Instantons on stacky ALE spaces

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Topological solitons, nonperturbative gauge dynamics and confinement

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The resolved instanton moduli space

$$\begin{tikzcd}
\boxed{\text{Moduli space of framed vector bundles on } \mathbb{P}^2} & \mathcal{M}^{\text{loc.lib.}}(r,n) \arrow[l, leftsquigarrow] \arrow[r, hook] \arrow[d, equal, "\text{(Donaldson)}"] & \color{blue}{\mathcal{M}(r,n)} \arrow[r, rightsquigarrow] \arrow[d, "\pi"', "\text{(Nakajima)}"] & \boxed{\text{Moduli space of framed torsion-free sheaves on } \mathbb{P}^2 \ \color{red}{\text{(smooth)}}} \\
\boxed{\text{Moduli space of framed instantons on } S^4} & \color{red}{\mathcal{M}_0^{\text{reg}}(r,n)} \arrow[l, leftsquigarrow] \arrow[r, hook] & \mathcal{M}_0(r,n) \arrow[r, rightsquigarrow] & \boxed{\text{Moduli space of ideal instantons on } S^4 \ \color{red}{\text{(singular)}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\text{Instantons on } \mathbb{R}^4 \text{ with a gauge fixing at infinity}} \arrow[r, "\substack{\text{conformal compact. } + \\ \text{Uhlenbeck sing. removal}}"] & \boxed{\text{Instantons on } S^4 \text{ with a framing at one point}} \arrow[d, "\text{Atiyah-Ward}"] \\
& \boxed{\text{Framed vector bundles on } \mathbb{P}^2}
\end{tikzcd}$$

+ degenerations

Computing invariants

$\mathcal{M}(r, n)$ carries the action of an algebraic torus

$$\mathbb{T}^{2+r} = \mathbb{C}^* \times \cdots \times \mathbb{C}^* \quad (2 + r \text{ factors}) \begin{cases} 2 \text{ factors from the toric action on } \mathbb{P}^2 \\ r \text{ factors act on the framing} \end{cases}$$

with a finite number of isolated fixed points

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$$P_t(X) = \sum_{i=0}^{\dim X} (-1)^i b_i(X)\, t^i$$

$$\sum_{n=0}^{\infty} q^n \, P_t(\mathcal{M}(r,n)) = \prod_{b=1}^{r} \prod_{m=1}^{\infty} \frac{1}{1 - t^{(2r(m-1)+b-1)}\, q^m}$$

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No-go theorem: According to a result of Bando, if $\bar{X}$ is a smooth compactification of a manifold X obtained by adding a divisor D (with positive normal bundle), bundles on $\bar{X}$ framed on D correspond to instantons on X with trivial holonomy at infinity

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Way out: Let's go stacky!!

A premonition

U.B., R. Poghossian and A. Tanzini, *Poincaré polynomials of moduli spaces of framed sheaves on (stacky) Hirzebruch surfaces*, Comm. Math. Phys. **304** (2011) 395–409

where we studied framed sheaves on Hirzebruch surfaces $\mathbb{F}_p$ ($\mathbb{P}^1$-bundles over $\mathbb{P}^1$, compactifications of the total spaces X_p of the line bundles $\mathcal{O}(-n) \to \mathbb{P}^1$).

(NB X_2 is the ALE space A_1 - the Eguchi-Hanson gravitational instanton)

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In analyzing the Young tableaux associated with the fixed point of the torus action on the moduli space of framed sheaves, one finds tableaux that do not correspond to possible fixed points — they correspond to sheaves with "fractional Chern classes."

$$P_t(\widetilde{\mathfrak{M}}^p(r,k,n)) =$$

$$\sum_{\substack{\text{fixed}\\ \text{points}}} \prod_{\alpha=1}^{r} t^{2(|Y_\alpha|-l(Y_\alpha))} \prod_{i=1}^{\infty} \frac{t^{2(m_i^{(\alpha)}+1)}-1}{t^2-1} \prod_{\alpha<\beta} t^{2(l'_{\alpha,\beta}+|Y_\alpha|+|Y_\beta|-n'_{\alpha,\beta})}$$

Stacks

Loosely speaking, a stack is the algebraic version of an orbifold.

It retains information about the local isotropy groups. Example: a "football" (stacky $\mathbb{P}^1 = \mathbb{P}^1/\mathbb{Z}_2$)

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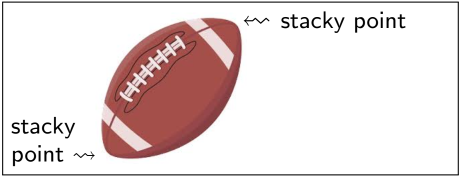

We shall introduce a root stack: given a smooth space X and a divisor D in it, we take in some sense "a k-th root of D"

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All (simplicial) toric varieties can be uniformized (i.e., written as a quotient Y/G) $\rightsquigarrow$ smooth toric stack $\mathscr{X}_k^{\mathrm{can}}$

$\rightarrow$ (NB - regarded as a stack, any quotient $[X/G]$ is smooth !!) $\leftarrow$

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Root construction: smooth stack $\mathscr{X}_k \to \mathscr{X}_k^{\mathrm{can}}$

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The compactification (stacky) divisor $\mathscr{D}_k$ has a rich structure and allows the encoding of nontrivial holonomies at infinity (indeed, Bando's no-go theorem does not apply!)

Moduli spaces of framed sheaves on projective 2-dimensional stacks were built in

U.B., F. Sala, *Framed sheaves on projective stacks.* Adv. Math. **272** (2015) 20–95.

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Gauge theory on these stacks was studied in

U.B., M. Pedrini, F. Sala and R. J. Szabo, *Framed sheaves on root stacks and supersymmetric gauge theories on ALE spaces,* Adv. Math. **288** (2016) 1175–1308

U.B., F. Sala and R. J. Szabo, $\mathcal{N}=2$ *quiver gauge theories on A-type ALE spaces,* Lett. Math. Phys. **105** (2015) 401–445.

(various sorts of partition functions à la Nekrasov, blowup formulas, prepotentials, ...)

Partition functions: $N = 2$ SYM on $\mathbb{R}^4$

$$Z^{\mathrm{inst}}_{\mathbb{C}^2,r}(\varepsilon_1, \varepsilon_2, \vec{a}; q) = \sum_{n=0}^{\infty} q^n \int_{\mathcal{M}(r,n)} \mathbf{1}$$

Partition functions for ALE spaces

$T_\mu = \mathbb{C}^*$, $H^*_{T_\mu}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}[\mu]$

$\boldsymbol{G}$ a T-equivariant vector bundle of rank n on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)$

$$\mathrm{E}_\mu(\boldsymbol{G}) := \mu^n + c_1(\boldsymbol{G})_T\,\mu^{n-1} + \cdots + c_n(\boldsymbol{G})_T \in H^*_{T\times T_\mu}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)\,;\,\mathbb{Q}\big)\ .$$

$\vec{v} \in \frac{1}{k}\,\mathbb{Z}^{k-1}$ s.t. $k\,v_{k-1} = \sum_{i=0}^{k-1} i\,w_i \mod k$ for fixed rank r and holonomy at infinity $\vec{w}$.

$$Z^*_{\vec{v}}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\ldots,\vec{t}^{\,(k-1)}\big) = \sum_{\Delta\in\frac{1}{2r\,k}\mathbb{Z}} \mathsf{q}^{\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}} \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)\big)$$
$$\cdot\ \exp\Big(\sum_{s=0}^{\infty}\Big(\sum_{i=1}^{k-1} t_s^{(i)}\,\big[\mathrm{ch}(\mathcal{E})_T/[\mathscr{D}_i]\big]_s + \tau_s\,\big[\mathrm{ch}(\mathcal{E})_T/[X_k]\big]_{s-1}\Big)\Big)\ .$$

(fixed rank, hololonomy at infinity and c_1, sum over Δ)

Comments and perspectives (as of 2016)

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- For $\mu \to 0$ we obtain the Vafa-Witten partition function for $\mathcal{N} = 4$ SYM on ALE spaces; we get the same result as Fuji-Minabe (they computed the Euler characteristic of the moduli space of $\mathbb{Z}_k$-invariant framed sheaves on $\mathbb{P}^2$)

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- Generalization: compute the Poincaré polynomials of our moduli spaces

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$$P_t(\mathscr{M}_{r,c,n}(\mathscr{X}_k, \mathscr{D}_\infty)) =$$

$$\sum_{(\vec{\Upsilon},\vec{\kappa})} \left[\prod_{\alpha=1}^{r} \left(\prod_{j\geq 1} \frac{t^{2(\nu_{\alpha,j}+1)} - 1}{t^2 - 1} \prod_{h=1}^{k-1} t^{2(|\Upsilon_\alpha^{(h)}| - \ell\Upsilon_\alpha^{(h)})} \right) \right]$$

$$\left[\prod_{\alpha<\beta} \prod_{h=1}^{r} t^{2(\sum_{d=0}^{1}(-1)^d \bar{\imath}_{\alpha\beta}^{(h),d} + |\Upsilon_\alpha^{(h)} + |\Upsilon_\beta^{(h)}| - n_{\alpha\beta}^{(h)})} \right]$$

(in A. Celotto, PhD thesis, SISSA 2017)

Congratulations Ken!!