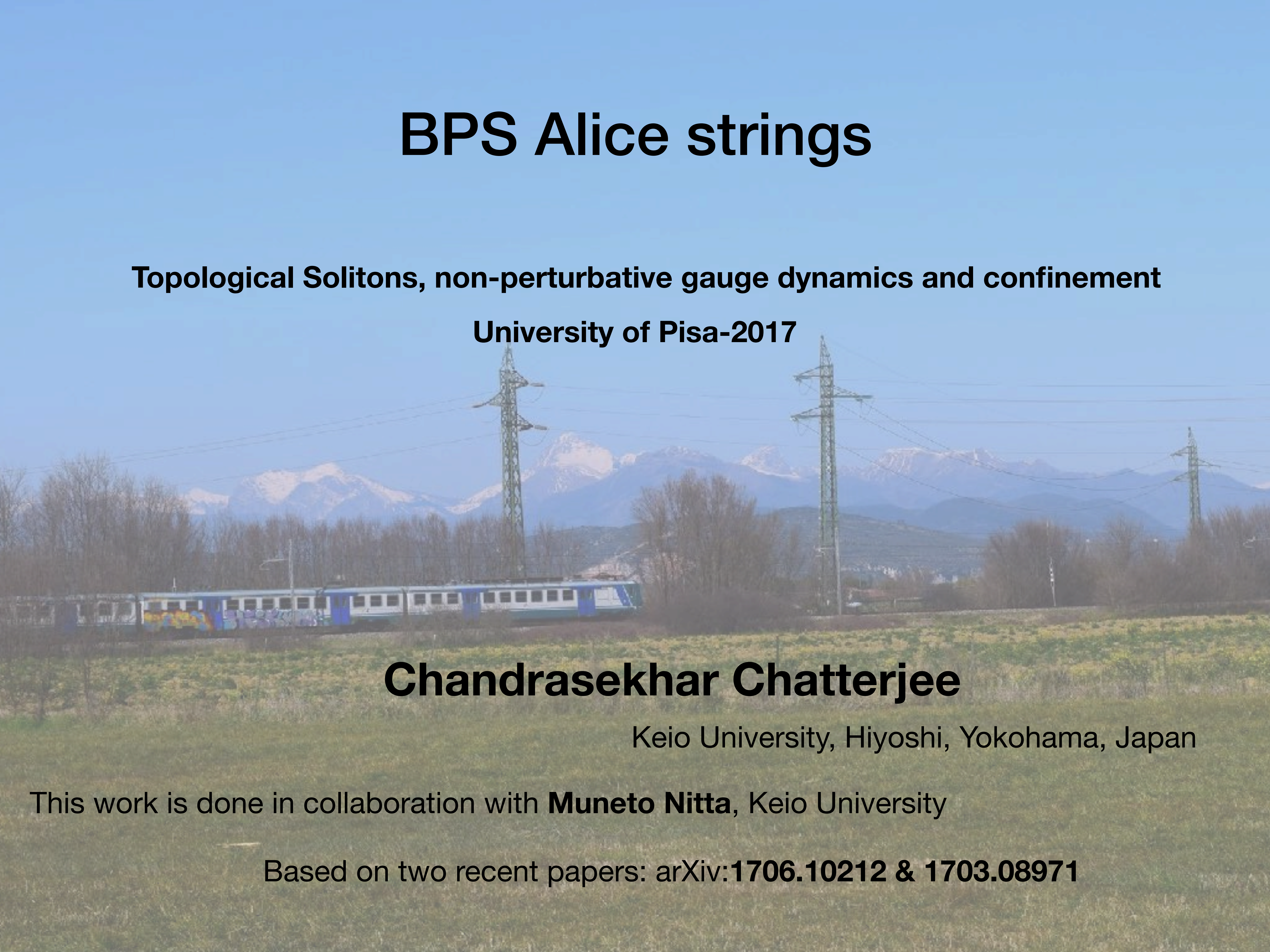

BPS Alice strings

Topological Solitons, non-perturbative gauge dynamics and confinement

University of Pisa-2017

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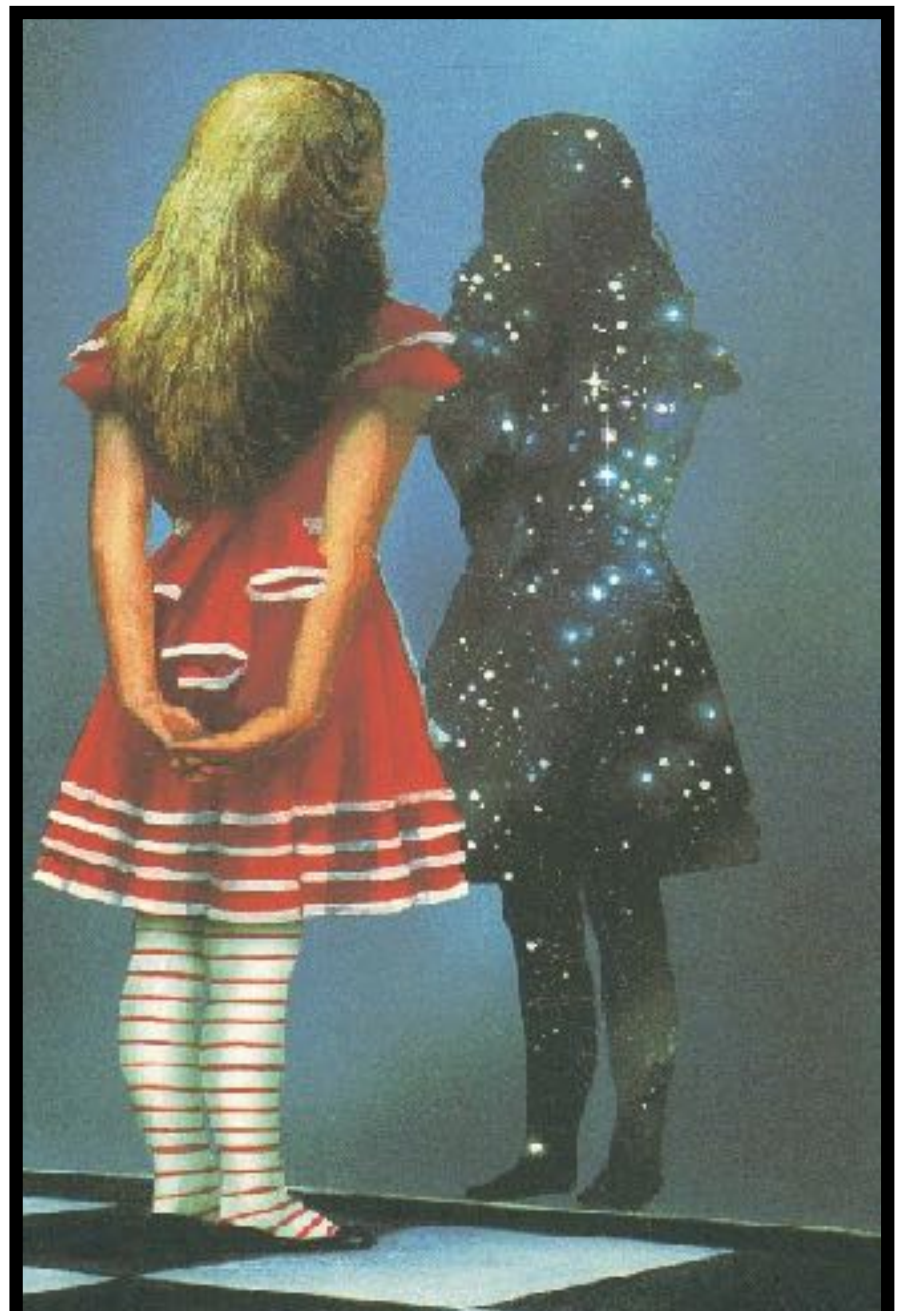
This work is done in collaboration with **Muneto Nitta**, Keio University

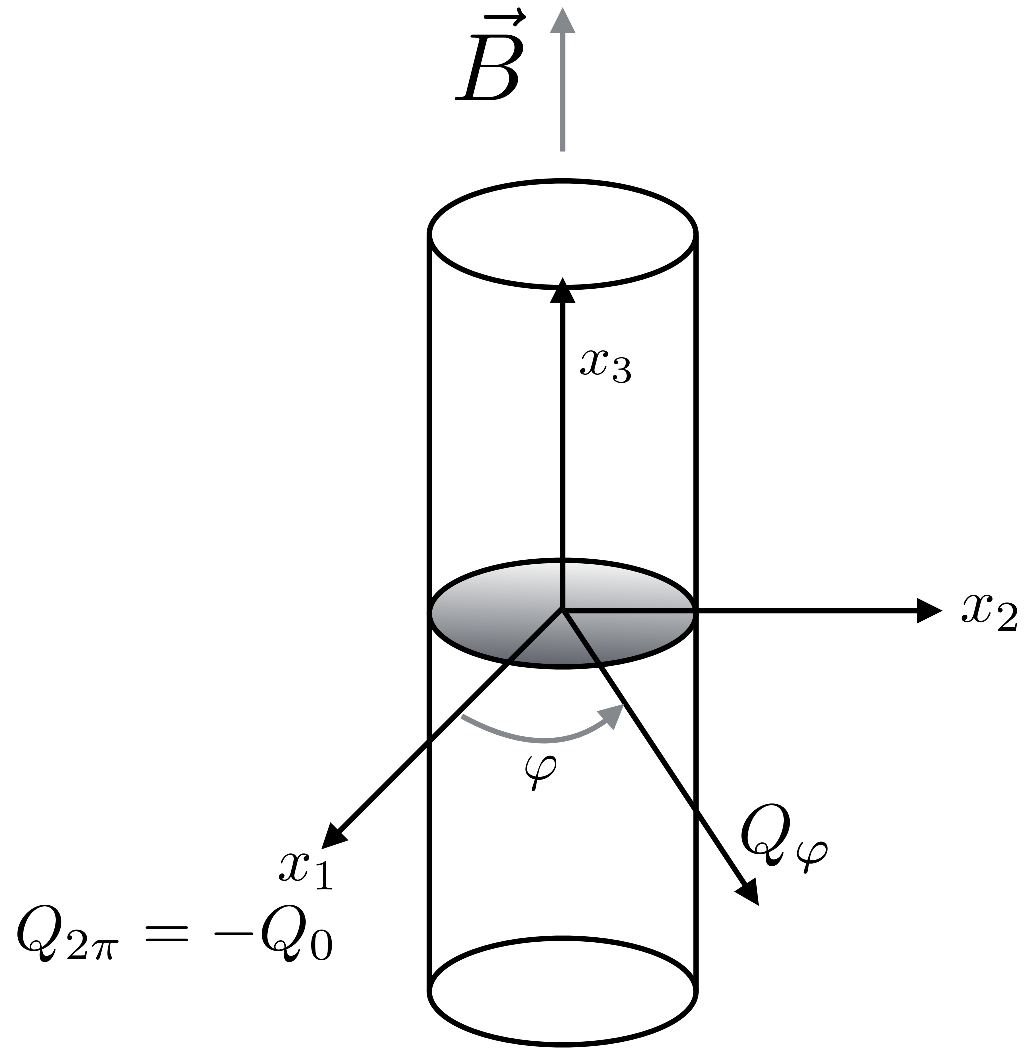
Based on two recent papers: arXiv:**1706.10212 & 1703.08971**

‘Alice string’ have the property to change the mirrority of the object as a result of the object going around a string along a closed path. It is related to Alice, the heroine of the book by L. Carroll, since she could come ‘through looking glass’ to the mirror world which is equivalent to passing around this string.



Alice string

- When a charged particle encircles around an **Alice string**, it changes the sign of the electric charge.



A.S.Schwarz,``Field Theories With No Local Conservation Of The Electric Charge,'' Nucl Phys B 208, 141 (1982).

- The theory admitting an Alice string is sometime called as "**Alice electrodynamics**", which is a U(1) gauge theory that includes the charge conjugation as a local symmetry.

J. E. Kiskis, "Disconnected Gauge Groups and the Global Violation of Charge Conservation," Phys. Rev. D 17, 3196 (1978)

- **Chashire Charge:** A delocalized charge present in the multi-vortex system.

- Example of a typical non-BPS Alice string in SU(2) 5 representation.
- Alice string in Spin-1 BEC
- A BPS construction of Alice string in U(1) X SU(2) gauge theory
- U(1) zero mode and Effective action construction
- The SUSY model and 1/2 BPS Alice string

Discussion

Example of Alice string

- A typical Alice string was found in an SO(3) gauge theory with scalar fields in spin-2 (traceless symmetric tensor) representation of SO(3).

- The ground state is described by

$$\Phi_0 = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- The ground state is invariant under the rotation around z-axis by $h(\varphi) = e^{i\varphi J_3}, J_3 \in SO(3).$

- Ground state is also invariant under a reflection around x or y axis

$$I_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Example of Alice string

- The SO(3) gauge group is spontaneously broken to O(2)

- The order parameter space is found to be

$$OPS = \frac{SO(3)}{O(2)} \simeq \frac{S^2}{\mathbb{Z}_2} \simeq \mathbb{R}P^2$$

$$\Pi_1(G/H) = \mathbb{Z}_2$$

- An Alice string can annihilate itself when two of them collide, contradict to a BPS property that energy is proportional to the topological charge. Therefore, Alice strings are naturally non-BPS.

Example of Alice string

- A vortex configuration can be written as

$$\Phi(\theta) = e^{i\frac{\theta}{2}J_1} \; \Phi_0 \; e^{-i\frac{\theta}{2}J_1}$$

- In the presence of vortices the unbroken generator becomes

$$Q(\theta) = e^{i\frac{\theta}{2}J_1} \; Q_0 \; e^{-i\frac{\theta}{2}J_1}, \quad Q_0 = J_3$$

- So the embedding of unbroken group changes within the full group and the generator is parallel transported along a circle around the vortex.

- The generator changes its sign after the completion of one full circle around the vortex .

$$Q(2\pi) = -Q_0$$

Global Alice string in triplet representation

- We can consider the case of half quantized vortices in spin-1 BEC. In this case a **global** Alice string can be constructed.

- Order parameter is the complex spin triplet

$$\Phi = \xi \, e^{i\varphi} \, \hat{d}$$

$\hat{d}$ is the real unit vector.

- Order parameter is invariant under any rotation around $\hat{d}$ and the transformation

$$\hat{d} \longrightarrow -\hat{d}, \quad \varphi \longrightarrow \varphi + \pi$$

$$U(1) \times SO(3) \longrightarrow O(2) \qquad OPS = \frac{G}{H} = \frac{U(1) \times SO(3)}{O(2)} \simeq \frac{S^1 \times S^2}{\mathbb{Z}_2}$$

$$\pi_1 \left(G/H \right) = \mathbb{Z}$$

Possibility of multi-vortex configurations

U. Leonhardt and G. E. Volovik, "**How to create Alice string (half quantum vortex) in a vector Bose-Einstein condensate**," Pisma Zh. Eksp. Teor. Fiz. 72, 66 (2000) [JETP Lett. 72, 46 (2000)] [cond-mat/0003428].

BPS-Alice string in triplet representation

- We consider an $\boxed{SU(2) \times U(1)}$ gauge theory coupled with one charged complex scalar field in the adjoint representation.

$$\boxed{\Phi = \phi^a \tau^a, A_\mu = A_\mu^a \tau^a, \tau^a = \frac{1}{2}\sigma^a}$$

$$I = \int d^4x \left[-\frac{1}{2}\mathrm{Tr}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu} - \frac{1}{4}\mathrm{f}_{\mu\nu}\mathrm{f}^{\mu\nu} + \mathrm{Tr}|\mathrm{D}_\mu \Phi|^2 - \frac{\lambda_\mathrm{g}}{4}\mathrm{Tr}[\Phi, \Phi^\dagger]^2 - \frac{\lambda_\mathrm{e}}{2}\left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2\right)^2 \right].$$

$$D_\mu \Phi = \partial_\mu \Phi - iea_\mu \Phi - ig\,[A_\mu, \Phi] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],\ f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

- The vacuum configuration

$$\langle \Phi \rangle_{Vac} = 2\xi\tau^1$$

- The unbroken group elements

$$\boxed{H = \left\{ \left(1,\ e^{\frac{i\alpha}{2}\sigma^1}\right), \left(-1,\ i\left(c_2\sigma^2 + c_3\sigma^3\right) e^{i\frac{\alpha}{2}\sigma^1}\right) \right\}}, \quad c_2^2 + c_3^2 = 1$$

$$\boxed{H \sim \mathbb{Z}_2 \ltimes U(1) \simeq O(2)}$$

Alice string in triplet representation

$$\langle \Phi \rangle_{Vac} = 2\xi\tau^1$$

$$\downarrow$$

$$G = U(1) \times \frac{SU(2)}{\mathbb{Z}_2} \simeq U(1) \times SO(3) \longrightarrow H \simeq \mathbb{Z}_2 \ltimes U(1) \simeq O(2)$$

$$\downarrow$$

$$OPS = \frac{G}{H} = \frac{U(1) \times SO(3)}{O(2)} \simeq \frac{S^1 \times S^2}{\mathbb{Z}_2}$$

$$\downarrow$$

$$\pi_1\left(G/H\right) = \mathbb{Z}$$

Alice string in triplet representation

- Large distance behavior of the order parameter for a vortex can be obtained by

$$\mathcal{D}_i \Phi \to 0, \quad R \to \infty$$

$$\Phi(\infty, \varphi) \sim \xi \begin{pmatrix} 0 & e^{i\varphi} \\ 1 & 0 \end{pmatrix}$$

$$\Phi(\infty, \varphi) \sim \xi e^{i\frac{\varphi}{2}} \begin{pmatrix} 0 & e^{i\frac{\varphi}{2}} \\ e^{-i\frac{\varphi}{2}} & 0 \end{pmatrix} = \xi e^{i\frac{\varphi}{2}} e^{i\frac{\varphi}{4}\sigma^3} \sigma^1 e^{-i\frac{\varphi}{4}\sigma^3}$$

$$U_0(\varphi) = e^{ie\int \mathbf{a}\cdot d\mathbf{l}}, \quad U_3(\varphi) = P e^{ig\int \mathbf{A}\cdot d\mathbf{l}}$$

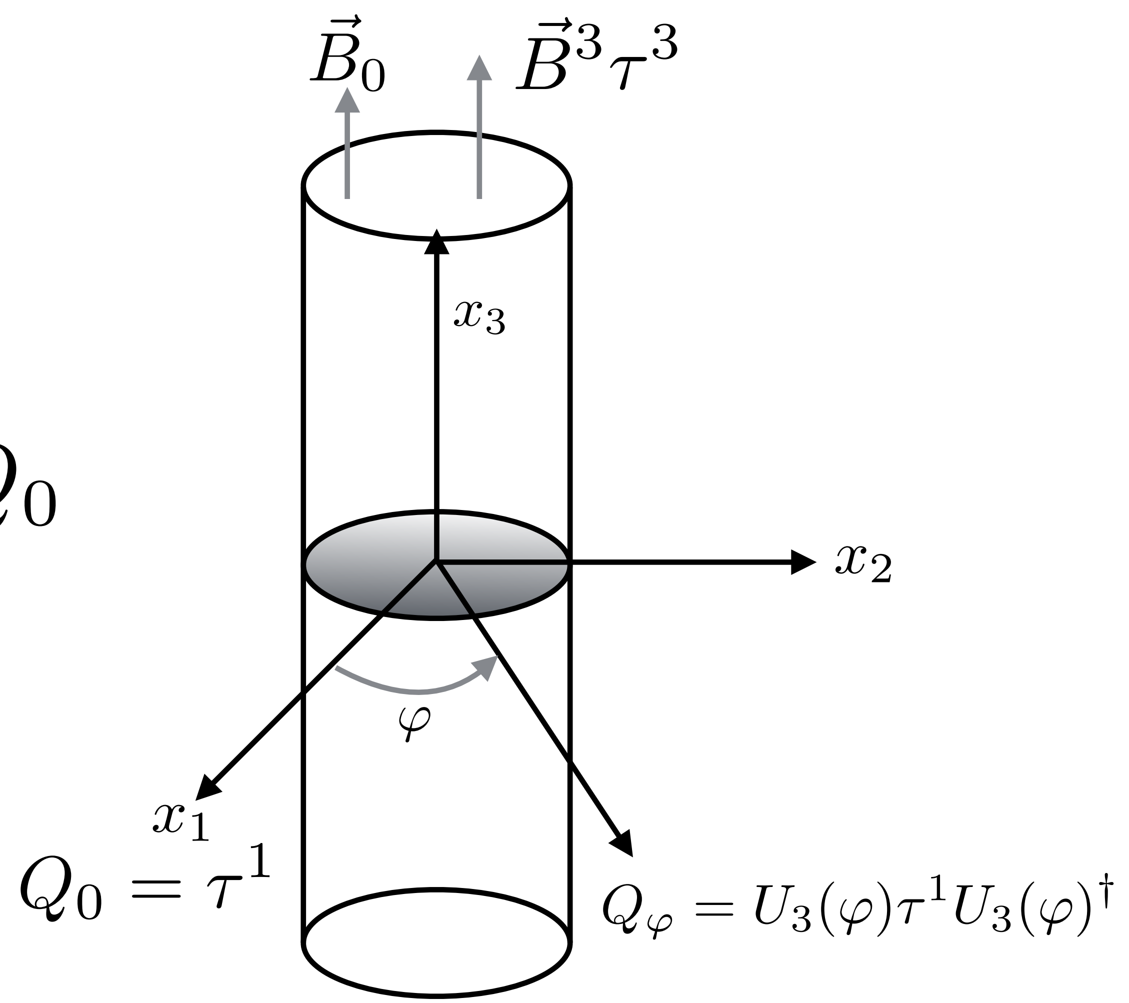
$$\Phi(\infty, \varphi) = U_0(\varphi) U_3(\varphi)\, \Phi(\infty, 0) U_3^{-1}(\varphi), \qquad \Phi(\infty, 0) = \xi\sigma^1$$

Alice string in triplet representation

- The unbroken group generator can be written as

$$Q_\varphi = U_3(\varphi) Q_0 U_3(\varphi)^{-1}, \quad U_3(\varphi) = e^{\frac{i\varphi}{4}\sigma^3}, Q_0 = \frac{1}{2}\sigma^1$$

$$Q_{2\pi} = -Q_0$$



Alice string in triplet representation

- The static hamiltonian is expressed as

$$\mathcal{H} = \int d^3x \left[\frac{1}{2}\mathrm{Tr}\mathrm{F}_{\mathrm{ij}}^2 + \frac{1}{4}\mathrm{f}_{\mathrm{ij}}^2 + \mathrm{Tr}|\mathrm{D_i}\Phi|^2 + \frac{\lambda_{\mathrm{g}}}{4}\mathrm{Tr}[\Phi, \Phi^\dagger]^2 + \frac{\lambda_{\mathrm{e}}}{2}\left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2\right)^2\right].$$

- We consider critical coupling $\lambda_e = e^2, \quad \lambda_g = g^2$

- We may perform the Bogomol'nyi completion, the tension (energy per the unit length) of a vertex along the x_3 coordinate looks

$$\mathcal{T} = \int d^2x \left[\mathrm{Tr}\left[\mathrm{F}_{12} \pm \frac{\mathrm{g}}{2}[\Phi, \Phi^\dagger]\right]^2 + \mathrm{Tr}|\mathcal{D}_\pm\Phi|^2 + \frac{1}{2}\left[\mathrm{f}_{12} \pm \mathrm{e}\left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2\right)\right]^2 \pm 2\mathrm{e}\mathrm{f}_{12}\xi^2\right]$$

Alice string in triplet representation

$$\mathcal{T} = \int d^2x \left[\mathrm{Tr} \left[\mathrm{F}_{12} \pm \frac{\mathrm{g}}{2}[\Phi, \Phi^\dagger] \right]^2 + \mathrm{Tr}|\mathcal{D}_\pm \Phi|^2 + \frac{1}{2} \left[\mathrm{f}_{12} \pm \mathrm{e} \left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2 \right) \right]^2 \pm 2\mathrm{e}\mathrm{f}_{12}\xi^2 \right]$$

$$\geq 2e\xi^2 \left| \int d^2x \; f_{12} \right|,$$

- At the saturation point the first order BPS equations are,

$$f_{12} \pm e \left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2 \right) = 0,$$

$$F_{12} \pm \frac{g}{2}[\Phi, \Phi^\dagger] = 0,$$

$$\mathcal{D}_\pm \Phi = \mathcal{D}_\mp \Phi^\dagger = 0$$

A BPS Alice string solution

- To solve BPS equations we consider the ansatz

$$\Phi(r,\varphi) = \xi \begin{pmatrix} 0 & f_1(r)e^{i\varphi} \\ f_2(r) & 0 \end{pmatrix},$$

$$a_i(r,\varphi) = -\frac{1}{2e}\frac{\epsilon_{ijx_j}}{r^2}a(r), \qquad A_i(r,\varphi) = -\frac{1}{4g}\frac{\epsilon_{ijx_j}}{r^2}\sigma^3 A(r).$$

- Here $f_1(r), f_2(r), A(r), a(r)$ are four profile functions with boundary condition.

$$\begin{aligned} f_1(0) = f_2'(0) = 0, \qquad & f_1(\infty) = f_2(\infty) = 1, \\ A(0) = a(0) = 0, \qquad & A(\infty) = a(\infty) = 1. \end{aligned}$$

- The equations satisfied by the profile functions are same as non-Abelian vortices.

A BPS Alice string solution

$$f_1(r) = r e^{-\frac{1}{2}(\psi_0(r)+\psi_1(r))}, \; f_2(r) = e^{\frac{1}{2}(\psi_1(r)-\psi_0(r))}.$$

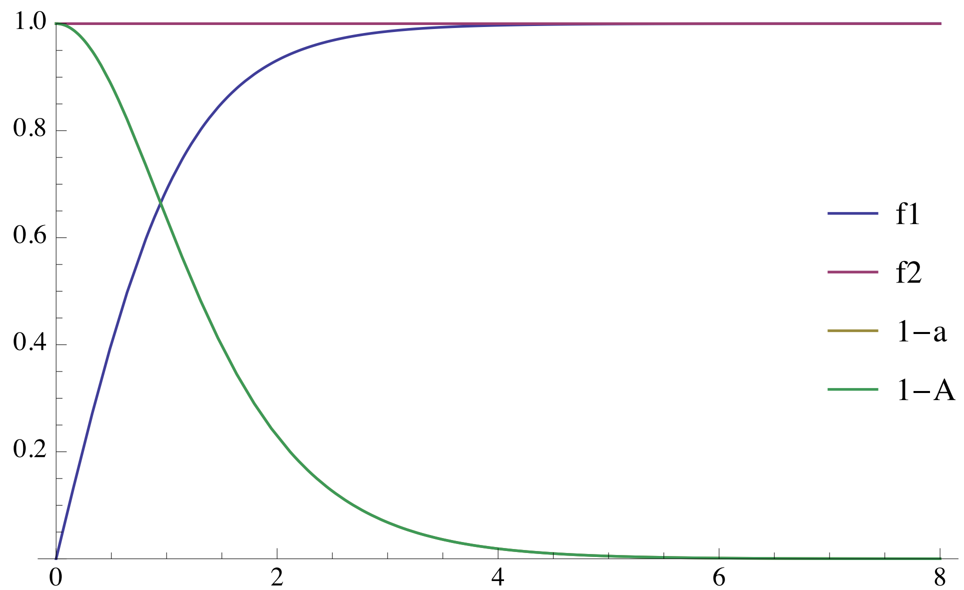
$$A(r) = r\psi_1'(r), \; a(r) = r\psi_0'(r).$$

$$\boxed{\begin{aligned} &\frac{1}{\rho}\partial_\rho\left[\rho\psi_0'(\rho)\right] + l^2\left[e^{-\psi_0}\left(\rho^2 e^{-\psi_1} + e^{\psi_1}\right) - 2\right] = 0, \\ &\frac{1}{\rho}\partial_\rho\left[\rho\psi_1'(\rho)\right] + e^{-\psi_0}(e^{-\psi_1} - e^{\psi_1}) = 0, \end{aligned}}$$

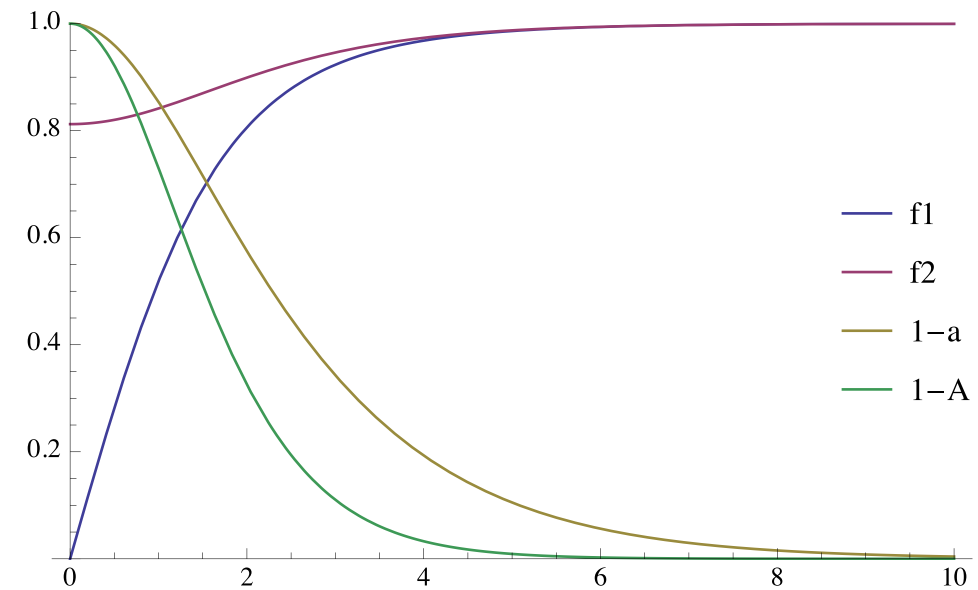
$$\psi_0'(0) = \psi_1'(0), \; \psi_0(R) = \psi_1(R) = \log R,$$

$$\rho^2 = 2g^2\xi^2 r^2, \; l = e/g$$

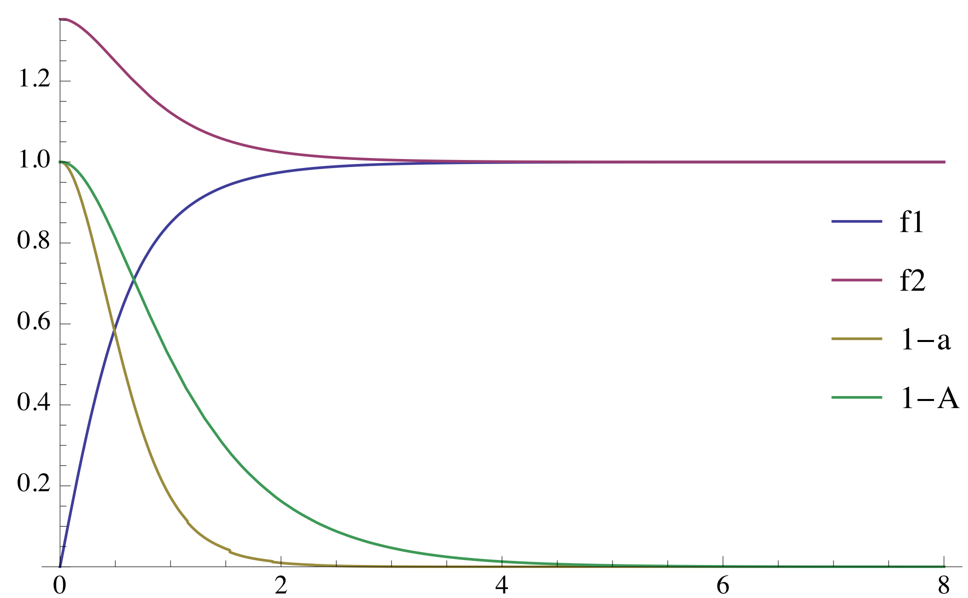
A BPS Alice string solution



$a)\ e/g = 1$



$b)\ e/g = 0.5$



$c)\ e/g = 5$

U(1) Zero mode

- Zero mode exists when a soliton solution is not invariant under a continuous unbroken symmetry of Lagrangian.
- Any small change due to the action of the U(1) group element $e^{i\frac{\varphi}{2}\sigma^1}$

$$\Phi(r,\theta) = \xi \begin{pmatrix} 0 & f_1(r)e^{i\theta} \\ f_2(r) & 0 \end{pmatrix}$$

$$\delta\Phi(r,0) = i\frac{\varphi}{2}\Big[\sigma^1, \Phi(r,0)\Big] = i\frac{\varphi\xi}{2}\Big(f_2(r) - f_1(r)\Big)\sigma^3.$$

$$f_1(r) \neq f_2(r) \qquad\qquad f_1(\infty) = f_2(\infty) = 1$$

- U(1) is broken inside the vortex core.

U(1) Zero mode and Effective action

- We want solutions whose energies vanish when there is no (t, z) dependence.
- We start with a gauge where the vortex solution looks like

$$\Phi(\varphi(t,z),r,\theta) = e^{i\frac{\varphi(t,z)}{2}\sigma^1}\begin{pmatrix} 0 & f_1(r)e^{i\theta} \\ f_2(r) & 0 \end{pmatrix} e^{-i\frac{\varphi(t,z)}{2}\sigma^1}$$
$$= U_\varphi(t,z)\ \Phi(r,\theta)\ U^\dagger_\varphi(t,z)$$
$$A_i(\varphi(t,z),x,y) = -\frac{1}{4g}\frac{\epsilon_{ij}x_j}{r^2}A(r)\ \ U_\varphi(t,z)\,\sigma^3\,U^\dagger_\varphi(t,z).$$

- This (t, z) dependence actually generates electric field in the system and the effective action would look like

$$I_{\rm eff} = \int dtdz\left[\int d^2x\left\{-\frac{1}{2}f_{i\alpha}f^{i\alpha} - \mathrm{TrF_{i\alpha}F^{i\alpha}} + \mathrm{Tr}|\mathrm{D}_\alpha\Phi(\varphi)|^2\right\}\right], \quad \alpha = \{0,3\}$$

$$F_{i\alpha} = \partial_i A_\alpha - D_\alpha A_i(\varphi), D_\alpha = \partial_\alpha - ig[A_\alpha,\,\cdot\,], D_\alpha\Phi(\varphi) = \partial_\alpha\Phi(\varphi) - iea_\alpha\Phi(\varphi) - ig[A_\alpha,\Phi(\varphi)]$$

U(1) Zero mode and Effective action

- We choose an ansatz for the generated gauge field as

$$A_\alpha(t,z,r,\theta) = \frac{1}{g}\left[\left(1-\Psi_1(r,\theta)\right)\tau^1 + \Psi_2(r,\theta)T^2\right]\partial_\alpha\varphi(t,z),$$

$$a_\alpha = 0, \qquad T^2 = U_\varphi(t,z)\,\tau^2\,U_\varphi^\dagger(t,z), \qquad \tau^i = \frac{1}{2}\sigma^i$$

$$\boxed{\begin{aligned}\mathcal{I}_{\text{eff}} &= I_\Psi \int dtdz\left\{\frac{\partial_\alpha\varphi\partial^\alpha\varphi}{2g^2}\right\}, \quad \Delta_g^2 = \xi^2 g^2, \\ I_\Psi &= \int d^2x\left[\left|D_i\Psi\right|^2 + \Delta_g^2\left|\Psi^* q_1 - \Psi q_2\right|^2\right]\end{aligned}}$$

$$\Psi(r,\theta) = \Psi_1(r,\theta) + i\,\Psi_2(r,\theta)$$

$$q_1 = f_1 e^{i\frac{1}{2}\theta}, \quad q_2 = f_2 e^{-i\frac{1}{2}\theta}$$

$$D_i\Psi = (\partial_i - i\zeta b_i)\Psi,\ \zeta = \frac{1}{2},\ b_i = -\frac{\epsilon_{ij}x_j}{r^2}A(r)$$

U(1) Zero mode and Effective action

We extremize I_Ψ by varying Ψ and find the equation for Ψ as

$$D_i^2 \Psi - \Delta_g^2 \left[\left(f_1^2 + f_2^2 \right) \Psi - 2 f_1 f_2 e^{i\theta} \Psi^* \right] = 0.$$

$$\Psi(\rho, \theta) = \sum_{m \geq 0} \left[\psi_m^+(\rho)\, e^{im\theta} + \psi_m^-(\rho)\, e^{-im\theta} \right], \qquad \psi_0^+(\rho) = 0,$$

$\rho = \Delta_g r, \quad \zeta = \frac{1}{2}$

$$\boxed{\begin{aligned} \psi_m^+(\rho) &= \rho^{m-\zeta} \left(\frac{f_1}{f_2} \right)^{\zeta}, \qquad \text{for } m > 0, \\ \psi_m^-(\rho) &= \rho^{m+\zeta} \left(\frac{f_1}{f_2} \right)^{-\zeta}, \qquad \text{for } m \geq 0. \end{aligned}}$$

U(1) Zero mode and Effective action

- The large distance behavior of the solution

$$\psi_m^+(\rho) \longrightarrow \rho^{m-\zeta}, \qquad \text{for } m > 0,$$
$$\psi_m^-(\rho) \longrightarrow \rho^{m+\zeta}, \qquad \text{for } m \geq 0,$$

- At zero:

$$\psi_m^{\pm}(0) = 0 \ \forall m \neq 0, \quad \psi_0^-(0) = f_2(0)^{\frac{1}{2}}.$$

- For a zero mode of wavelength λ in the z-direction the energy (m =0) behaves as

$$\mathcal{E}_\lambda = \frac{2\pi R}{4g^2\lambda^2}$$

The SUSY model and 1/2 BPS Alice string

- Since our Alice string is BPS it can be embedded in supersymmetric theories. In fact it can be shown that it preserves two super charges.

- We may write down an N=1 action as

$$\mathcal{I} = -\frac{1}{4}\Re\int d^4x d^2\theta\left(2\mathrm{Tr}\mathrm{W}^{\alpha}\mathrm{W}_{\alpha} + \mathrm{W}_0^{\alpha}\mathrm{W}_{\alpha}^0\right)$$

$$+\frac{1}{4}\int d^4x d^2\theta d^2\bar{\theta}\mathrm{Tr}\left(\Psi_+^\dagger \mathrm{e}^{-2\mathrm{eV}_0-2\mathrm{gV}}\Psi_+ + \Psi_-^\dagger \mathrm{e}^{2\mathrm{eV}_0-2\mathrm{gV}}\Psi_- - \xi^2\mathrm{V}_0\right)$$

- The auxiliary fields can be solved as

$$D_0 = 2\xi^2 - \mathrm{Tr}\left(\Phi_+^\dagger\Phi_+ - \Phi_-^\dagger\Phi_-\right), \quad \mathrm{D} = -\frac{1}{2}\sum_{\mathrm{i}=\pm}\left[\Phi_{\mathrm{i}}, \Phi_{\mathrm{i}}^\dagger\right], \quad \mathrm{F}^{\pm} = 0.$$

- If we set $\Phi_- = 0$ and $\Phi_+ = \Phi$, we then can recover the action in bosonic fields, for which the potential term is

$$V(\Phi) = \frac{g^2}{4}\mathrm{Tr}[\Phi, \Phi^\dagger]^2 + \frac{\mathrm{e}^2}{2}\left(\mathrm{Tr}\Phi\Phi^\dagger - 2\xi^2\right)^2.$$

The SUSY model and 1/2 BPS Alice string

- The SUSY transformations of the fermions can be written as

$$\delta_\epsilon \psi_\pm = i\sqrt{2}\sigma^\mu \bar{\epsilon}\mathcal{D}_\mu \Phi_\pm + \sqrt{2}\epsilon F_\pm,$$

$$\delta_\epsilon \lambda^a = \sigma^{\mu\nu}\epsilon F^a_{\mu\nu} + i\epsilon D^a,$$

$$\delta_\epsilon \lambda^0 = i\epsilon D_0 + \sigma^{\mu\nu}\epsilon f_{\mu\nu}$$

$$\Big\{ \left(D_1 + iD_2\right)\Phi_+ \Big\}\bar{\epsilon}^{\dot{1}} = \Big\{ \left(D_1 - iD_2\right)\Phi_+ \Big\}\bar{\epsilon}^{\dot{2}} = 0$$

$$\left(F_{12} + \frac{1}{2}\left[\Phi_+, \Phi_+^\dagger\right]\right)\epsilon_1 = \left(F_{12} - \frac{1}{2}\left[\Phi_+, \Phi_+^\dagger\right]\right)\epsilon_2 = 0,$$

$$\Big\{ f_{12} + e\left(\mathrm{Tr}\Phi_+\Phi_+^\dagger - 2\xi^2\right) \Big\}\epsilon_1 = \Big\{ f_{12} - e\left(\mathrm{Tr}\Phi_+\Phi_+^\dagger - 2\xi^2\right) \Big\}\epsilon_2 = 0.$$

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$$\delta_\epsilon \lambda^a = \sigma^{\mu\nu} \epsilon F^a_{\mu\nu} + i\epsilon D^a,$$
$$\delta_\epsilon \lambda^0 = i\epsilon D_0 + \sigma^{\mu\nu} \epsilon f_{\mu\nu}$$

$$\underline{\left\{ (D_1 + iD_2)\, \Phi_+ \right\}} \bar{\epsilon}^{\dot{1}} = \left\{ (D_1 - iD_2)\, \Phi_+ \right\} \bar{\epsilon}^{\dot{2}} = 0$$

$$\underline{\left(F_{12} + \frac{1}{2}\left[\Phi_+, \Phi_+^\dagger\right] \right)} \epsilon_1 = \left(F_{12} - \frac{1}{2}\left[\Phi_+, \Phi_+^\dagger\right] \right) \epsilon_2 = 0,$$

$$\underline{\left\{ f_{12} + e\left(\mathrm{Tr}\Phi_+\Phi_+^\dagger - 2\xi^2\right) \right\}} \epsilon_1 = \left\{ f_{12} - e\left(\mathrm{Tr}\Phi_+\Phi_+^\dagger - 2\xi^2\right) \right\} \epsilon_2 = 0.$$

Discussion

- The excitation of zero mode excite bulk gauge field which generates singularity. However. The singularity is expected to be logarithmic in general. However in the case of systems like us power law singularity is generated.

- However, in singular gauge the bulk system will have a brunch cut.   The effective action can be written as

$$\mathcal{I}_{2d-4d} = -\int d^4x \ \frac{1}{4} F_{\mu\nu}^2 + M_r^2 \int dt dz \left(\partial_\alpha \varphi + A_\alpha\right)^2$$

where M_r is the regularized mass and the field strength can be defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - 2\pi\zeta\Sigma_{\mu\nu}.$$

Obstruction parameter


Thank you