm{n}}\,\rangle\,\langle\,\psi_{\mathrm{n}}|\;=\;\mathbb{1}$$

$$(p^2 - m^2) = (\not{p} - m)(\not{p} + m)$$

$$\varepsilon^{\mu\nu} = \varepsilon^{\mu}\varepsilon^{\nu}$$

Completeness Relations: cutting “1”

- the richness of factorization —> ideas for workshop organisation



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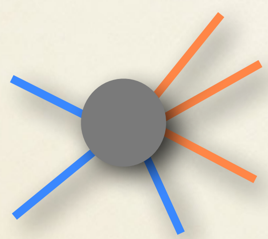
Parco dei Colli Euganei

Amplitudes Decomposition:

the algebraic way

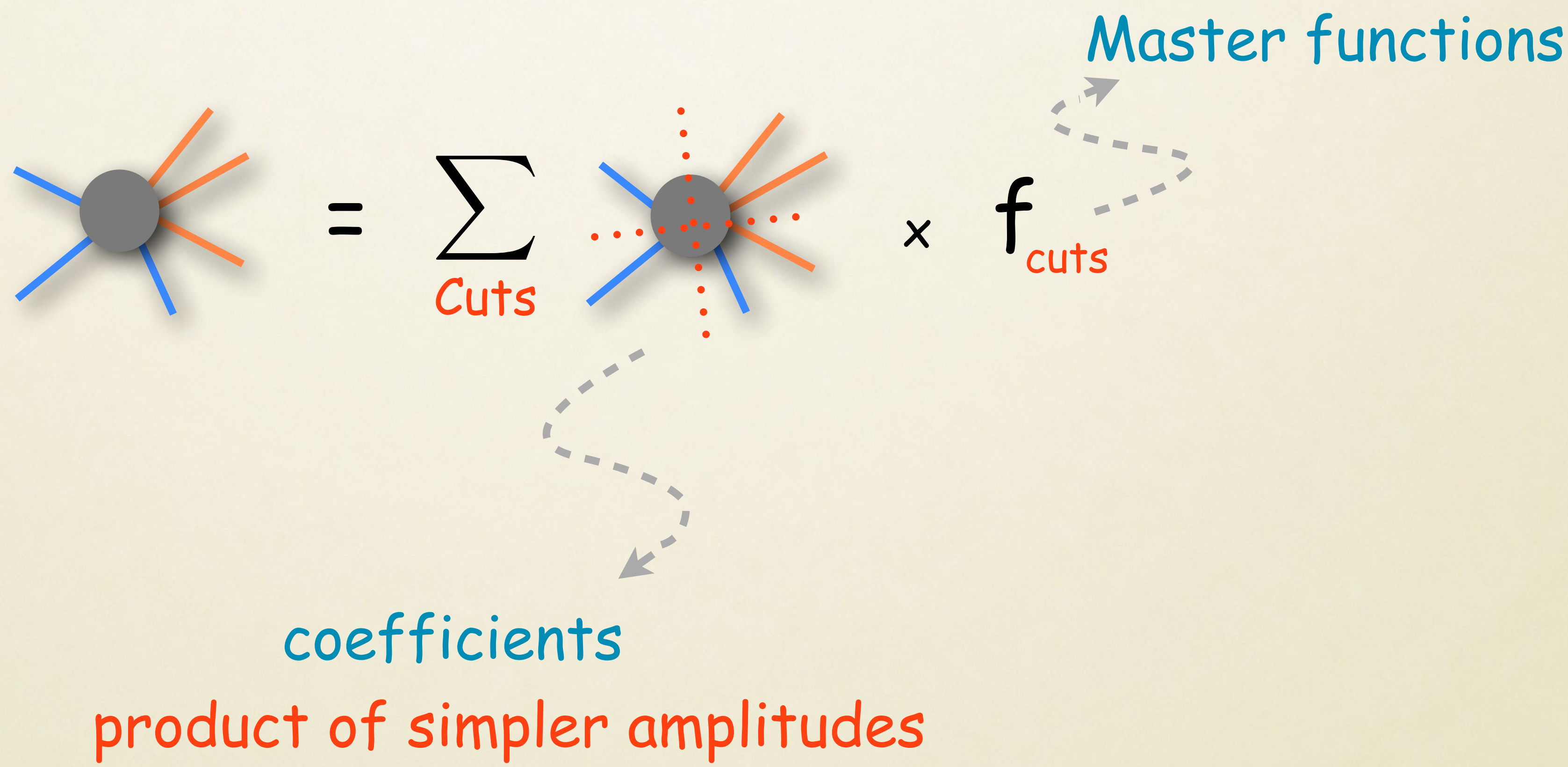
Master functions

 $$= \sum_i c_i \times f_i$$

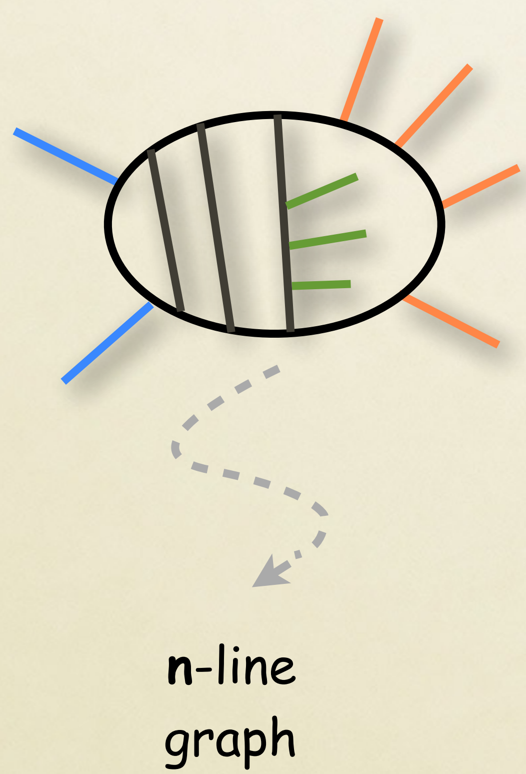
coefficients

Amplitudes Decomposition:

the algebraic way



Multi-loop Integrand Decomposition



$$= \iint \cdots \int \frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}}$$

Multi-loop Integrand Decomposition

Ossola & **P.M.** (2011);
Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

- **Polynomial Division**

$$\mathcal{N}_{i_1 \ldots i_n} = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1 \ldots i_{\kappa-1} i_{\kappa+1} \ldots i_n} \; D_{i_\kappa} + \Delta_{i_1 \ldots i_n}$$

$$\frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1 \ldots i_{\kappa-1} i_{\kappa+1} \ldots i_n} \; D_{i_\kappa}}{D_{i_1} \cdots D_{i_{\kappa-1}} D_{i_\kappa} D_{i_{\kappa+1}} \cdots D_{i_n}} + \frac{\Delta_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}}$$

Multi-loop Integrand Decomposition

Ossola & **P.M.** (2011);
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Polynomial Division

$$\mathcal{N}_{i_1\ldots i_n} = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\ldots i_{\kappa-1} i_{\kappa+1}\ldots i_n}\, D_{i_\kappa} + \Delta_{i_1\ldots i_n}$$

$$\frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1\ldots i_{\kappa-1} i_{\kappa+1}\ldots i_n}\, \cancel{D_{i_\kappa}}}{D_{i_1}\cdots D_{i_{\kappa-1}} \cancel{D_{i_\kappa}} D_{i_{\kappa+1}}\cdots D_{i_n}} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}}$$

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \,.$$

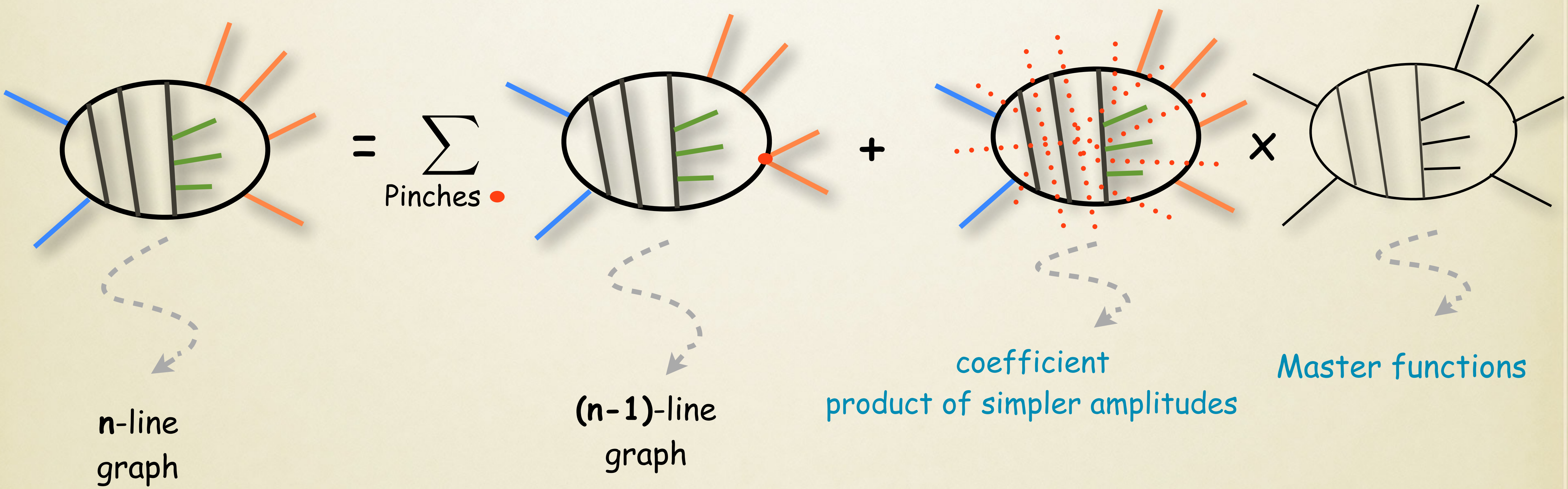
n-denominator integrand

(n-1)-denominator integrand

Multi-Loop Integrand Recurrence

Ossola & **P.M.** (2011);
Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

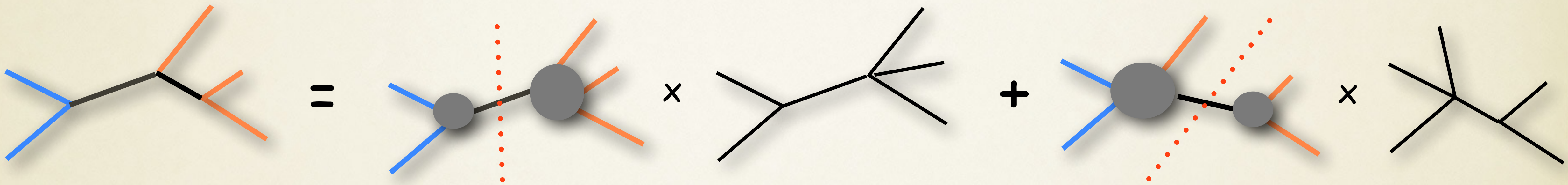
ℓ-Loop Recurrence Relation



- ☑ **all orders** (any number of loops and legs)
- ☑ **any topology** (planar and non-planar)
- ☑ **all kinematics** (massless and massive)
- ☑ **high-power of denominators**

Unique Methodology: Multiple-cuts as *Projectors*

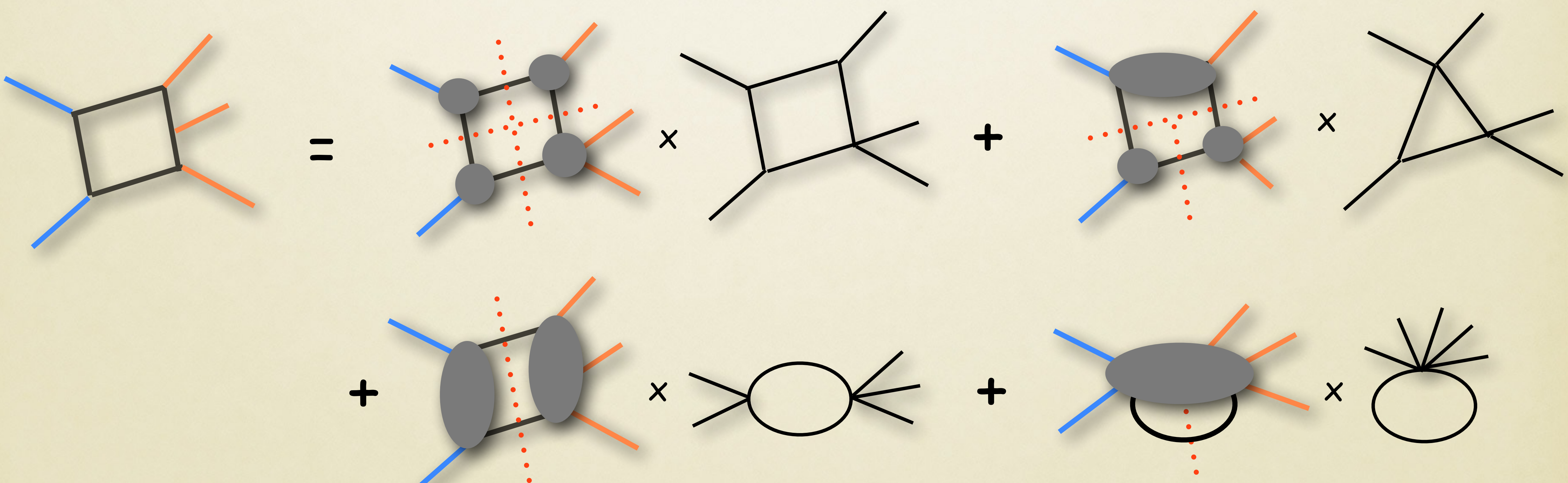
- **Tree-level** Britto Cachazo Feng Witten '04



- **One-Loop** (Revolution)

Bern Dixon Kosower 1997 Britto Cachazo Feng '04 Britto Feng **PM** '06-08

Ossola Papadopoulos Pittau '06 Ossola Papadopoulos Pittau **PM** '08 ...and many more...



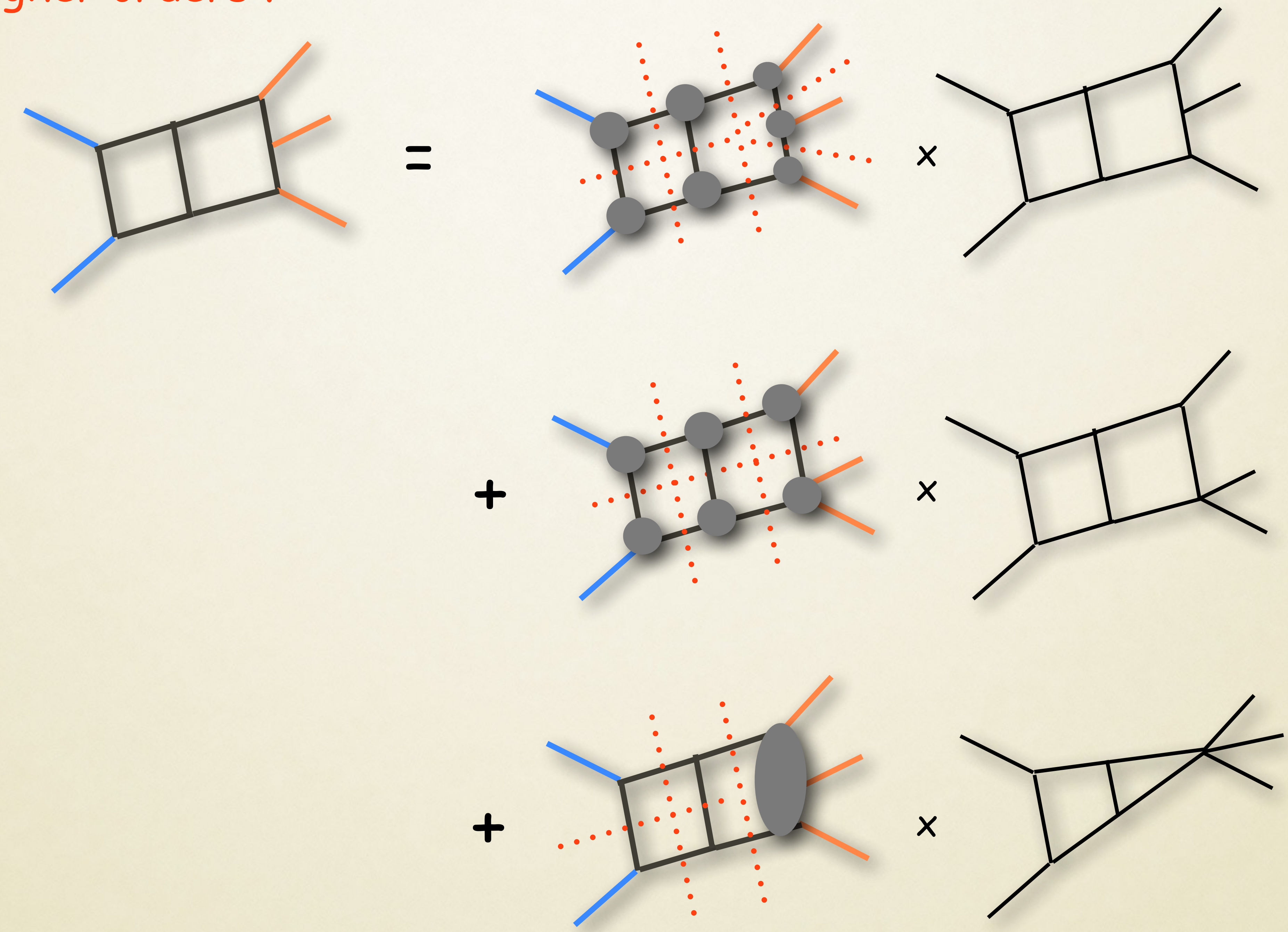
Unique Methodology: Multiple-cuts as *Projectors*

What about higher orders ?

- **Two-Loop**

  Evolution

  ...in progress...



+ ... ... all other cuts

Longitudinal and Transverse Space

Peraro Primo & ***P.M.*** (2016)

- **Dimensional Regularization**

$$d = 4 - 2\epsilon$$

- **if n-legs < 5**

$$d = d_{//} + d_{\perp}$$

Collins; Kreimer

Transverse Space

Longitudinal space
spanned by the
(independent) legs

☑ Denominators **do not depend on** "the angular variables" of the Transverse Space

☑ Numerators **depend on** "all" loop variables

$$\boxed{d = 4 - 2\epsilon}$$

$$q^\alpha = q^\alpha_{[4]} + \mu^\alpha, \qquad q^\alpha_{[4]} = \sum_{i=1}^{4} x_i e_i^\alpha, \qquad q^2 = q^2_{[4]} + \mu^2.$$

$$\mathcal{D}_i = \Big(q_{[4]} + \sum_{j=0}^{i} p_j\Big)^2 + \mu^2 + m_i^2,$$

$$\boxed{d = d_{//} + d_\perp}$$

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e_j^\alpha, \qquad q^2 = q^2_{[k]} + \lambda^2,$$

k-dimensional the space spanned by the external momenta

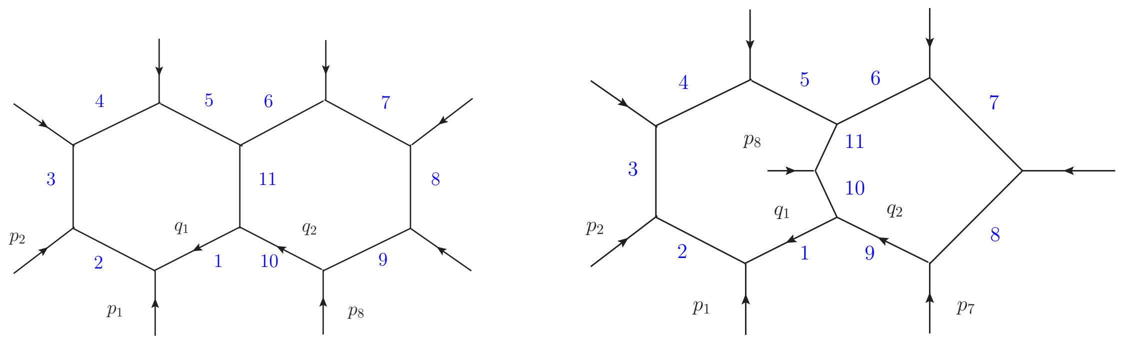
$$\lambda^\alpha = \sum_{j=k+1}^{4} x_j e_j^\alpha + \mu^\alpha, \qquad \lambda^2 = \sum_{j=k+1}^{4} x_j^2 + \mu^2,$$

$(d-k)$-dimensional orthogonal subspace.

$$\mathcal{D}_i = \Big(q_{[k]} + \sum_{j=0}^{i} p_j\Big)^2 + \lambda^2 + m_i^2.$$

Adaptive Unitarity @ 2-loop

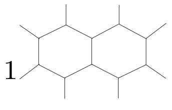
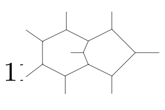
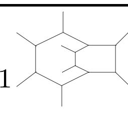
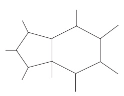
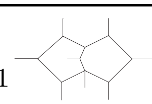
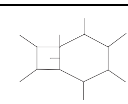
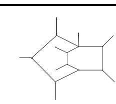
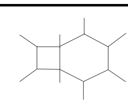
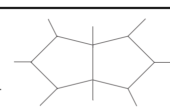
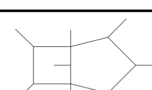
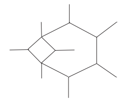
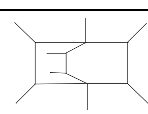
- **Novel Integrand red'n**

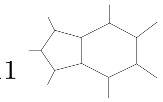
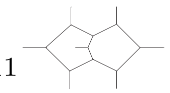
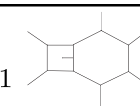
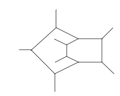
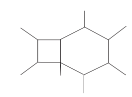
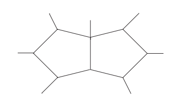
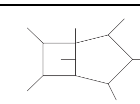
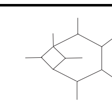
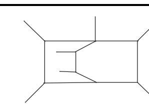
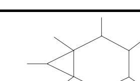
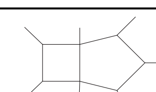
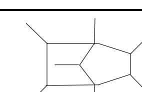
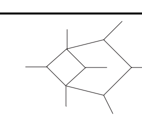


(a) $\mathcal{I}^{\mathrm{P}}_{12345678910\,11}$ (b) $\mathcal{I}^{\mathrm{NP1}}_{12345678910\,11}$ (c) $\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$

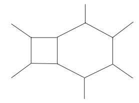
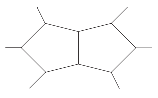
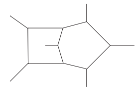
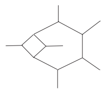
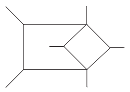
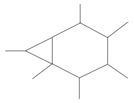
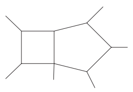
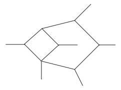
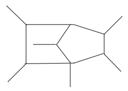
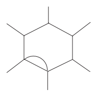
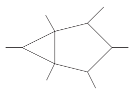
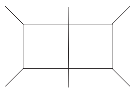
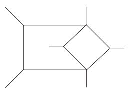
☑ Arbitrary (external and internal) kinematics!

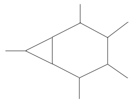
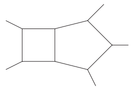
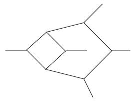
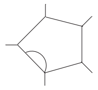
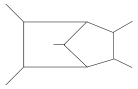
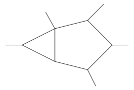
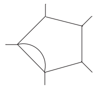
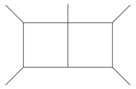
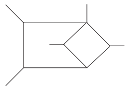
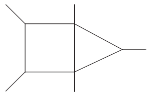
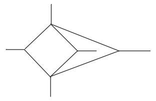
8 and 7 legs

$\mathcal{I}_{i_1\cdots i_n}$		$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{12345678910\,1}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345678910\,1}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{P}}_{2345678910\,11}$		6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2345678910\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234578910\,11}$		6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234678910\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{234678910\,11}$		15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{234578910\,11}$		33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{234578910\,11}$		39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$		15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{2346789\,10\,11}$		45 $\{1, x_{41}, x_{42}\}$

$\mathcal{I}_{i_1\cdots i_n}$		$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{12456789\,10\,11}$		6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12456789\,10\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1234568910\,11}$		6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{12456789\,10\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{245678910\,11}$		15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP2}}_{234567910\,11}$		33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{124568910\,11}$		39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456810\,11}$		15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{124678910\,11}$		45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2478910\,11}$		20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23478910\,11}$		76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24578910\,11}$		116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12457810\,11}$		80 $\{1, x_{31}, x_{41}, x_{42}\}$

6 and 5 legs

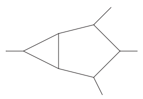
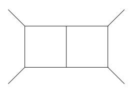
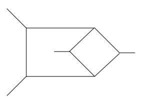
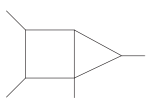
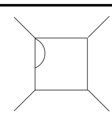
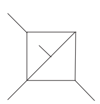
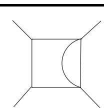
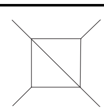
$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{135678910\,11}$ 	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{124567910\,11}$ 	62 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23568910\,11}$ 	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$ 	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{135678910\,11}$ 	45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{25678910\,11}$ 	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{23568910\,11}$ 	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{25678910\,11}$ 	80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24568910\,11}$ 	116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{3678910\,11}$ 	15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{2578910\,11}$ 	94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{2357910\,11}$ 	160 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2457910\,11}$ 	185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{15678910\,11}$ 	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{13567910\,11}$ 	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{15678910\,11}$ 	80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678910\,11}$ 	15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{13568910\,11}$ 	116 $\{1, x_{31}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1467910\,11}$ 	94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678911}$ 	66 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256910\,11}$ 	160 $\{1, x_{31}, x_{41}, y_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1357910\,11}$ 	185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256911}$ 	180 $\{1, x_{11}, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{246910\,11}$ 	246 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$

4 legs: *divide-integra-divide*

λ_{ij}

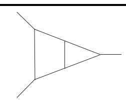
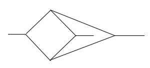
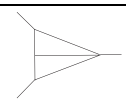
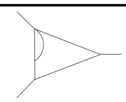
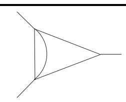
reducible

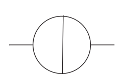
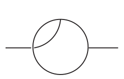
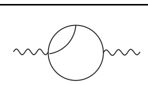
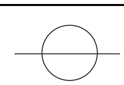
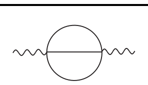
$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\rm P}_{1567910\,11}$ 	94 $\{1,x_{21},x_{31},x_{41},x_{42}\}$	53 $\{1,x_{21},x_{31},\lambda_{11},\lambda_{22},\lambda_{12}\}$	10 $\{1,x_{21},x_{31}\}$
$\mathcal{I}^{\rm P}_{12256910\,11}$ 	160 $\{1,x_{31},x_{41},x_{32},x_{42}\}$	93 $\{1,x_{31},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	22 $\{1,x_{31},x_{32}\}$
$\mathcal{I}^{\rm NP1}_{1356910\,11}$ 	184 $\{1,x_{31},x_{42},x_{32},x_{42}\}$	105 $\{1,x_{31},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	25 $\{1,x_{31},x_{32}\}$
$\mathcal{I}^{\rm P}_{1356811}$ 	180 $\{1,x_{31},x_{41},x_{22},x_{32},x_{42}\}$	101 $\{1,x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	39 $\{1,x_{31},x_{22},y_{32}\}$
$\mathcal{I}^{\rm P}_{168910\,11}$ 	66 $\{1,x_{11},x_{21},x_{31},x_{41},x_{42}\}$	35 $\{1,x_{11},x_{21},x_{31},\lambda_{11},\lambda_{22},\lambda_{12}\}$	10 $\{1,x_{11},x_{21},x_{31}\}$
$\mathcal{I}^{\rm NP1}_{246910\,11}$ 	245 $\{1,x_{31},x_{41},x_{21},x_{32},x_{42}\}$	137 $\{1,x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	55 $\{1,x_{31},x_{22},y_{32}\}$
$\mathcal{I}^{\rm P}_{36810\,11}$ 	115 $\{1,x_{31},x_{41},x_{12},x_{22},x_{32},x_{42}\}$	66 $\{1,x_{31},x_{12},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	35 $\{1,x_{31},x_{12},x_{22},x_{32}\}$
$\mathcal{I}^{\rm P}_{136811}$ 	180 $\{1,x_{11},x_{31},x_{41},x_{22},x_{32},x_{42}\}$	103 $\{1,x_{11},x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	60 $\{1,x_{11},x_{31},x_{22},x_{32}\}$

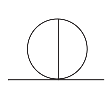
3, 2 , 1 legs: *divide-integra-divide*

λ_{ij}

reducible

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_k}$	$\Delta'_{i_1\cdots i_k}$
$\mathcal{I}^{\rm P}_{1356911}$ 	180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	22 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	4 $\{1, x_{22}\}$
$\mathcal{I}^{\rm NP1}_{156910\,11}$ 	240 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	30 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{22}\}$
$\mathcal{I}^{\rm P}_{15710\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	33 $\{1, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	13 $\{1, x_{21}, x_{12}\}$
$\mathcal{I}^{\rm P}_{16910\,11}$ 	115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	20 $\{1, x_{11}, x_{22}\lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{12}, x_{22}\}$
$\mathcal{I}^{\rm P}_{3610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{x_{11}, x_{21}, x_{22}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\rm P}_{15610\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	8 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{1610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{4}, x_{22}, y_{3}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{1310\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{1, x_{11}, x_{21}, x_{12}\}$
$\mathcal{I}^{\rm P}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	9 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\rm P}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	18 $\{1, x_{11}, x_{21}, x_{12}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	15 $\{1, x_{11}, x_{22}, x_{21}, x_{22}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\rm P}_{110\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$

Adaptive Integrand Decomposition

Peraro Primo & ***P.M.*** (2016)

- Integrand reduction beyond multivariate polynomial division

$$d = d_{//} + d_{\perp}$$

idea n.1 **Integrating over Transverse Space**

idea n.2 **Cutting in the Longitudinal Space**

1&2-loop Automation :: AIDA

Peraro Primo TorresBobadilla & ***P.M.*** (w.i.p.)

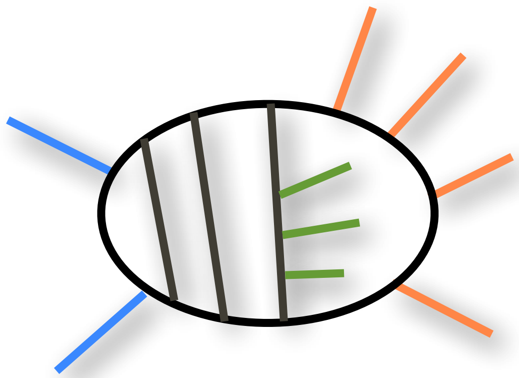
- MATHEMATICA implementation of AID reduction
  - Analytic 1- and 2-loop decomposition

- **Application to Mu-e scattering**

Ossola Peraro Primo TorresBobadilla Schubert & ***P.M.*** (w.i.p.)

Dimensionally Regulated Integrals

Graph Topology & Integrals



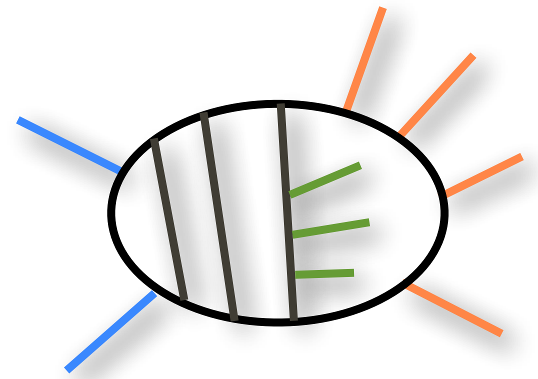
$e = \#$ legs :: p_i, $\quad (i = 1, \ldots, e)$;
$\ell = \#$ loops :: q_i $\quad (i = 1, \ldots, \ell)$;
$n = \#$ denominators :: D_i $\quad (i = 1, \ldots, n)$;

$N = \#$ scalar products (of types $q_i \cdot p_j$ and $q_i \cdot q_j$) $\qquad N = \ell(e-1) + \frac{\ell(\ell+1)}{2}$

$n = \#$ **reducible** scalar products (expressed in terms of denominators);

$m = \#$ **irreducible** scalar products $= N - n$ :: S_i $\quad (i = 1, \ldots, m)$

Graph Topology & Integrals



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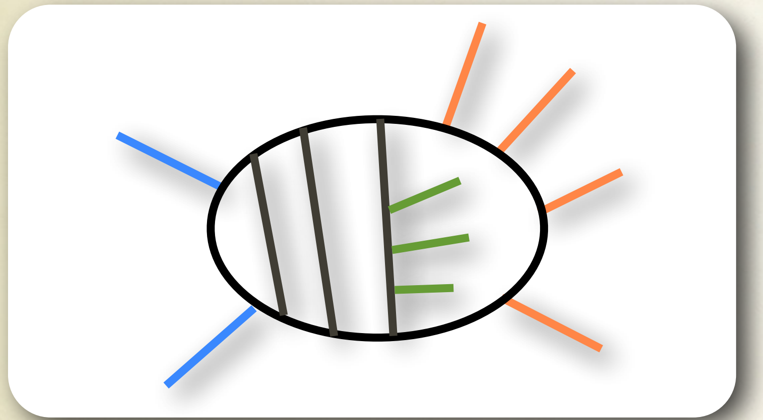
Graph Topology & Integrals



$e = \#\ \text{legs} :: p_i, \quad (i = 1, \ldots, e);$

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$n = \#$ **reducible** scalar products (expressed in terms of denominators);

$m = \#$ **irreducible** scalar products $= N - n :: S_i \quad (i = 1, \ldots, m)$

Associated Integrals ::

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \ , \qquad \int_{q_1 \ldots q_\ell} \equiv \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d q_\ell}{(2\pi)^d}$$

$$f_{n,m}(\mathbf{x}, \mathbf{y}) = \frac{S_1^{y_1} \cdots S_m^{y_m}}{D_1^{x_1} \cdots D_n^{x_n}}$$

Integration-by-parts Identities (IBPs)

Tkachov; Chetyrkin Tkachov; Laporta;

$$\int_{q_1 \ldots q_\ell} \frac{\partial}{\partial q_i^\mu} \Big(v^\mu \; f_{n,m}(\mathbf{x}, \mathbf{y}) \Big) = 0 \; , \qquad v = q_1, \ldots, q_\ell, \; p_1, \ldots, p_{e-1}.$$

$\forall (n,m)$, $N_{\mathrm{IBP}} = \#$ of IBP relations $= \ell(\ell + e - 1)$

Relations between integrals associated to the same topology (or subtopologies)

$$c_0 \; F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) + \sum_{i,j} c_{i,j} \; F^{[d]}_{n,m}(\mathbf{x_i}, \mathbf{y_j}) = 0 \; ,$$

$$\mathbf{x_i} = \{x_1, \ldots, x_i \pm 1, \ldots, x_n\}$$

$$\mathbf{y_j} = \{y_1, \ldots, y_j \pm 1, \ldots y_n\}$$

public codes :: AIR; Reduze2; FIRE; LiteRed;
private codes :: ... many authors ... Laporta, Sturm ...

Master Integrals (MIs)

Independent set of integrals $M_i^{[d]}$,

$$M_i^{[d]} \equiv \int_{q_1 \ldots q_\ell} m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \ ,$$

with a definite set of powers $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ such that

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_k c_k M_k^{[d]} \ , \qquad \forall (n,m)$$

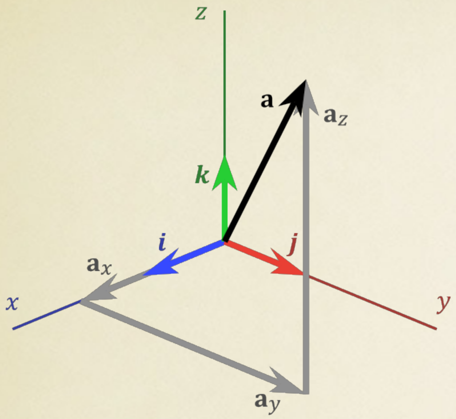
They form a *basis* for the integrals of the corresponding topology.

Two special cases

Two types of integrals generated from the **master integrands**

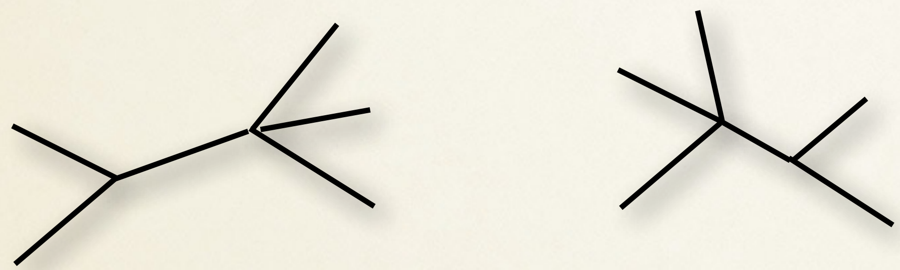
- Polynomial insertion: $$\int_{q_1 \ldots q_\ell} P(q_i \cdot p_j, q_i \cdot q_j) \ m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \alpha_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$

- External-leg derivatives: $$p_i^\mu \frac{\partial}{\partial p_j^\mu} M_k^{[d]} = \int_{q_1 \ldots q_\ell} p_i^\mu \frac{\partial}{\partial p_j^\mu} \ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \beta_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$



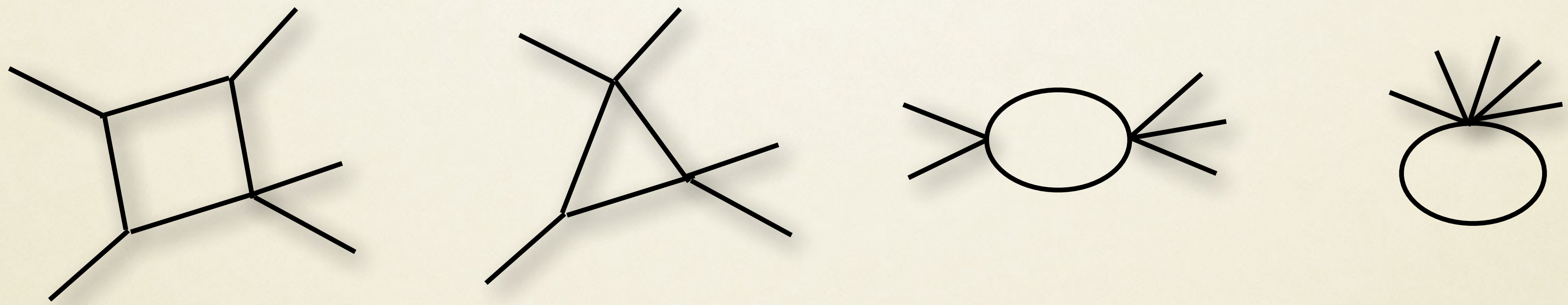
Basis :: Master Functions

- **Tree level**



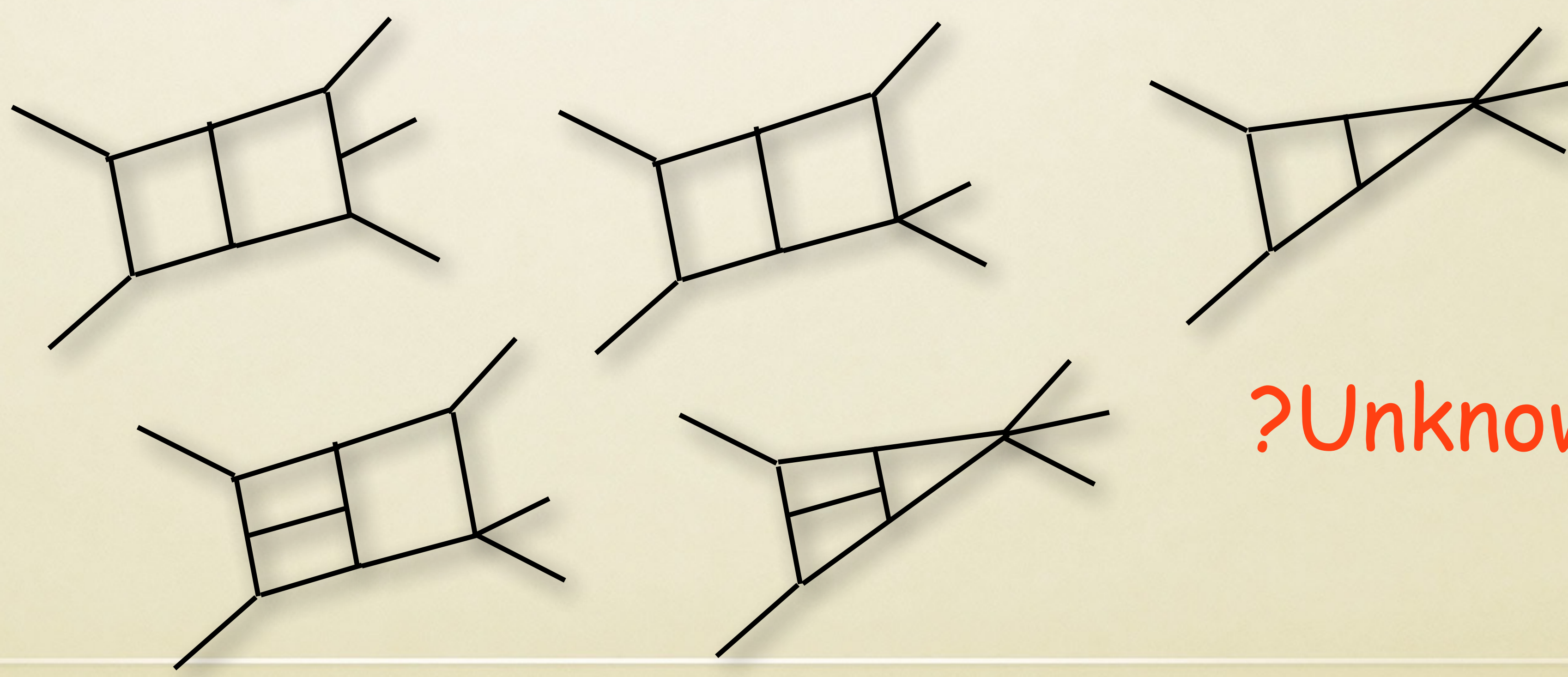
Known!

- **One Loop**



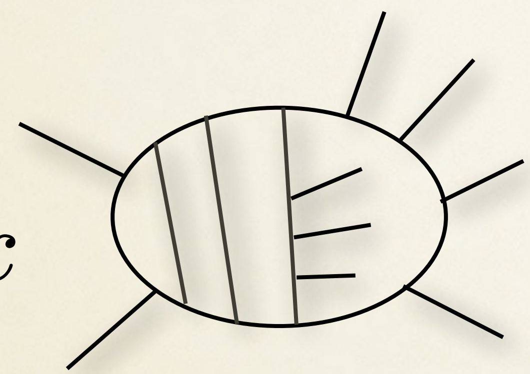
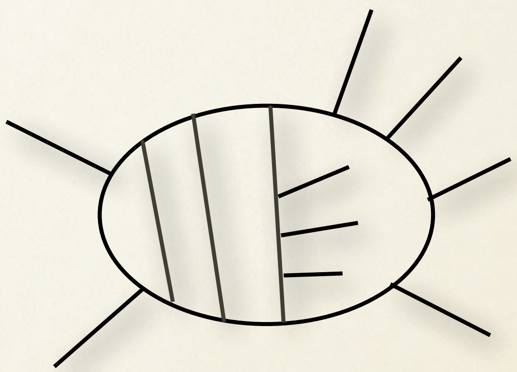
Known!

- **Higher Loops**



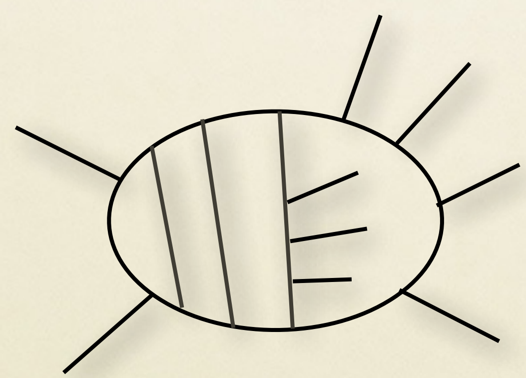
?Unknown?

Differential Equations for Master Integrals

∂_x  $= A(d, x)$ 

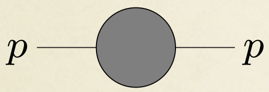
kinematic variable
(s,t,u, masses)

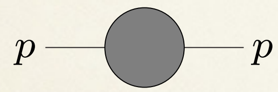
space-time
dimensions

 = ?

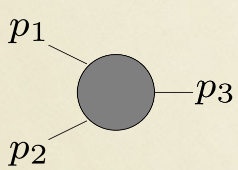
Differential Equations for Master Integrals

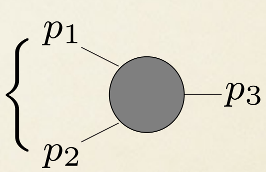
Kotikov; Remiddi;
Gehrmann Remiddi
Argeri Bonciani Ferroglia Remiddi **P.M.**
……
Henn; Henn Smirnov
Lee; Papadopoulos;
Argeri diVita Mirabella Schlenk Schubert Tancredi **P.M.**
diVita Schubert Yundin **P.M.**
Remiddi Tancredi; Primo Tancredi
Papadopoulos Frellesvig
Zheng
……

$p^2 \frac{\partial}{\partial p^2} \Big\{$ 

$\Big\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \Big\{$ 

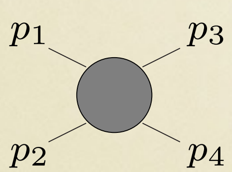
$\Big\}$

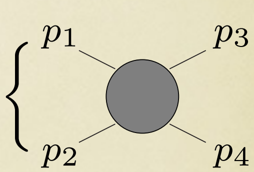
$P^2 \frac{\partial}{\partial P^2} \Big\{$ 

$\Big\} = \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \Big\{$ 

$\Big\}$

$$P = p_1 + p_2,$$

$P^2 \frac{\partial}{\partial P^2} \Big\{$ 

$\Big\} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \Big\{$ 

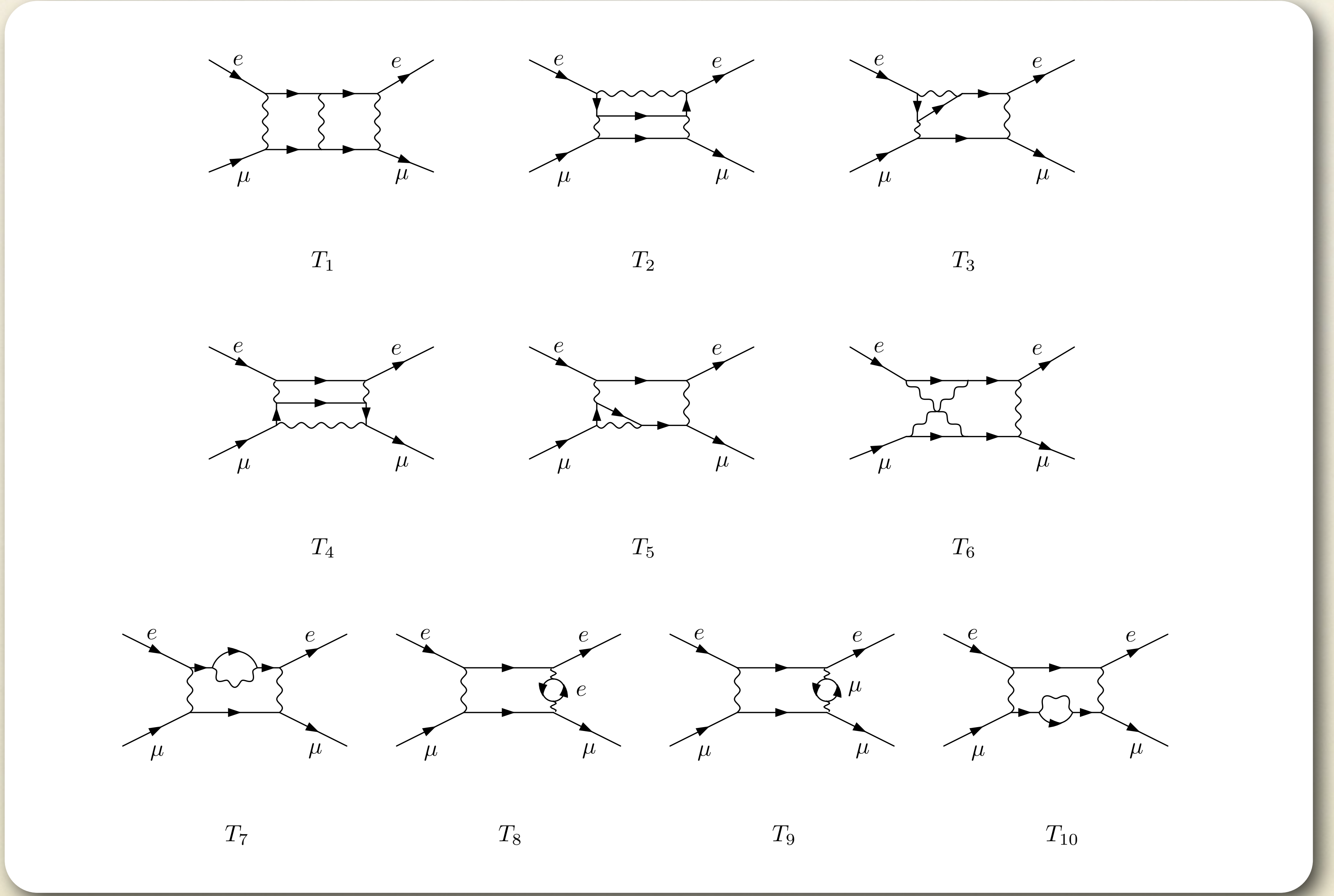
$\Big\}$

In general, n MIs obey a **system of 1st ODE**

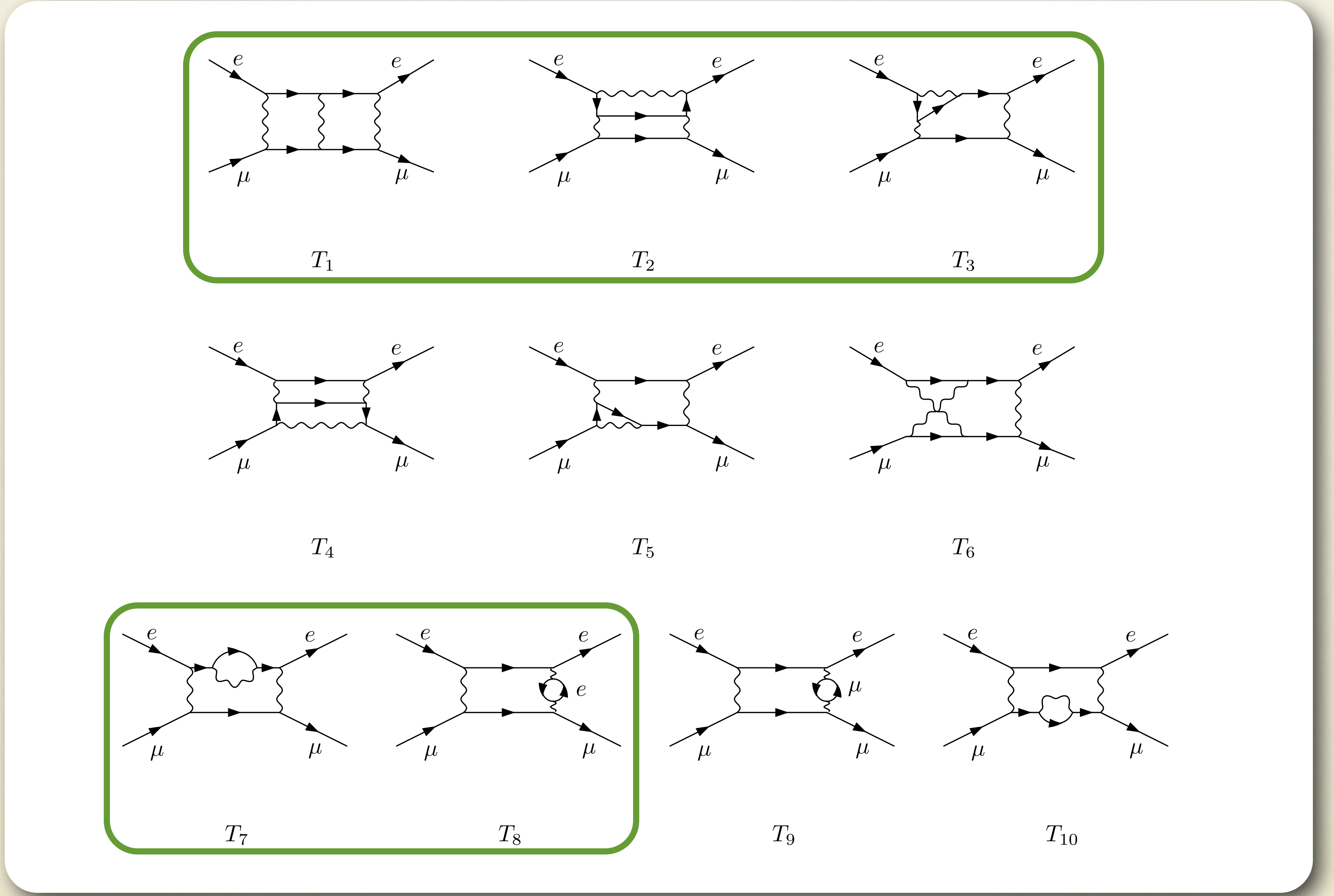
$$\partial_z \mathbf{M}^{[d]} = \mathbb{A}(d, z)\ \mathbf{M}^{[d]}$$

Two-Loop Integrals for Mu-E Scattering

Feynman diagrams @ 2-loop

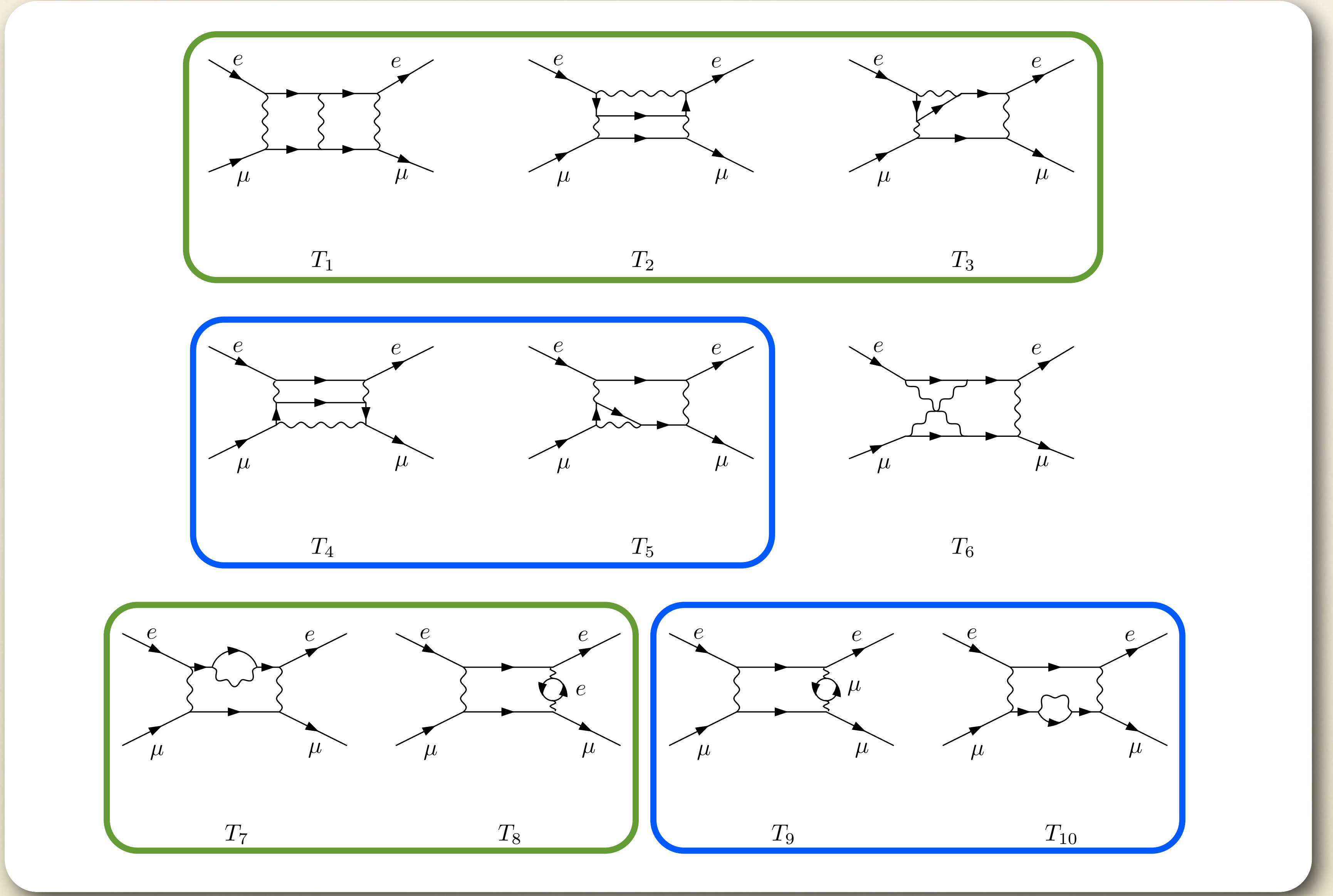


Feynman diagrams @ 2-loop



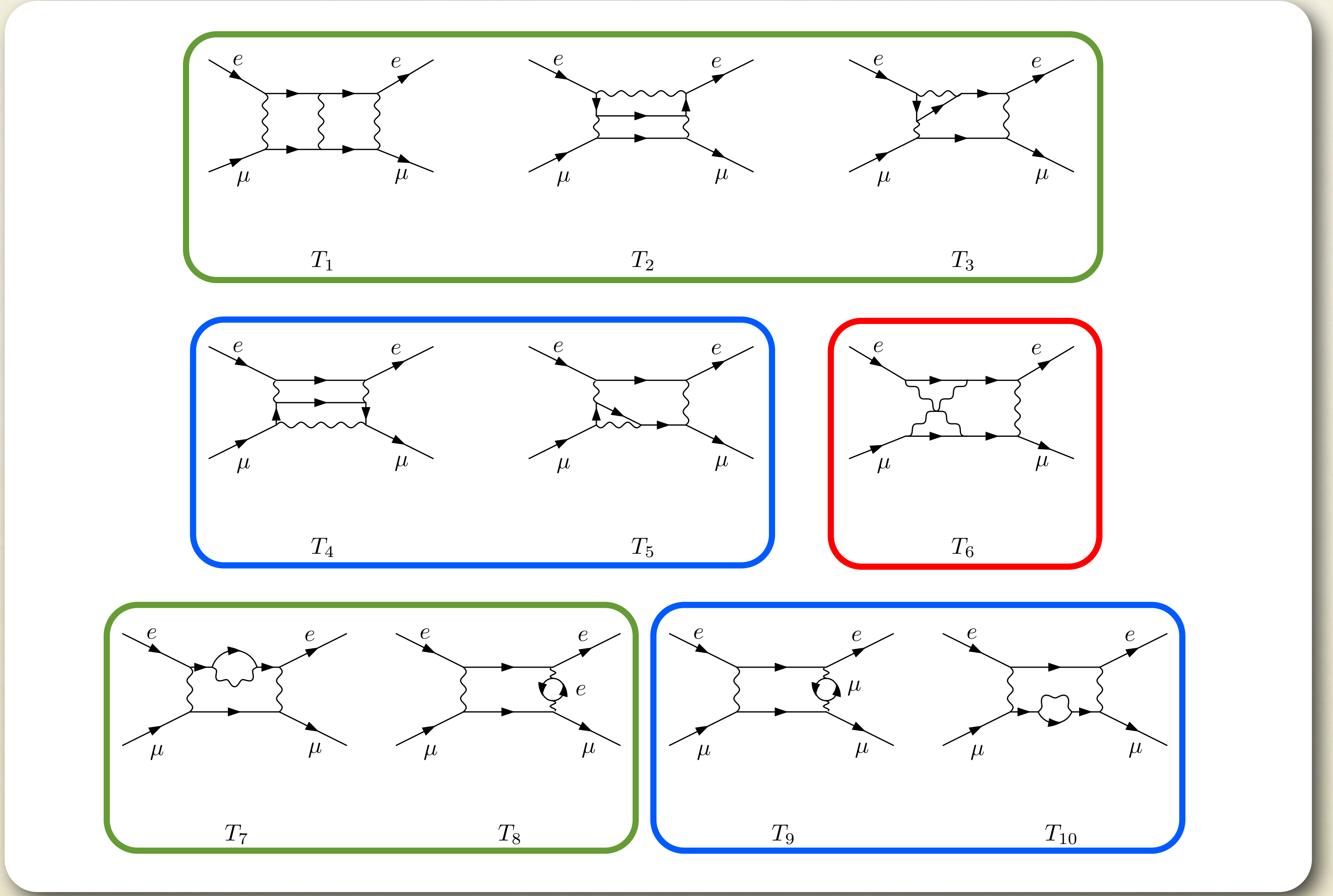
- **Planar Integrals :: Family-1**

Feynman diagrams @ 2-loop



- Planar Integrals :: Family-1
- Planar Integrals :: Family-2

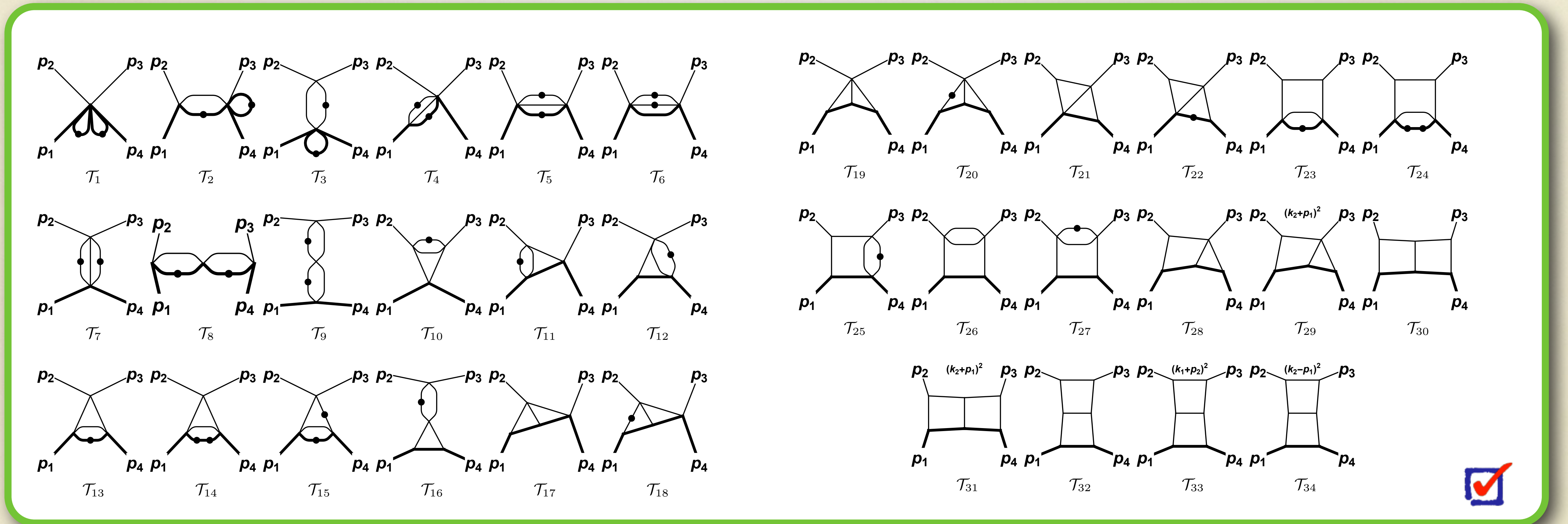
Feynman diagrams @ 2-loop



- Planar Integrals :: Family-1
- Planar Integrals :: Family-2
- Non-Planar Integrals

- massless ***electron***
- **massive *muon***

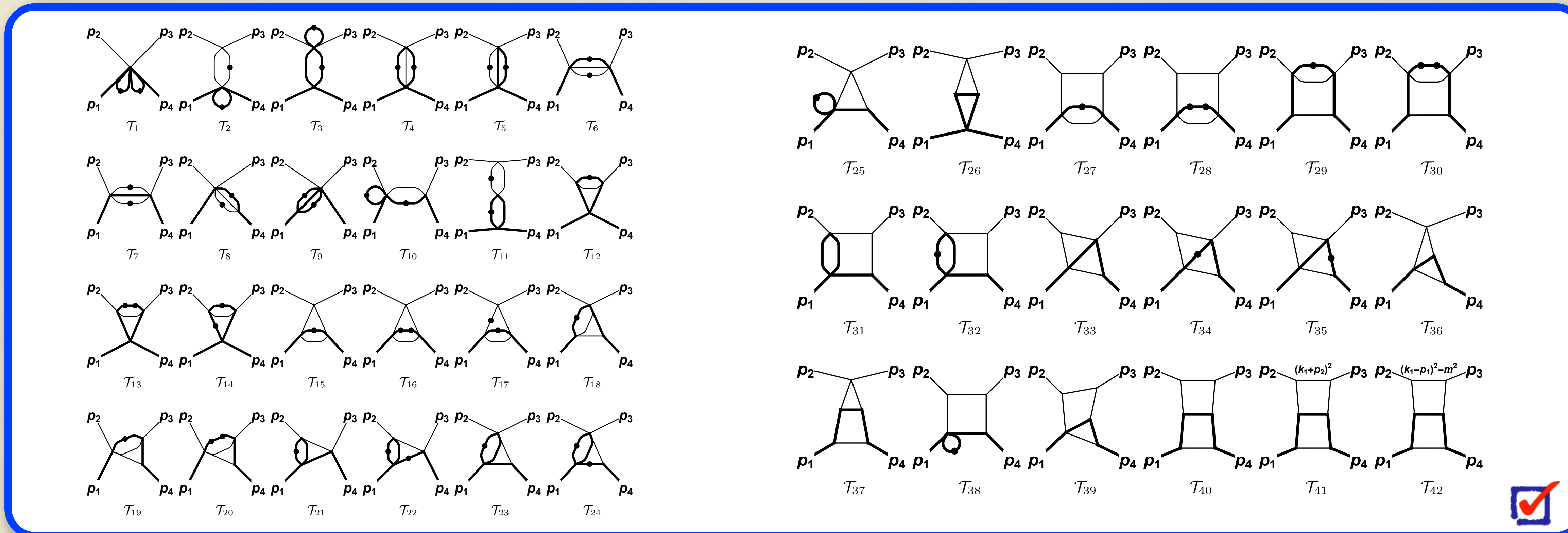
MIs for Family-1 Passera Primo Schubert & **P.M.** (coming soon)



- **A different set of MIs already known** Bonciani Ferroglia Gehrmann vonManteuffel
- **We recomputed them with an alternative/simpler way**

- **31 MIs**
- **alphabet:** 8 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **SecDec**

MIs for Family-2 Passera Primo Schubert & **P.M.** (coming soon)

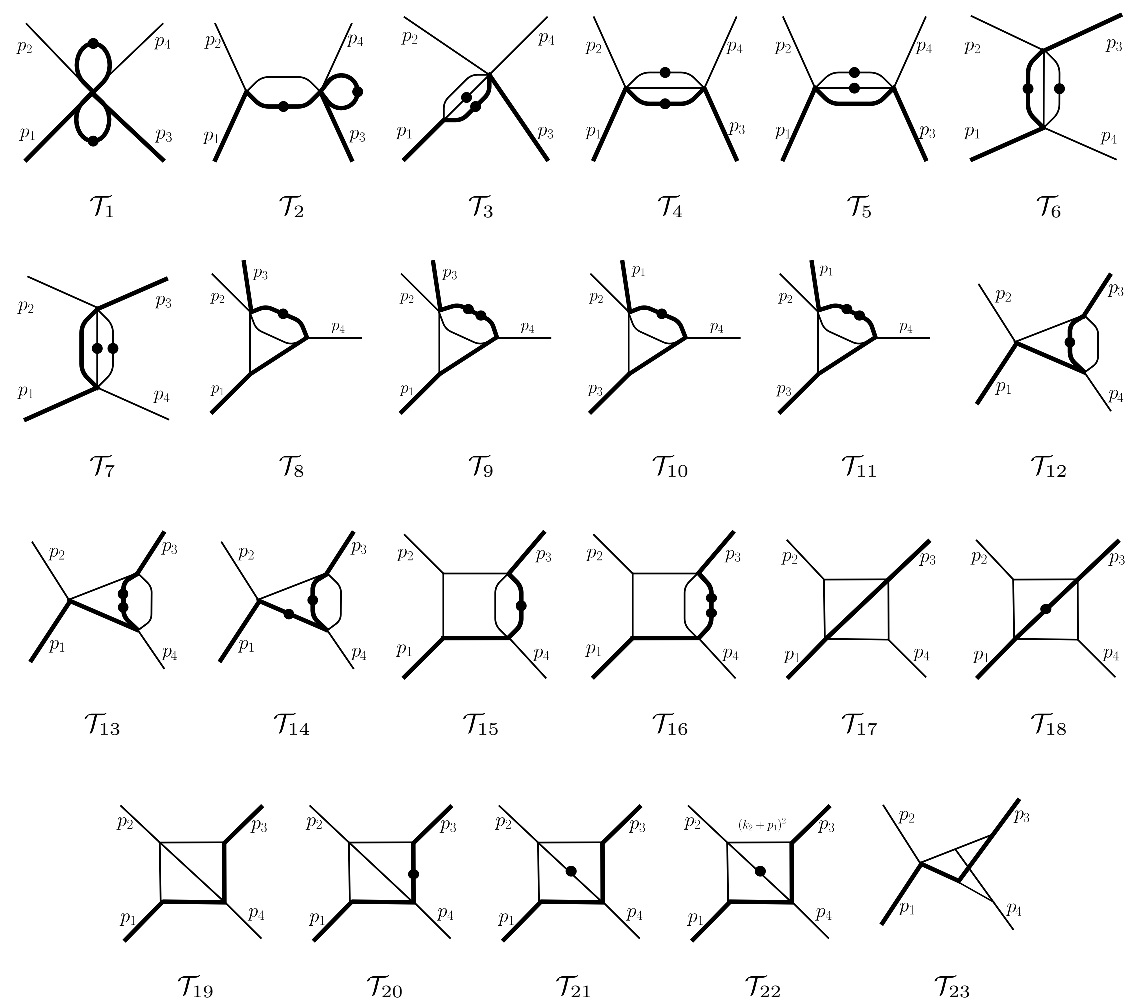


NEW

- **42 MIs**
- **alphabet:** 9 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **SecDec**

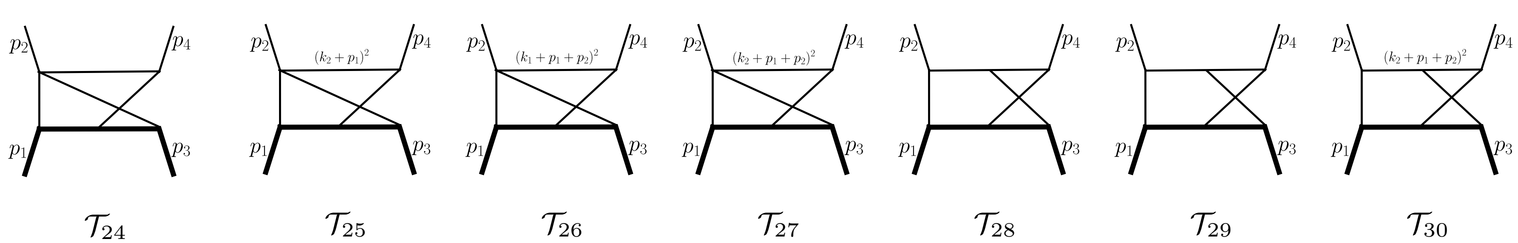
30 MIs for non-planar diagrams

Primo Schubert & **P.M.** (w.i.p.)



NEW

The last missing pieces



Quantum Mechanics

- **Schroedinger Eq'n (*eps-linear Hamiltonian*)**

$$i\hbar\, \partial_t |\Psi(t)\rangle = H(\epsilon, t)|\Psi(t)\rangle \ , \qquad H(\epsilon, t) = H_0(t) + \epsilon H_1(t)$$

- **Interaction Picture**

$$H_{i,I}(t) = B^\dagger(t)\ H_i(t)\ B(t)$$

- **Matrix Transform**

$$i\hbar\,\partial_t B(t) = H_0(t)B(t) \qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

- **Schroedinger Eq'n (*canonical form*)**

$$i\hbar\ \partial_t|\Psi_I(t)\rangle = \epsilon\, H_{1,I}(t)|\Psi_I(t)\rangle \,,$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

A(x) non-commutative

Magnus' series :: *Matrix Exponential*

$$Y(x) = e^{\Omega(x,x_0)} \, Y(x_0) \equiv e^{\Omega(x)} \, Y_0 \, ,$$

BCH-formula

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$

$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \ [A(\tau_1), A(\tau_2)] \ ,$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .$$

....

Iterated Integrals
(generalized polylogs)

Dyson's series

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x) \ ,$$

$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \ A(\tau_1) A(\tau_2) \cdots A(\tau_n)$$

Chen
Goncharov
Remiddi Vermaseren
Gehrmann Remiddi
Bonciani Remiddi **P.M.**
Vollinga Weinzierl
Brown
Duhr Gangl Rhodes
....

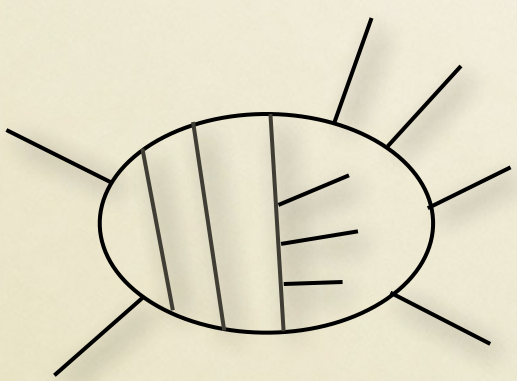
Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

Quantum Mechanics

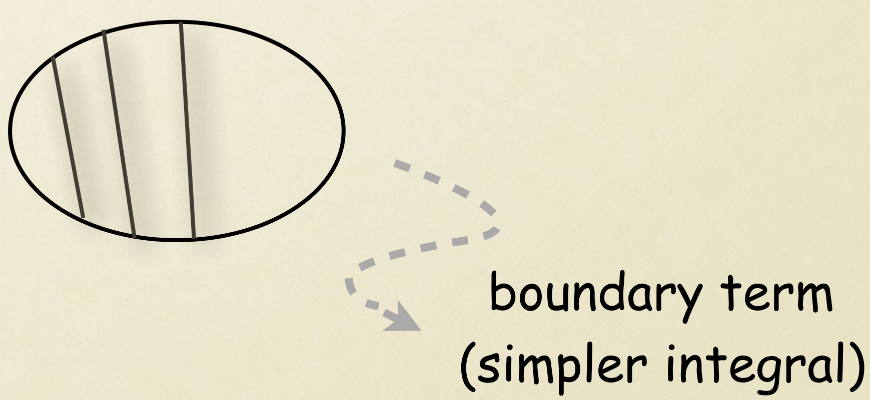
- Time-evolution in Perturbation Theory
- perturbation parameter: ε
- **Linear** Hamiltonian in ε
- **Unitary transform**
- Schroedinger Equation
  in the interaction picture (ε-factorization)
- solution: Dyson series

Feynman Integrals

- Kinematic-evolution in Dimensional Regularization
- space-time dimensional parameter: ε = (4-d)/2
- **Linear** system in ε
- non-Unitary **Magnus transform**
- System of Differential Equations
  in canonical form (ε-factorization) Henn (2013)
- solution: Dyson/Magnus series



$$= e^{\Omega(d,x)}$$



- Feynman integrals can be determined from differential equations that looks like **gauge transformations**

Argeri, Di Vita, Mirabella, Schlenk, Schubert, Tancredi, **P.M.** (2014)

- ***pre-Canonical form***
  Linear-**eps** Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$

- change of basis :: Magnus #1

$$f(\epsilon, x) = B_0(x)\ g(\epsilon, x)\ , \qquad B_0(x) \equiv e^{\Omega[A_0](x,x_0)} . \qquad \partial_x B_0(x) = A_0(x) B_0(x)\ ,$$

- ***Canonical form*** Henn (2013)

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x) .$$

(*)

- Solution :: Magnus #2 (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x,x_0)}$$

Feynman integrals can be determined from differential equations that looks like gauge transformations

- Kinematic variables

$$-\frac{s}{m^2}=x, \quad -\frac{t}{m^2}=\frac{(1-y)^2}{y}$$

- eps-linear basis

$$\partial_x\ f(x,y,\epsilon)=\Big(A_{10}(x,y)+\epsilon A_{11}(x,y)\Big)\ f(x,y,\epsilon)$$
$$\partial_y\ f(x,y,\epsilon)=\Big(A_{20}(x,y)+\epsilon A_{21}(x,y)\Big)\ f(x,y,\epsilon)$$

- canonical form: Magnus #1

$$\partial_x\ g(x,y,\epsilon)=\epsilon\hat{A}_1(x,y)\ g(x,y,\epsilon)$$
$$\partial_y\ g(x,y,\epsilon)=\epsilon\hat{A}_2(x,y)\ g(x,y,\epsilon)$$

- Total Differential dLog-form

$$dg(x,y,\epsilon)=\epsilon\ d\hat{\mathcal{A}}(x,y)\ g(x,y,\epsilon)\ , \qquad d\hat{\mathcal{A}}\equiv\hat{A}_1dx+\hat{A}_2dy$$

$$dA=\sum_{i=1}^{n}M_i\ \mathrm{dlog}\,\eta_i$$

Alphabet

Constant matrices

Canonical systems and Iterated Integrals Henn (2013)

- Canonical system of DE $\quad d\mathbf{I} = \epsilon\, d\mathbb{A}\, \mathbf{I} \qquad dA = \sum_{i=1}^{n} \mathbb{M}_i \; d\log \eta_i \qquad \vec{x} = (x_1, x_2, \ldots, x_n)$

- Solution as path-ordered exponential

$$\mathrm{I}(\epsilon, \vec{x}) = \mathcal{P} \exp \left\{ \epsilon \int_\gamma dA \right\} \mathrm{I}(\epsilon, \vec{x}_0)\,, \qquad \mathcal{P} \exp \left\{ \epsilon \int_\gamma dA \right\} = \mathbb{1} + \epsilon \int_\gamma dA + \epsilon^2 \int_\gamma dA\, dA + \epsilon^3 \int_\gamma dA\, dA\, dA \ldots\,,$$

- Path invariance $\quad \gamma : [0,1] \ni t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))\,, \qquad \gamma(0) = \vec{x}_0 \text{ and } \gamma(1) = \vec{x}.$

- Taylor expansion and Dyson/Magnus series

$$\mathrm{I}(\epsilon, \vec{x}) = \mathrm{I}^{(0)}(\vec{x}) + \epsilon\, \mathrm{I}^{(1)}(\vec{x}) + \epsilon^2 \mathrm{I}^{(2)}(\vec{x}) + \ldots$$

$$\begin{aligned}
\mathrm{I}^{(0)}(\vec{x}) &= \mathrm{I}^{(0)}(\vec{x}_0)\,, \\
\mathrm{I}^{(1)}(\vec{x}) &= \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,, \\
\mathrm{I}^{(2)}(\vec{x}) &= \mathrm{I}^{(2)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,, \\
\mathrm{I}^{(3)}(\vec{x}) &= \mathrm{I}^{(3)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(2)}(\vec{x}_0)\,, + \int_\gamma dA\, dA\, \mathrm{I}^{(1)}(\vec{x}_0) \; + \int_\gamma dA\, dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,, \\
\mathrm{I}^{(4)}(\vec{x}) &= \mathrm{I}^{(4)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(3)}(\vec{x}_0) + \int_\gamma dA\, dA\, \mathrm{I}^{(2)}(\vec{x}_0) \; + \int_\gamma dA\, dA\, dA\, \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, dA\, dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,.
\end{aligned}$$

- Chen's Iterated integral

$$\int_\gamma \underbrace{dA \ldots dA}_{\text{k times}}, \longleftrightarrow \quad \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \equiv \int_\gamma d\log \eta_{i_k} \ldots d\log \eta_{i_1} \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k) \ldots g^\gamma_{i_1}(t_1)\, dt_1 \ldots dt_k\,, \qquad g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))\,.$$

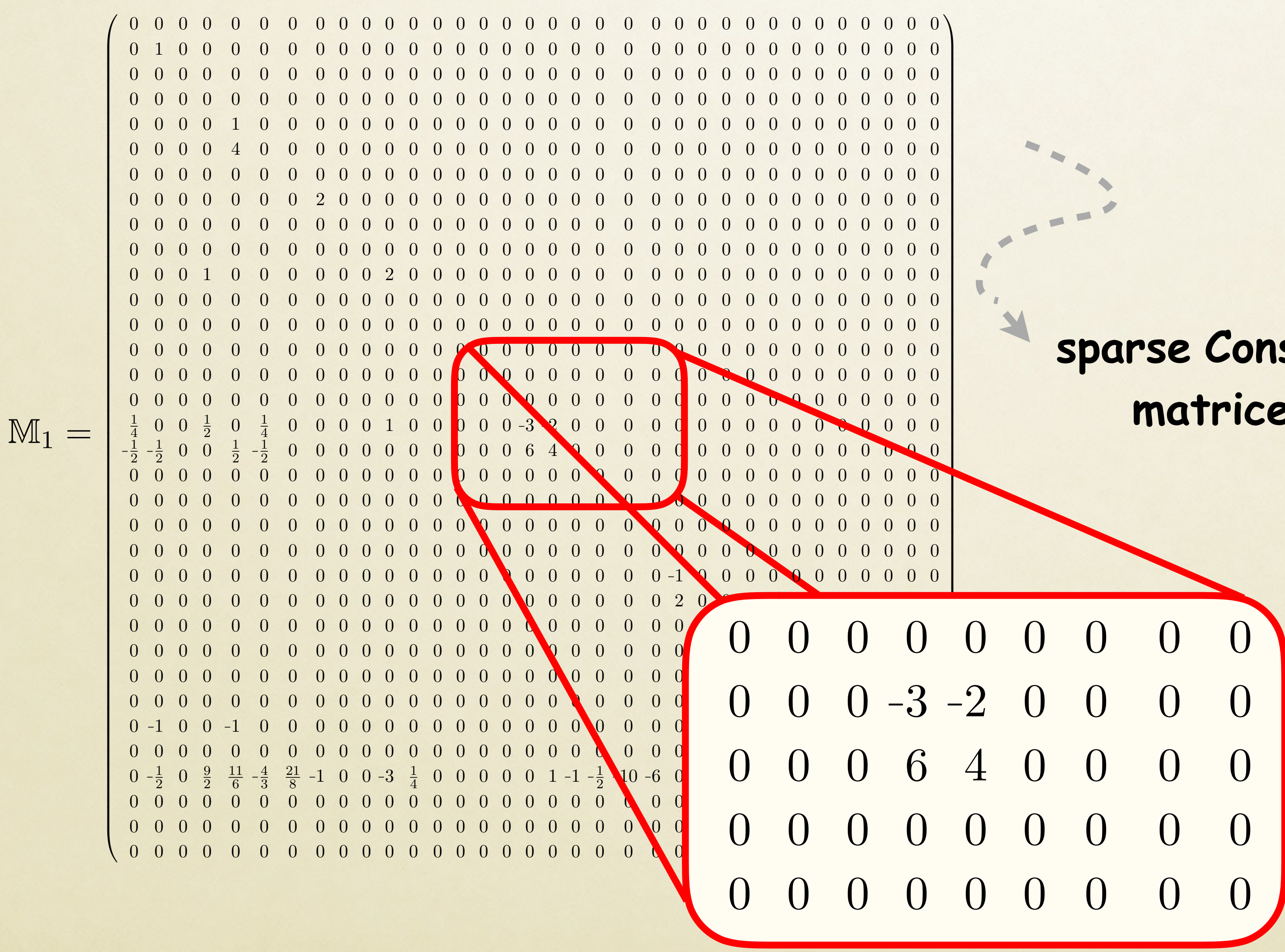
- a special case :: Goncharov's polylogs

$$G(\vec{w}_n; x) \equiv G(w_1, \vec{w}_{n-1}; x) \equiv \int_0^x dt \frac{1}{t - w_1} G(\vec{w}_{n-1}; t), \qquad G(\vec{0}_n; x) \equiv \frac{1}{n!} \log^n(x), \qquad \partial_x\, G(\vec{w}_n; x) = \partial_x\, G(w_1, \vec{w}_{n-1}; x) = \frac{1}{x - w_1} G(\vec{w}_{n-1}; x).$$

Ex :: Family-1

- Total Differential dLog-form

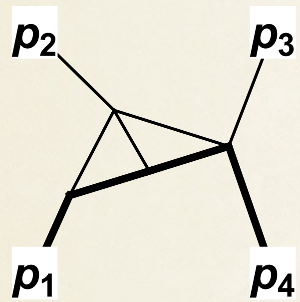
$$
\begin{aligned}
d\mathbb{A} = \mathbb{M}_1\ d\log(x) + \mathbb{M}_2\ d\log(1+x) + \mathbb{M}_3\ d\log(1-x) \\
+ \mathbb{M}_4\ d\log(y) + \mathbb{M}_5\ d\log(1+y) + \mathbb{M}_6\ d\log(1-y) \\
+ \mathbb{M}_7\ d\log(x+y) + \mathbb{M}_8\ d\log\left(1+x\,y\right) \\
+ \mathbb{M}_9\ d\log\left(1-y(1-x-y)\right)\ ,
\end{aligned}
$$

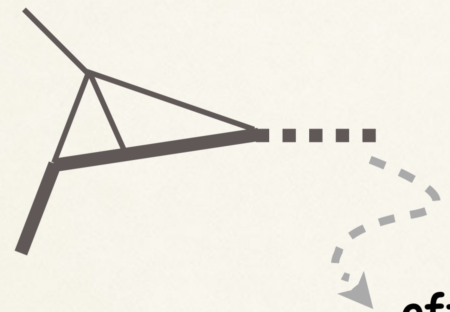


$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
$$

sparse Constant matrices

Ex :: Boundary conditions

$\mathrm{F(s)} =$ 

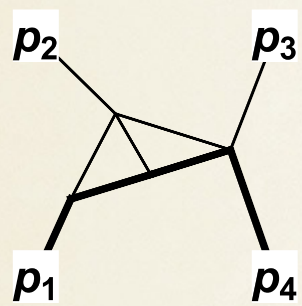
$=$ 

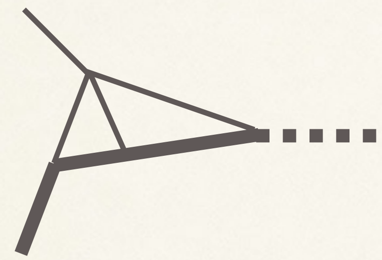
→ $\mathrm{F(0)} =$ 

$= ?$

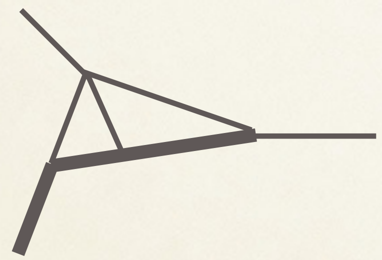
- regular @ s = 0
- trivial conditions !

Ex :: Boundary conditions

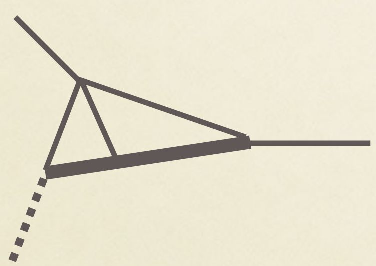
$F(s) =$ 

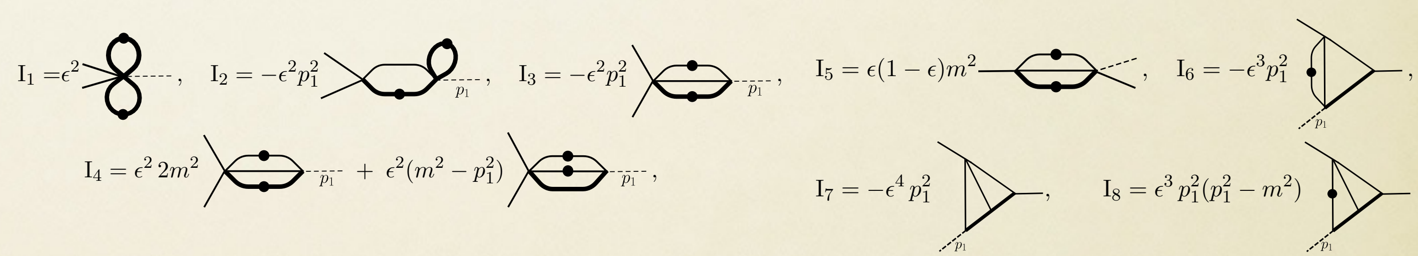
$=$ 

- regular @ s = 0
- trivial conditions !

$\longrightarrow$ $F(0) =$  $= ?$

- Auxiliary system of differential equation for:

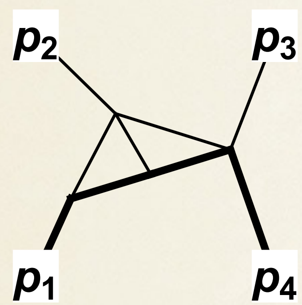
$I(p_1^2) =$ 

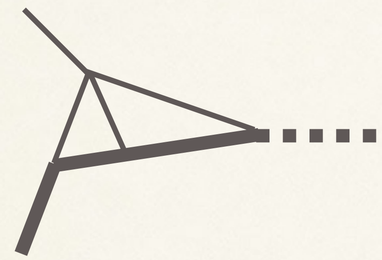


$$d\mathbb{A} = \mathbb{M}_0 \; d\log x + \mathbb{M}_{-1} \; d\log(x+1)$$

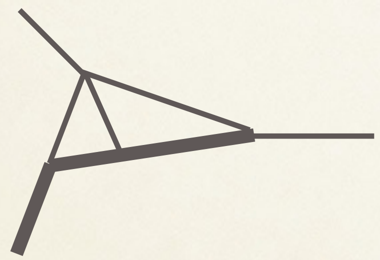
$$\mathbb{M}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & -1 & \frac{5}{2} \end{pmatrix}, \quad \mathbb{M}_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -\frac{1}{2} & -3 & 0 & -4 \end{pmatrix}.$$

Ex :: Boundary conditions

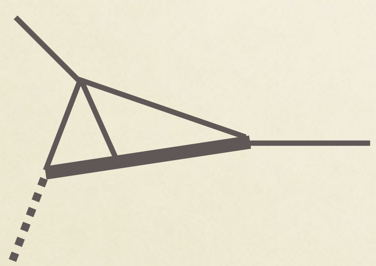
$F(s) =$ 

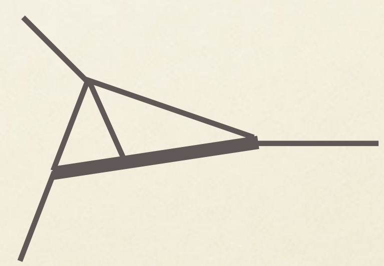
$=$ 

- regular @ s = 0
- trivial conditions !

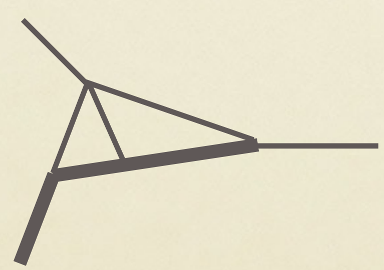
$\longrightarrow$ $F(0) =$  $= ?$

- Auxiliary system of differential equation for:

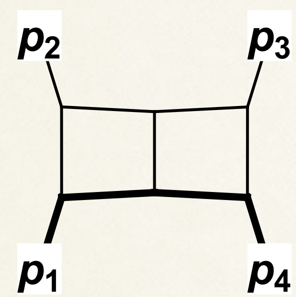
$I(p_1^2) =$ 

$I(0) =$ 

- **regular @** p_1^2 **= 0**
- **non-trivial** b.c. conditions !

$\longrightarrow$ $F(0) =$  $= \mathrm{I(m\hat{\ }2)}$

Ex :: Family-1 :: MI I_{30}



$$= \sum_{k=0}^{4} I_{30}^{(k)}(x,y)\ \epsilon^k$$

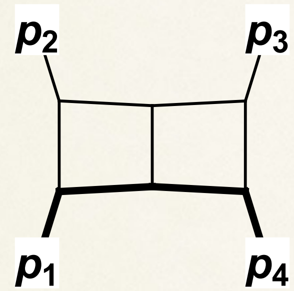
$$I_{30}^{(0)}(x,y) = -\frac{4}{3}$$

$$I_{30}^{(1)}(x,y) = \frac{7}{3} G[\{-1\}, x] - \frac{3}{2} G[\{0\}, y] + 3\, G[\{1\}, y]$$

$$I_{30}^{(2)}(x,y) = \frac{11\,\pi^2}{9} + G[\{-1\}, x]\,(4\, G[\{0\}, y] - 8\, G[\{1\}, y]) - \frac{4}{3} G[\{-1, -1\}, x] - G[\{0, 0\}, y] + 2\, G[\{0, 1\}, y] + 2\, G[\{1, 0\}, y] - 4\, G[\{1, 1\}, y]$$

$$\begin{aligned}
I_{30}^{(3)}(x,y) = {} & \pi^2 \left(-\frac{23}{9} G[\{-1\}, x] + \frac{7}{6} G[\{0\}, y] - \frac{11}{3} G[\{1\}, y] + \frac{4}{3} G\left[\left\{-\frac{1}{y}\right\}, x\right]\right) + \\
& G[\{-y\}, x]\,(-2\, G[\{1, 0\}, y] + 4\, G[\{1, 1\}, y]) + \\
& G\left[\left\{-\frac{1}{y}\right\}, x\right]\,(2\, G[\{0, 0\}, y] - 4\, G[\{0, 1\}, y] - 2\, G[\{1, 0\}, y] + 4\, G[\{1, 1\}, y]) + \\
& G[\{-1\}, x]\,(4\, G[\{0, 0\}, y] - 8\, G[\{0, 1\}, y] - 8\, G[\{1, 0\}, y] + 16\, G[\{1, 1\}, y]) + \\
& G[\{1\}, y] \left(16\, G[\{-1, -1\}, x] - 4\, G\left[\left\{-\frac{1}{y}, -1\right\}, x\right] - 4\, G[\{-y, -1\}, x]\right) + \\
& G[\{0\}, y] \left(-8\, G[\{-1, -1\}, x] + 2\, G\left[\left\{-\frac{1}{y}, -1\right\}, x\right] + 2\, G[\{-y, -1\}, x]\right) - \\
& \frac{32}{3} G[\{-1, -1, -1\}, x] + 4\, G[\{-1, 0, -1\}, x] - 2\, G[\{0, 1, 0\}, y] + \\
& 4\, G[\{0, 1, 1\}, y] - 2\, G[\{1, 0, 0\}, y] + 4\, G[\{1, 0, 1\}, y] + 4\, G[\{1, 1, 0\}, y] - \\
& 8\, G[\{1, 1, 1\}, y] + 4\, G\left[\left\{-\frac{1}{y}, -1, -1\right\}, x\right] - 2\, G\left[\left\{-\frac{1}{y}, 0, -1\right\}, x\right] + \\
& 4\, G[\{-y, -1, -1\}, x] - 2\, G[\{-y, 0, -1\}, x] + \frac{29\, \mathrm{Zeta}[3]}{3}
\end{aligned}$$

Ex :: Family-1 :: MI I_{30}



$$= \sum_{k=0}^{4} I_{30}^{(k)}(x,y)\ \epsilon^k$$

$$I_{30}^{(4)}(x,y) =$$

$$-\frac{4051\,\pi^4}{7200} +$$
$$G[\{-1,-1\},x]\ (-16\,G[\{0,0\},y] + 32\,G[\{0,1\},y] + 32\,G[\{1,0\},y] - 64\,G[\{1,1\},y])$$
$$G[\{0,0\},y]\ \left(2\,G\left[\left\{-\frac{1}{y},-\frac{1}{y}\right\},x\right] + 4\,G[\{-y,-1\},x] - 2\,G\left[\left\{-y,-\frac{1}{y}\right\},x\right]\right) +$$
$$\pi^2\ \Big(G[\{-1\},x]\ \left(-\frac{14}{3}G[\{0\},y] + \frac{28}{3}G[\{1\},y]\right) -$$
$$\frac{4}{3}G[\{-1\},y]\ G\left[\left\{-\frac{1}{y}\right\},x\right] + G[\{0\},y]\ G\left[\left\{-\frac{1}{y}\right\},x\right] + \frac{4}{3}G[\{-1\},y]\ G[\{-y\},x] +$$
$$G[\{1\},y]\ \left(-\frac{11}{3}G\left[\left\{-\frac{1}{y}\right\},x\right] + \frac{5}{3}G[\{-y\},x]\right) + \frac{8}{9}G[\{-1,-1\},x] +$$
$$\frac{4}{3}G[\{0,-1\},y] - G[\{0,1\},y] - G[\{1,0\},y] + \frac{22}{3}G[\{1,1\},y] -$$
$$3\,G\left[\left\{-\frac{1}{y},-1\right\},x\right] + \frac{4}{3}G\left[\left\{-\frac{1}{y},-\frac{1}{y}\right\},x\right] - \frac{1}{3}G[\{-y,-1\},x] - \frac{4}{3}G\left[\left\{-y,-\frac{1}{y}\right\},x\right]\Big) +$$
$$G[\{0,1\},y]\ \left(-4\,G\left[\left\{-\frac{1}{y},-\frac{1}{y}\right\},x\right] - 8\,G[\{-y,-1\},x] + 4\,G\left[\left\{-y,-\frac{1}{y}\right\},x\right]\right) +$$
$$G[\{1,0\},y]\ \Big(-4\,G\left[\left\{-\frac{1}{y},-1\right\},x\right] - 2\,G\left[\left\{-\frac{1}{y},-\frac{1}{y}\right\},x\right] + 2\,G\left[\left\{-\frac{1}{y},-y\right\},x\right] -$$
$$4\,G[\{-y,-1\},x] + 2\,G\left[\left\{-y,-\frac{1}{y}\right\},x\right] - 2\,G[\{-y,-y\},x]\Big) +$$
$$G[\{1,1\},y]\ \Big(8\,G\left[\left\{-\frac{1}{y},-1\right\},x\right] + 4\,G\left[\left\{-\frac{1}{y},-\frac{1}{y}\right\},x\right] - 4\,G\left[\left\{-\frac{1}{y},-y\right\},x\right] +$$
$$8\,G[\{-y,-1\},x] - 4\,G\left[\left\{-y,-\frac{1}{y}\right\},x\right] + 4\,G[\{-y,-y\},x]\Big) + G[\{-y\},x]$$
$$(2\,G[\{-1,0,0\},y] - 4\,G[\{-1,0,1\},y] - 6\,G[\{0,1,0\},y] + 12\,G[\{0,1,1\},y] -$$
$$2\,G[\{1,0,0\},y] + 4\,G[\{1,0,1\},y] + 12\,G[\{1,1,0\},y] - 24\,G[\{1,1,1\},y]) +$$
$$G\left[\left\{-\frac{1}{y}\right\},x\right]\ (-2\,G[\{-1,0,0\},y] + 4\,G[\{-1,0,1\},y] + 6\,G[\{0,0,0\},y] -$$

$$12\,G[\{0,0,1\},y] - 6\,G[\{0,1,0\},y] + 12\,G[\{0,1,1\},y] - 10\,G[\{1,$$
$$20\,G[\{1,0,1\},y] + 12\,G[\{1,1,0\},y] - 24\,G[\{1,1,1\},y]) +$$
$$G[\{1\},y]\ \Big(8\,G\left[\left\{-\frac{1}{y},-1,-1\right\},x\right] - 4\,G\left[\left\{-\frac{1}{y},-\frac{1}{y},-1\right\},x\right] + 4\,G\left[\left\{-\frac{1}{y},-y\right.\right.$$
$$8\,G[\{-y,-1,-1\},x] + 4\,G\left[\left\{-y,-\frac{1}{y},-1\right\},x\right] - 4\,G[\{-y,-y,-1\},x]$$
$$G[\{0\},y]\ \Big(-4\,G\left[\left\{-\frac{1}{y},-1,-1\right\},x\right] + 2\,G\left[\left\{-\frac{1}{y},-\frac{1}{y},-1\right\},x\right] - 2\,G\left[\left\{-\frac{1}{y},-\right.\right.$$
$$4\,G[\{-y,-1,-1\},x] - 2\,G\left[\left\{-y,-\frac{1}{y},-1\right\},x\right] + 2\,G[\{-y,-y,-1\},x]$$
$$\frac{128}{3}G[\{-1,-1,-1,-1\},x] - 16\,G[\{-1,-1,0,-1\},x] -$$
$$24\,G[\{-1,0,-1,-1\},x] +$$
$$12\,G[\{-1,0,0,-1\},x] + 2\,G[\{0,-1,0,0\},y] -$$
$$4\,G[\{0,-1,0,1\},y] - 10\,G[\{0,0,1,0\},y] +$$
$$20\,G[\{0,0,1,1\},y] - 6\,G[\{0,1,0,0\},y] +$$
$$12\,G[\{0,1,0,1\},y] + 20\,G[\{0,1,1,0\},y] -$$
$$40\,G[\{0,1,1,1\},y] - 6\,G[\{1,0,0,0\},y] +$$
$$12\,G[\{1,0,0,1\},y] + 20\,G[\{1,0,1,0\},y] -$$
$$40\,G[\{1,0,1,1\},y] + 20\,G[\{1,1,0,0\},y] -$$
$$40\,G[\{1,1,0,1\},y] - 40\,G[\{1,1,1,0\},y] + 80\,G[\{1,1,1,1\},y] -$$
$$24\,G\left[\left\{-\frac{1}{y},-1,-1,-1\right\},x\right] + 8\,G\left[\left\{-\frac{1}{y},-1,0,-1\right\},x\right] +$$
$$12\,G\left[\left\{-\frac{1}{y},0,-1,-1\right\},x\right] - 6\,G\left[\left\{-\frac{1}{y},0,0,-1\right\},x\right] +$$
$$4\,G\left[\left\{-\frac{1}{y},-\frac{1}{y},-1,-1\right\},x\right] - 2\,G\left[\left\{-\frac{1}{y},-\frac{1}{y},0,-1\right\},x\right] -$$
$$4\,G\left[\left\{-\frac{1}{y},-y,-1,-1\right\},x\right] + 2\,G\left[\left\{-\frac{1}{y},-y,0,-1\right\},x\right] -$$
$$24\,G[\{-y,-1,-1,-1\},x] + 8\,G[\{-y,-1,0,-1\},x] +$$
$$12\,G[\{-y,0,-1,-1\},x] - 6\,G[\{-y,0,0,-1\},x] -$$
$$4\,G\left[\left\{-y,-\frac{1}{y},-1,-1\right\},x\right] + 2\,G\left[\left\{-y,-\frac{1}{y},0,-1\right\},x\right] +$$
$$4\,G[\{-y,-y,-1,-1\},x] - 2\,G[\{-y,-y,0,-1\},x] +$$
$$\left(-\frac{62}{3}G[\{-1\},x] + 9\,G[\{0\},y] - 28\,G[\{1\},y] + 10\,G\left[\left\{-\frac{1}{y}\right\},x\right]\right)\ \mathrm{Zeta}[3]$$

Summary …

- **Feasibility of Two-Loop QED Corrections *analytically***
  - Master Integrals via Differential Equations + Magnus Series
  - Amplitude decomposition via Adaptive Integrand Decomposition (AID)

    Mu-e scattering :: a first example of 2-loop automation for massive amplitudes

…Outlook

- Building the 2-loop amplitude (Form Factors and AID)
- Analytic continuation and Numerical Evaluation of 2-loop MIs
- The 1-Loop amplitude and 2-loop renorm. counterterms (GoSam, AID)
- Implementing a Subtraction Scheme for NNLO (*hinc sunt leones*)
- MonteCarlo Integration

  >> see Piccinini's talk

- Progress in **Mu-e Scattering** @ 2-loop QED
  ==> benefit for **e+ e- —> Mu+ Mu-** @ 2-loop QED
  ==> benefit for **p p —> t T** @ 2-loop QCD

Muon-electron scattering: Theory kickoff workshop 4-5 September 2017 Padova

https://agenda.infn.it/internalPage.py?pageId=0&confId=13774

Organizing Committee

Carlo Carloni Calame - INFN Pavia
Pierpaolo Mastrolia - U. Padova
Guido Montagna - U. Pavia
Oreste Nicrosini - INFN Pavia
Paride Paradisi - U. Padova
Massimo Passera - INFN Padova (Chair)
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Luca Trentadue - U. Parma

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