Quantisation of Skyrmions

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Topological Solitons, Nonperturbative Gauge Dynamics and Confinement
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related to work in collaboration with
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Motivation

- The Skyrme model is a classical field theory which can be motivated from QCD by a $\frac{1}{N_c}$-expansion.
- The Skyrme models is built of pion fields, and has soliton solutions known as *Skyrmions* that are viewed as "classical" atomic nuclei.
- The first step is to understand static classical solutions.
- By restricting to minimal energy configurations we can perform a **zero-mode quantisation** (FR constraints).
- If we want to go beyond we need to know how Skyrmions deform when they vibrate and scatter.
- This leads to a **vibrational quantisation.**
- Further improvements are also expected by taking into account how Skyrmions deform when they are spinning and isospinning.

The Skyrme model

- The Skyrme energy can be written as $E = \int \mathcal{E}\, \mathrm{d}x^3$, with

$$\mathcal{E} = \sum_i \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} + \frac{1}{2} \sum_i (\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi})^2 - \frac{1}{2} \sum_{i,j} (\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi})^2 + 2m^2(1 - \phi_0)\,,$$

  where the vector

$$\boldsymbol{\phi} = \left(\phi^0, \phi^1, \phi^2, \phi^3\right)$$

  is a function of space $\mathbf{x}$.

- The vector $\boldsymbol{\phi}$ satisfies $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$ or in components $\phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$. Hence, the fields $\boldsymbol{\phi}$ parametrize a $3-$sphere.
- For $|\mathbf{x}|$ the vector $\boldsymbol{\phi}$ tends to $\boldsymbol{\phi} = (1, 0, 0, 0)$. So, we can do a "one-point compactification" and think of physical space $\mathbb{R}^3 \cup \{\infty\}$ as a $3-$sphere.

Skyrmions

- Mathematically, we have a map $S^3 \to S^3$ and the degree B of this map counts the number of solitons.
- These solitons are known as *Skyrmions*.
- How do we visualize them?
- The degree B can be calculated as

$$B = \int \mathcal{B} \, \mathrm{d}x^3,$$

  where $\mathcal{B}$ is the topological density.
- One option is to plot level sets of constant topological density $\mathcal{B}$.

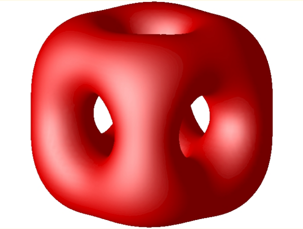


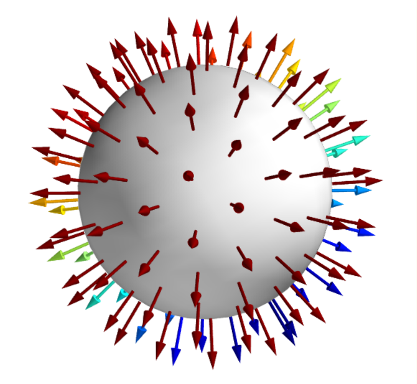
Figure : $B = 4$ Skyrmion

The $B = 1$ Skyrmion

- Spherically symmetric configuration

  $$\phi_0 = \cos f(r), \quad \phi_j = \sin f(r) x_j,$$

  where $r = |\mathbf{x}|$, $f(0) = \pi$ and $f(\infty) = 0$.
- Note if we rotate the hedgehog in space (x_j) and then rotate the ϕ_i in the right way, then can we get back where we started. The hedgehog is symmetric under rotations.

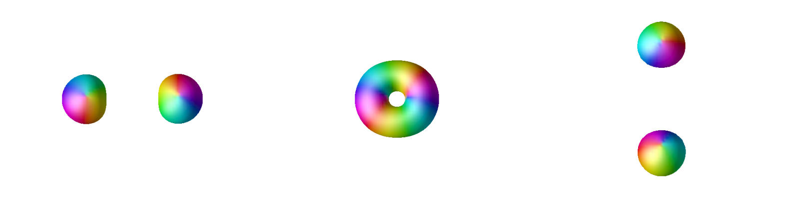


A colour scheme

- We want to label the direction of ϕ_j in terms of colours.
- The figures shows the hedgehog looking down the $z-$axis.
- Again the surface corresponds to $\mathcal{B} = const$. For the hedgehog, this is the surface of a sphere.
- The direction of ϕ_j can be expressed in polar coordinates (θ, φ). Then we can colour the surface according to value of φ along the colour circle:

  red, **yellow**, **green**, **cyan**, **blue**, **magenta**.

Classical Skyrmion scattering: Rotation without Rotating



- The initial and the final Skyrmions are oriented such that one Skyrmion is rotated in π which is known as the **attractive channel.**
- During scattering the Skyrmions do not rotate.
- However, after scattering the Skyrmions are rotated by $\frac{\pi}{2}$.

Quantisation

- In quantum field theory, there are two types of particles: **Bosons** and **Fermions**.
- When a **Boson** wavefunction is rotated by 2π, it remains invariant. However, if a **Fermion** wavefunction is rotated by 2π, then it changes by a factor of (-1).
- If two identical particles are exchanged then nothing happens to **Bosons**, whereas the wavefunction of the **Fermions** changes by a factor of (-1).
- In quantum field theory, **Bosons** are usually described by scalar, vector or tensor fields, whereas **Fermions** are represented by spinors.

Finkelstein-Rubinstein constraints

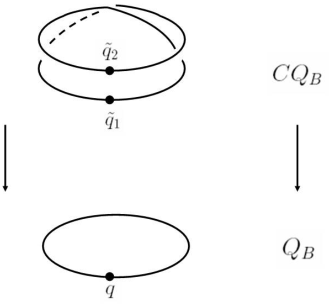
- Key observation:

$$\pi_1(Q_B) = \mathbb{Z}_2,$$

  where Q_B is the space of Skyrme configurations with charge B.
- Define wavefunctions ψ on the covering space of configuration space:

$$\psi : CQ_B \to \mathbb{C}.$$

- Impose $\psi(\tilde{q}_1) = -\psi(\tilde{q}_2)$.
- Symmetries of Skyrmions induce loops in configuration space.



Finkelstein-Rubinstein constraints

- Induced action of $SO(3) \times SO(3)$ symmetries on ψ :

$$\exp\left(-i\alpha\,\mathbf{n}\cdot\mathbf{L}\right)\exp\left(-i\beta\,\mathbf{N}\cdot\mathbf{K}\right)\psi(\tilde{q}) = \chi_{FR}\psi(\tilde{q}),$$

$$\text{where } \chi_{FR} = \begin{cases} 1 & \text{if the induced loop is contractible,} \\ -1 & \text{otherwise.} \end{cases}$$

- Here $\mathbf{L}$ and $\mathbf{K}$ are the body-fixed angular momentum operators in space and target space, respectively.

- Can we calculate $\chi_{FR} \in \pi_1(Q_B)$?

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- Can we calculate $\chi_{FR} \in \pi_1(Q_B)$?
- Yes, there is a simple formula:

$$\chi_{FR} = (-1)^{\mathcal{N}} \quad \text{where} \quad \mathcal{N} = \frac{B}{2\pi}\left(B\alpha - \beta\right).$$

(under some technical assumptions)

Rigid Body Quantisation — Key idea

- Calculate a minimal energy Skyrmion for a given charge B.
- Derive its symmmetries.
- Use Finkelstein-Rubinstein constraints to find allowed states with given spin J and isospin I.
- The energy of a state $|J\rangle|I\rangle$ can be calculated (roughly) via

$$E = \mathcal{M} + \frac{\hbar^2 J(J+1)}{2\Theta_J} + \frac{\hbar^2 I(I+1)}{2\Theta_I},$$

  where $\mathcal{M}$ is the classical mass of the Skyrmion, and Θ_J and Θ_I are spin and isospin moments of inertia.

Rigid Body Quantisation - Discussion

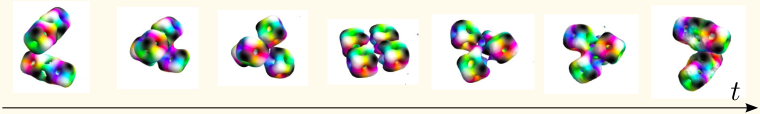
- This approach is successful for calculating ground states for small nuclei, for $B = 1$ to 4, and for most small nuclei with even B.
- More detailed studies also showed that excitation spectra can be reproduced fairly well, eg for ${}^{6}_{3}\mathrm{Li}$ and other small even nuclei.
- For ${}^{12}_{6}\mathrm{C}$ the ground state and the Hoyle state have been calculated from two different Skyrme configurations.
- There are predictions for ground and excited states when symmetries are imposed, e.g. for T_d and O_h symmetric Skyrmions.
- Electromagnetic form factors have been calculated.

Beyond Rigid Body Quantisation

- While there has been a lot of progress, there are also drawbacks. For example, the ground states of nuclei with odd B are not described very well at all. Furthermore, Skyrmions are generally too tightly bound.
- **Key idea:** Allow for deformations of Skyrmions during quantisation:
  - Classically, Skyrmions deform when they rotate or isorotate.
  - Classically, Skyrmions also deform when they vibrate or scatter.

Vibrational Quantisation

- Manton, Leese and Schroers quantisd the attractive channel of two $B = 1$ Skyrmions in the instanton approximation and calculated various properties of the deuteron ${}^{2}_{1}\mathrm{H}$.
- Halcrow quantised $B = 7$ and derived the correct ground state and a good match to excited states of ${}^{7}_{3}\mathrm{Li}$.
- Halcrow, King and Manton quantised a two-dimensional scattering space of $B = 8$ skyrmions to quantise ${}^{16}_{8}\mathrm{O}$.



Vibrational Quantisation II

- When we go beyond the zero mode quantization we need to construct a manifold N of Skyrme configuration parametrized by coordinates y_i.
- Then the Quantum Hamiltonian becomes

$$\hat{H} = -\frac{\hbar^2}{2}\triangle + V(y_i),$$

where the kinetic energy operator is the Laplace-Beltrami operator

$$\triangle = \det(g)^{-\frac{1}{2}}\partial_i\left(\det(g)^{\frac{1}{2}}g^{ij}\partial_j\right)$$

and g is the metric on N, and $V(y_i)$ is the potential on N.

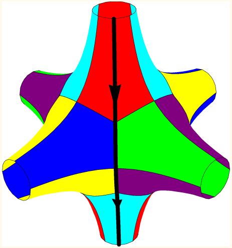
- For the $B = 2$ a natural choice of vibrational parameter is the separation of two $B = 1$ Skyrmions in the attractive channel.

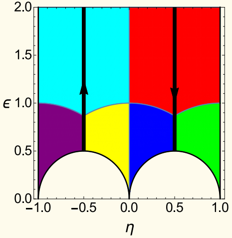
$^{16}_{8}$O quantisation

- For $^{16}_{8}$O the manifold N is $M \times SO(3)$, where M is the six punctured sphere.
- This can be mapped to the hyperbolic plane.
- Then the vibrational wave function can be found by solving

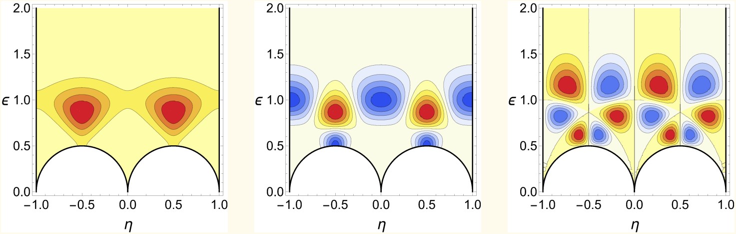
$$-\triangle_{\mathrm{vib.}}\phi + V\phi = (E - E_J)\phi,$$

and taking the relevant symmetries into account.



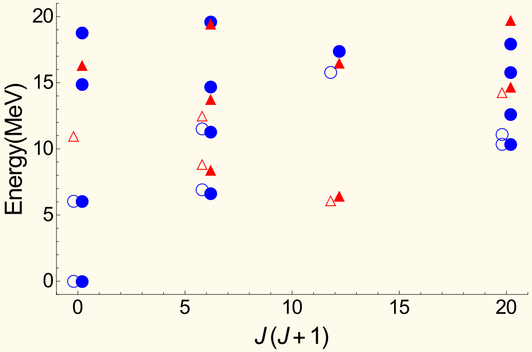


Some wave functions



- Here we show the ground state, the first excited state and the lowest state with negative parity.
- The wave function of the ground state is localised around the minimum energy solution (tetrahedron).
- The first excited state is localised around two different minima (tetrahedron and square).
- The negative parity state actually vanishes at the minimum energy configuration.

The energy spectrum of $^{16}_{8}O$



- This is energy spectrum (for $I = 0$) by Halcrow, King and Manton.
- The calculated states of positve/negative parity are displayed as solid circles/triangles, while hollow symbols correspond to the relevant experimentally observed states.

Isospinning Skyrmions

- If we want to understand how Skyrmions deform when isorotating in colour space, we need to minimize the total energy

$$E = \mathcal{M} + \frac{1}{2} U_{33}^{-1} K_3^2,$$

where K_3 is the conserved body-fixed isospin angular momentum and

$$\begin{aligned} U_{33} &= 2 \int \left(\phi_1^2 + \phi_2^2\right)\left(1 + \partial_k \phi_l \partial_k \phi_l\right) - \left(\phi_1 \partial_k \phi_2 - \phi_2 \partial_k \phi_1\right)^2 \mathrm{d}^3\mathbf{x}, \\ &= 2 I_{1,2} + 2 I_{1,2} \times e_2 - U_{33}^{\mathrm{cross}}. \end{aligned} \tag{1}$$

- To minimise E we need increase U_{33} while keeping Skyrmion mass $\mathcal{M}$ as small as possible.
- It can be shown that

$$\int_{\mathbb{R}^3} \left(\phi_1^2 + \phi_2^2\right) \mathcal{B} \, \mathrm{d}^3\mathbf{x} = B \int_{S^3} \left(\phi_1^2 + \phi_2^2\right) \mathrm{d}^3\boldsymbol{\phi} = \frac{B}{2},$$

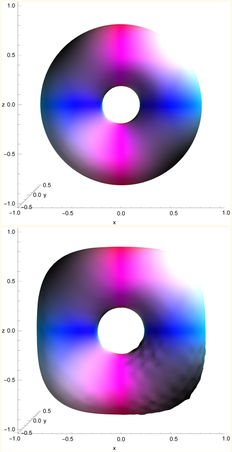
where $\mathcal{B}$ can be viewed as the Jacobian of the map ϕ.

Isospinning Skyrmions II

- We want to develop an intuition of how isospinning Skyrmions deform as we increase K_3.
- First, we amend our colouring scheme. The polar angle φ still represents the colour circle. Now, the angle θ measures how much "colour" there is, where $\theta \approx 0$ is **black**, $\theta \approx \frac{\pi}{2}$ is **colourful,** and $\theta \approx \pi$ is **white.**
- **Conjecture:** The main contribution comes from increasing the integral
$$I_{1,2} = \int \left(\phi_1^2 + \phi_2^2\right) \mathrm{d}^3\mathbf{x},$$
which corresponds to the amount of "colour" in space.
- However, since
$$\int_{\mathbb{R}^3} \left(\phi_1^2 + \phi_2^2\right) \mathcal{B} \, \mathrm{d}^3\mathbf{x} = \frac{B}{2}$$
we cannot freely, decrease the amount of black and white in space. Since $I_{1,2}$ does not depend on the baryon density, the baryon density will decrease in regions that are coloured and increase in regions that are black or white.

$B = 2$: Isospinning and colour

- Here we consider the $B = 2$ Skyrmion oriented such that the hole in the middle is blue/purple.
- For zero isospin, we have the familiar torus.
- For fast isospin the "colourful" hole in the middle expands.
- Black and white regions are around the equation, so baryon density moves there, flattening the torus slightly.
- Finally, the baryon density increases around black and white regions, breaking axial symmetry to D_4.

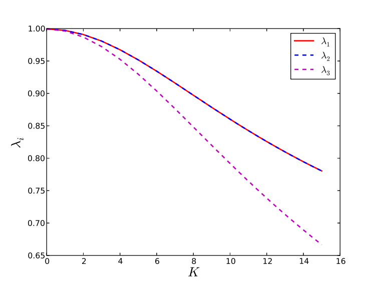


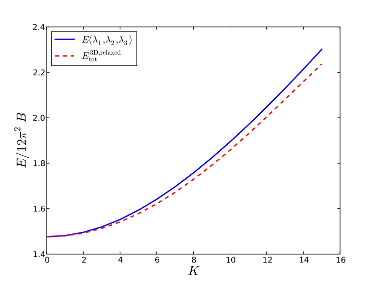
$B = 2$: Isospinning and rescaling

- The flattening of the torus can be captured by rescaling in space, namely,

$$x_1 \mapsto \lambda_1 x_1, \quad x_2 \mapsto \lambda_2 x_2, \quad x_3 \mapsto \lambda_3 x_3.$$

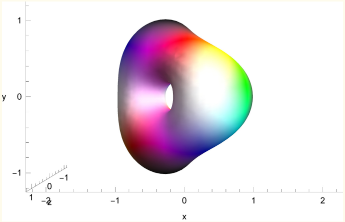
- The top graph shows that a rescaling to flatten the torus decreases the energy.
- The bottom graph shows how well the rescaling approximates the energy of the numeric calculation.
- The "squaring up" is not captured by rescaling. It has only a small effect on the energy.

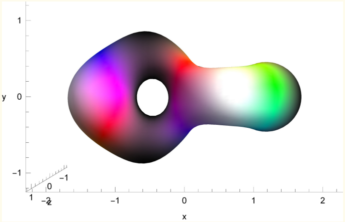




$B = 3$: Isospinning and colour

- For $B = 3$ we consider a tetrahedron, where two corners are white/black, with two opposing edge also white/black.
- Initially, $\phi_1^2 = \phi_2^2 = \phi_3^2$ in the hole.
- The triangle distorts such ϕ_3^2 in the hole decreases.
- The white edge takes baryon density away from the corners that connects it, and moves away from the hole. The same happens for the black edge.
- The four "colourful" edges move towards the hole.
- Thereby, the two triangular faces with a white or black edge transform into two joined kites.



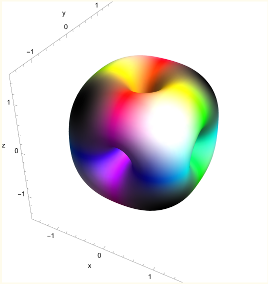


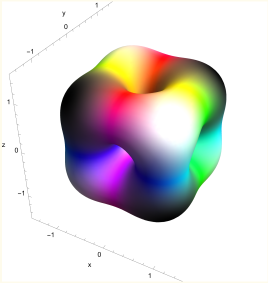
Conclusion

- The Skyrme model is a classical model for the strong interaction which needs to be quantised to make predictions in nuclear physics.
- The simplest approximation is the zero-mode quantisation which calculates states based on a single minimal energy configuration and its zero-modes.
- Currently there are various approaches to improve the predictions of the Skyrme model:
  - Modify the Skyrme model such that the classical energies are more degenerate, i.e. closer to the energies of atomic nuclei (near-BPS models, alternative mass terms,...)
  - Vibrational quantisation
  - Spinning and **isospinning** Skyrmions
- We discussed vibrational quantization.
- We also discussed how to obtain a better understanding of isospinning Skyrmions.

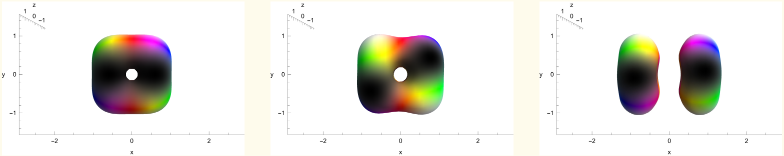
$B = 4$: White/Black Vertices

- For $B = 4$ we consider different cases.
- First, the vertices are black/white, and are already the highest density points. The hole of the cube has $\phi_3 = 0$.
- The density flows further to the vertices, the hole expands.
- This produces a "pointier" and larger cube.

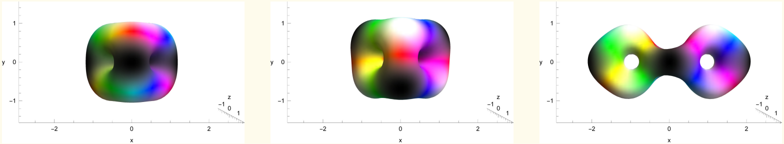




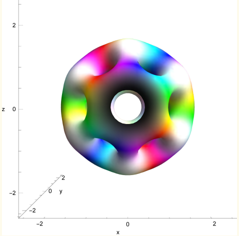
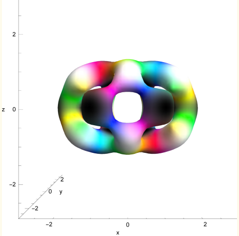
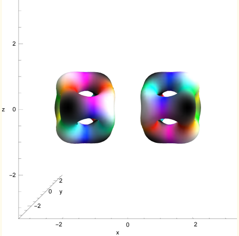
$B = 4$: Different colours



- Here are two examples where the cube splits into two tori, for two different colourings.
- Note that for fixed K_3, both have higher energy than the previous colouring.



$B = 8$: Different break-up configurations

- Here are two different colourings for $B = 8$ which split up into 2 cubes and 4 tori, respectively.
- All $B = 8$ configurations that we have considered split up.
- It is difficult to tell numerically, which one has lower energy.