Theta Dependence in QCD

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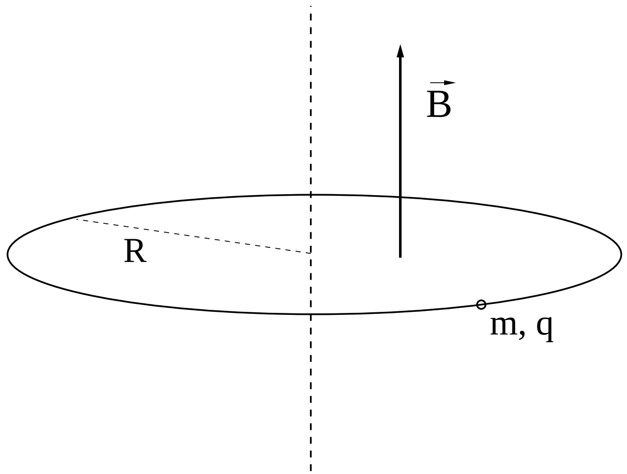
Topological Solitons, Nonperturbative Gauge Dynamics and Confinement - KenFest!

The free particle on a circle

alias: the simplest QM problem with non-trivial topological structure and numerical challenges

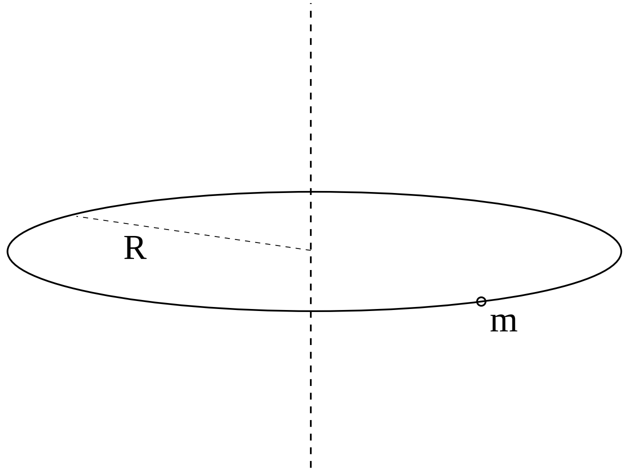


We will consider the path integral formulation for a free particle constrained on a circle of radius R

with and without a uniform magnetic field orthogonal to the circle

- This is a great example, where basic issues concerning topology and θ-dependence in gauge theories can be discussed in a simplified framework

- Even if everything is analitically computable here, as we try to study it by Monte-Carlo simulations, we face the same problems and failures as in QCD



In the standard approach Z is written a sum over energy/angular momentum eigenstates

$$Z = \sum_{n=-\infty}^{\infty} \exp\left(-\beta \frac{\hbar^2 n^2}{2mR^2}\right)$$

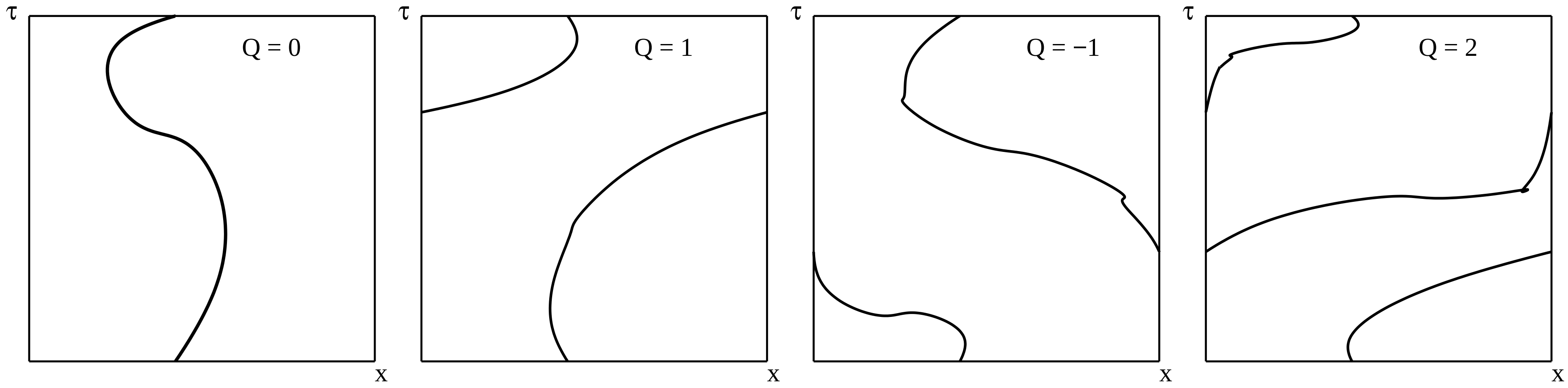
in the path integral approach

$$Z = \mathcal{N} \int_{x(0)=x(\beta\hbar)} \mathcal{D}x(\tau) \exp\left(\frac{-S_E[x(\tau)]}{\hbar}\right) \quad ; \quad S_E[x(\tau)] = \int_0^{\beta\hbar} d\tau \, \frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2$$

New feature: paths divide in topological classes

Boundary conditions in space $\Longrightarrow$ each path $x(\tau)$ contributing to Z is a continuous application from the temporal circle to the spatial circle.

how many times does the path wind around the circle before closing in eucl. time?



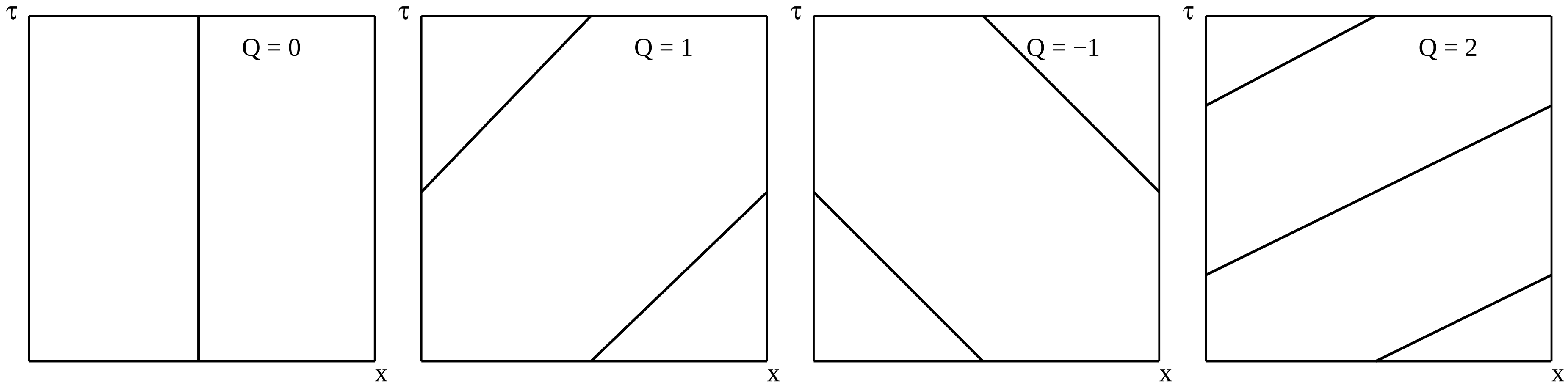
Paths are divided in homotopy classes according to their winding number Q which cannot be changed without cutting the path.

On the other hand, discontinuous paths have zero measure in the path integral

Wiener measure: first derivative divergent, but continuity is guaranteeed

The homotopy group is $\pi_1(S^1) = \mathbb{Z}$

Can we compute the contribution of each topological sector to the path integral? YES



- The path integral over each sector can be done by integrating over classical solutions, which are minima of the Euclidean action

- In this simple case the integration can be done exactly, yielding a result proportional to $\exp(-S_Q/\hbar)$ where S_Q is the action of the classical path

$$S_Q = \frac{1}{2} m \frac{(2\pi R Q)^2}{\beta\hbar}$$

- We have therefore an expression for the weight of each sector, which is nothing but the probability distribution $P(Q)$ over the winding number Q

$$P(Q) \propto \exp\left(-\frac{Q^2}{2\beta\hbar\chi}\right) \quad ; \quad \chi \equiv \frac{\hbar}{4\pi^2 m R^2}$$

Low and high T limits

$$Z = \sum_{n=-\infty}^{\infty} \exp\left(-\beta \frac{\hbar^2 n^2}{2mR^2}\right) \propto \sum_{Q=-\infty}^{\infty} \exp\left(-\frac{1}{\beta}\frac{Q^2}{2\hbar\chi}\right) = \sum_{Q=-\infty}^{\infty} \exp\left(-\frac{1}{\beta}\frac{2\pi^2 mR^2Q^2}{\hbar^2}\right)$$

the partition function can be written in terms of two different series, which are sort of dual to each other (β vs $1/\beta$ in the exponential)

- **low T (ground state physics)** ($\beta\hbar^2/(mR^2) \sim \hbar\beta\chi \gg 1$)

  - only lowest energy levels (lowest $|n|$) contribute

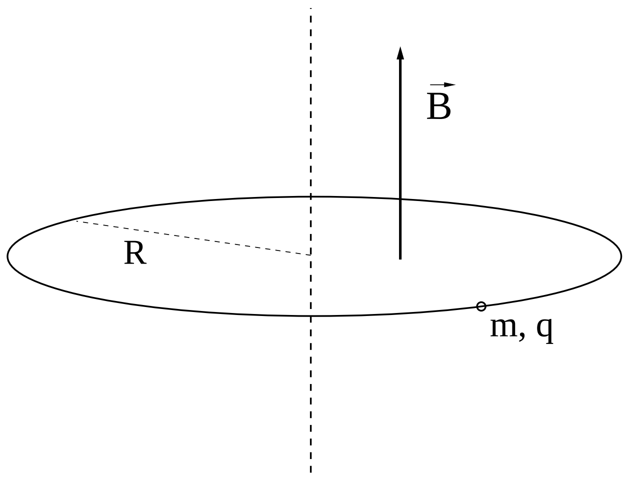
  - all Q values contribute and they are $\sim$ Gaussian distributed with variance $\sigma = \hbar\beta\chi$

- **high T:** ($\beta\hbar^2/(mR^2) \sim \hbar\beta\chi \ll 1$)

  - all energy levels n contribute, they are $\sim$ Gaussian distributed with variance $\sigma = 1/(4\pi^2\beta\hbar\chi)$

  - only lowest winding numbers contribute

And now the magnetic field, alias the θ-term



A uniform magnetic field across the circle implies a tangential gauge potential $A = BR/2$, hence

$$L = \frac{1}{2}mv^2 + qAv = \frac{1}{2}mv^2 + \frac{qBR}{2}v$$

while the energy levels change into

$$E_n = \frac{(\hbar^2 n^2 - qB^2R^4/2)^2}{(2mR^2}$$

In the Euclidean path integral formalism ($t \to -i\tau$) that amounts to adding the following term to the Euclidean action S_E:

$$i\frac{qBR}{2\hbar}\int_0^{\beta\hbar} d\tau \frac{dx}{d\tau} = i\frac{qBR}{2\hbar}\, 2\pi R Q = i\theta Q \quad ; \quad \theta \equiv \frac{\pi q B R^2}{\hbar}$$

Notice: a total derivative in the Lagrangian, does not change the classical equations, but leads to a global contribution which is constant in each topological sector

How the partition function changes

$$Z = \sum_{n=-\infty}^{\infty} e^{-\beta E_n} = \sum_{n=-\infty}^{\infty} e^{-\frac{\beta\hbar\chi}{2}(2\pi n-\theta)^2} = \mathcal{N}\int \mathcal{D}x(\tau) e^{-S_E[x(\tau)]/\hbar} e^{i\theta Q} \propto \sum_{Q=-\infty}^{\infty} e^{-\frac{1}{2\hbar\chi\beta}Q^2} e^{i\theta Q}$$

- **θ-dependence of the free energy $F(\theta) = -\log Z(\theta)/\beta$ here is related to the magnetic properties of the system. General features:**
  - $F(\theta+2\pi) = F(\theta)$ (**θ is an angular variable**) ; $F(-\theta) = F(\theta)$
  - $Z(\theta) \leq Z(0) \implies F(\theta) \geq F(0)$ **(sort of Vafa-Witten theorem)** $\implies$ **diamagnetism**

- **in the path integral formalism, a complex weight appears, which hinders the application of Monte-Carlo simulations. This is usually known as the sign problem.**

- **It afflicts other theories with a topological term. Here, it disappears when resumming Z in terms of other variables (n): such a rewriting is still a mirage in other cases**
  Then, how to investigate θ-dependence in the path-integral approach?

Taylor expansion approach

$$F(\theta) - F(0) = \frac{1}{2}F^{(2)}\theta^2 + \frac{1}{4!}F^{(4)}\theta^4 + \ldots \quad ; \qquad F^{(2n)} = \left.\frac{d^{2n}F}{d\theta^{2n}}\right|_{\theta=0}$$

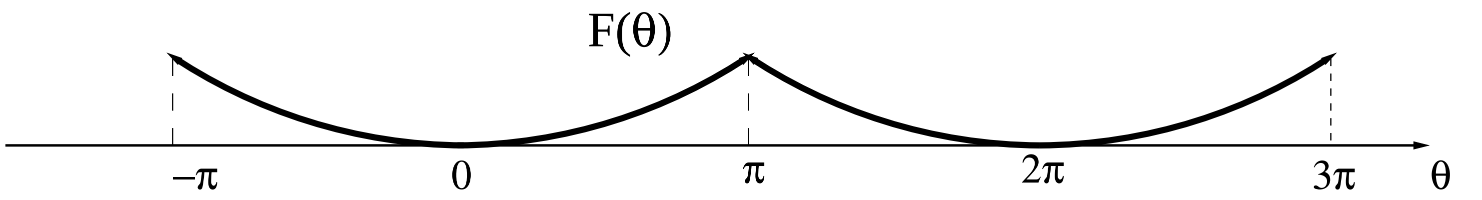
Taylor coefficients: cumulants of the Q distribution $P(Q) \propto e^{-Q^2/(2\hbar\beta\chi)}$ at $\theta = 0$

$$F^{(2)} = \frac{\langle Q^2\rangle_c}{\beta\hbar} = \frac{\langle Q^2\rangle - \langle Q\rangle^2}{\beta\hbar} \; ; \quad F^{(4)} = -\frac{\langle Q^4\rangle_c}{\beta\hbar} = -\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\beta\hbar} \; ; \quad F^{(2n)} = -(-1)^n\frac{\langle Q^{2n}\rangle_c}{\beta\hbar}$$

as $\hbar\beta\chi \to \infty$ (vacuum), Q is purely Gaussian, only $F^{(2)} \neq 0$ (topological susceptibility)

$$F(\theta) - F(0) = \frac{\chi}{2}\theta^2$$

that, when combined with the expected periodicity and symmetries, gives rise to a multibranched function with quantum phase transitions at $\theta = \pi$ or odd multiples of it



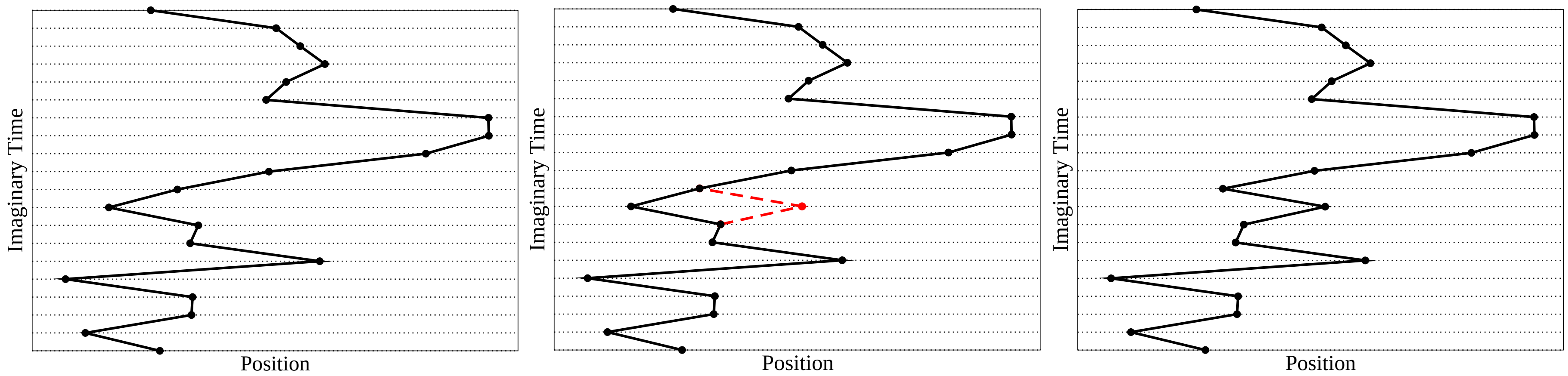
In terms of energy levels, at π we have a level crossing associated with the quantum phase transition, which disappears as soon as $T \neq 0$

In the opposite, hight T limit, $\hbar\beta\chi \ll 1$, only the lowest topological sectors contribute, taking just $Q = 0, 1, -1$:

$$Z(\theta) \propto 1 + 2e^{-1/(2\hbar\beta\chi)} \cos\theta \implies F(\theta) - F(0) \simeq -\frac{2}{\beta} e^{-1/(2\hbar\beta\chi)} \cos\theta$$

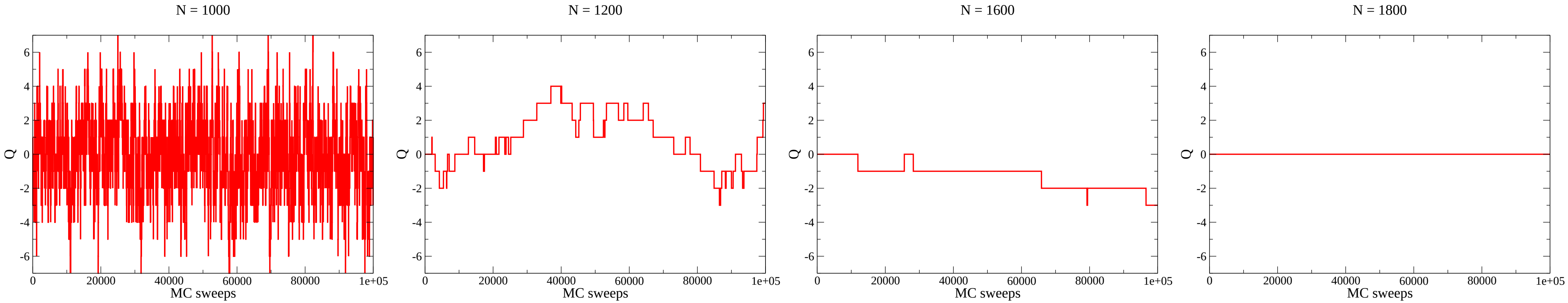
i.e. a smooth, periodic behavior in θ

Problems in numerical sampling of topological modes



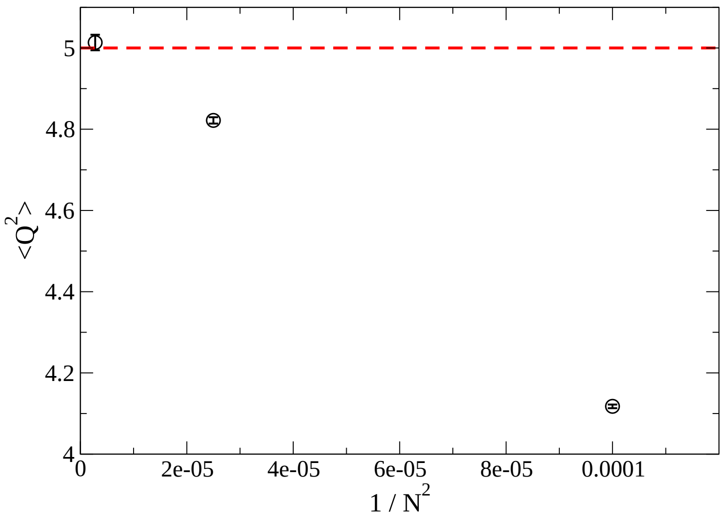
Apart from the sign problem, the study of θ-dependence faces other numerical challenges:

- **standard algorithms update the path configuration by small local steps**
- **in the continuum limit, paths evolve *almost continuously* in configuration space**
- **how is it possible, in this limit, to change topological sector? One should move across unlikely "discontinuous paths", which in the continuum limit have zero measure.**
- **standard algorithms become slower and slower in moving from one topological sector to the other, until they become completely non-ergodic.**



Set of MC histories (100K sweeps, Metropolis algorithm), obtained at fixed $\hbar\beta\chi = 5$ varying the number of temporal slices N

Critical slowing down proceeds fast towards complete freezing of topological modes



Luckily enough, in this case numerical results approach the continuum limit quite earlier (finest point $N = 600$)

Back to QCD

QCD is a bit less trivial than a particle on a circle, its Euclidean action reads:

$$S_{QCD} = \int d^4x \, \mathcal{L}_{QCD} = \int d^4x \left(\sum_f \bar{\psi}_f \left(D_\mu \gamma_\mu + m_f \right) \psi_f + \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} \right)$$

Also in this case relevant gauge field configurations divide in homotopy classes, characterized by an integer winding number $Q = \int d^4x \; q(x)$

$$q(x) = \frac{g^2}{64\pi^2} G^a_{\mu\nu}(x) \tilde{G}^a_{\mu\nu}(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu}(x) G^a_{\rho\sigma}(x)$$

$$GG \propto \vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a \;\; ; \quad G\tilde{G} \propto \vec{E}^a \cdot \vec{B}^a$$

Homotopy group: $\pi_3(SU(3)) = \mathbb{Z}$ **(actually,** $\pi_3(SU(N_c)) = \pi_3(SU(2)) \; \forall \; N_c$**)**

Classical solutions with non-trivial winding around the gauge group: instantons

characterized by various parameters: position, radiud ρ, ...

Effective action known only perturbatively. The 1-loop one-instanton contribution is

$$\exp\left(-\frac{8\pi^2}{g^2(\rho)}\right)$$

where $g(\rho)$ is the running coupling at the instanton scale ρ.

- **by asymptotic freedom, works well for small instantons, which are then exponentially suppressed, implying the validity of a dilute instanton gas approximation (DIGA)**

- **however, perturbation theory breaks down for large instantons ($1/\rho \lesssim \Lambda_{QCD}$), which become dominant, overlap with each other, and break DIGA**

QCD at non-zero θ

Also in this case, we can modify the theory introducing a θ-parameter coupled to Q:

$$Z(\theta) = \int [\mathcal{D}A][\mathcal{D}\bar{\psi}][\mathcal{D}\psi]\, e^{-S_{QCD}}\, e^{i\theta Q}$$

θ is a super-selection parameter: different θ, different Hilbert space

$G\tilde{G}$ is renormalizable, the theory at $\theta \neq 0$ is well defined, but presents explicit breaking of CP symmetry ($\tilde{G} \propto \vec{E} \cdot \vec{B}$)

As for the 1D-model, the free energy density $F(\theta) = -T \log Z/V$ is a periodic even function of θ, $F(\theta) \geq F(0)$, which can be expanded around $\theta = 0$ (assuming analyticity)

$$F(\theta) - F(0) = \frac{1}{2} F^{(2)} \theta^2 + \frac{1}{4!} F^{(4)} \theta^4 + \ldots \quad ; \qquad F^{(2n)} = \left.\frac{d^{2n} F}{d\theta^{2n}}\right|_{\theta=0} = -(-1)^n \frac{\langle Q^{2n} \rangle_c}{V_4}$$

$V_4 = V/T$ **is the 4D volume**

A common parametrization

$$F(\theta,T) - F(0,T) = \frac{1}{2}\chi(T)\theta^2\left[1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots\right]$$

$$\chi = \frac{1}{V_4}\langle Q^2\rangle_0 = F^{(2)} \qquad b_2 = -\frac{\langle Q^4\rangle_0 - 3\langle Q^2\rangle_0^2}{12\langle Q^2\rangle_0}$$

$$b_4 = \frac{\langle Q^6\rangle_0 - 15\langle Q^2\rangle_0\langle Q^4\rangle_0 + 30\langle Q^2\rangle_0^3}{360\langle Q^2\rangle_0}$$

The probability distribution $P(Q)$ of the different topological sectors now is not known: it is a non-perturbative property of QCD

Coefficients b_{2n} parametrize deviations of the distribution of topological charge from a Gaussian in the theory at $\theta = 0$.

A substantial difference with respect to the toy-model: we have fermions around

An axial $U(1)_A$ rotation of the fermion fields move θ from the gluon to the quark sector (same concept as for the axial anomaly). For any flavor:

$$\psi_f \to e^{i\alpha\gamma_5}\psi_f \qquad \text{and} \qquad \bar{\psi}_f \to \bar{\psi}_f e^{i\alpha\gamma_5}$$

$$\Longrightarrow \quad \theta \to \theta - 2\alpha \qquad \text{and} \qquad m_f \to m_f e^{2i\alpha}$$

- **should any quark be massless (this is not the case), θ could be rotated away and θ-dependence would be trivial**

- **in the presence of light quarks (this is the case), θ-dependence can be reliably studied within the framework of chiral perturbation theory (χPT)**

Experimental bounds on the electric dipole of the moment set stringent limits to the amount of CP-violation in strong interactions.

$$|\theta| \lesssim 10^{-10}$$

So: why do we bother with θ-dependence at all?

- **θ-dependence $\longleftrightarrow P(Q)$ at $\theta = 0$ $\quad\Longrightarrow$ it enters phenomenology anyway.**
  e.g., Witten-Veneziano mechanism: $\chi^{YM} = f_\pi^2 m_{\eta'}^2/(2N_f)$

- **Strong CP-problem: why is $\theta = 0$?** $\qquad m_f = 0$ **is ruled out.**
  A possible mechanism (Peccei-Quinn) invokes the existence of a new scalar field (axion) whose properties are largely fixed by θ-dependence

- **Axions are popular dark matter candidates, so the issue is particularly important**

The QCD axion

Main idea: add a new scalar field a, with only derivative terms acquiring a VEV $\langle a\rangle$ and coupling to the topological charge density. Low energy effective lagrangian:

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \frac{1}{2}\partial_\mu a \partial^\mu a + \left(\theta + \frac{a(x)}{f_a}\right)\frac{g^2}{32\pi^2}G\tilde{G} + \dots$$

- a is the Goldstone boson of a spontaneously broken (Peccei-Quinn) $U(1)$ axial symmetry (various high energy models exist)
- coupling to $G\tilde{G}$ involves the decay constant f_a, supposed to be very large
- shifting $\langle a\rangle$ shifts θ by $\langle a\rangle/f_a$. However θ-dependence of QCD breaks global shift symmetry on $\theta_{eff} = \theta + \langle a\rangle/f_a$, and the system selects $\langle a\rangle$ so that $\theta_{eff} = 0$.
- Assuming f_a very large, a is quasi-static and its effective couplings (mass, interaction terms) are fixed by QCD θ-dependence. For instance

$$m_a^2(T) = \frac{\chi(T)}{f_a^2} = \frac{\langle Q^2\rangle_{T,\theta=0}}{V_4 f_a^2}$$

knowing $F(\theta, T)$ fixes axion parameters during the Universe evolution

Predictions about θ-dependence - I

Dilute Instanton Gas Approximation (DIGA) for high T (Gross, Pisarski, Yaffe 1981)

- **instantons - antiinstantons treated as uncorrelated (non-interacting) objects**

  Poisson distribution with an average probability density p per unit volume

$$Z_\theta \simeq \sum \frac{1}{n_+! n_-!} (V_4 p)^{n_+ + n_-} e^{i\theta(n_+ - n_-)} = \exp\left[2 V_4 p \cos\theta\right]$$

$$F(\theta, T) - F(0, T) \simeq \chi(T)(1 - \cos\theta) \quad \Longrightarrow \quad b_2 = -1/12\,; \quad b_4 = 1/360\,; \ldots$$

- **Instantons of size $\rho \gg 1/T$ suppressed by thermal fluctuations, for high T instantons of effective perturbative action $8\pi/g^2(T)$ dominate. Including also leading order suppression due to light fermions and zero modes:**

$$\chi(T) \sim T^4 \left(\frac{m}{T}\right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4 - \frac{11}{3}N_c - \frac{1}{3}N_f} \propto T^{-7.66} \quad (\text{for } N_f = 2)$$

Notice: perturbative limit implies diluteness, hence DIGA, however DIGA might be good before reaching the asymptotic perturbative behavior

Predictions about θ-dependence - II

Chiral Perturbation Theory (χPT) for low T

At low T, perturbation theory breaks down, however, by $U(1)$ axial rotations, θ can be moved to the light quark masses. Then, χPT can be applied as usual.

Result for the ground state energy (Di Vecchia, Veneziano 1980)

$$E_0(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2 \frac{\theta}{2}}$$

$$\chi = \frac{z}{(1+z)^2} m_\pi^2 f_\pi^2, \quad b_2 = -\frac{1}{12} \frac{1+z^3}{(1+z)^3}, \quad z = \frac{m_u}{m_d}$$

Explicitly

$$z = 0.48(3) \qquad \chi^{1/4} = 75.5(5)\,\mathrm{MeV} \qquad b_2 = -0.029(2)$$

$$z = 1 \qquad \chi^{1/4} = 77.8(4)\,\mathrm{MeV} \qquad b_2 = -0.022(1)$$

$$\Longrightarrow \quad m_a \sim 10^{-5} \left(\frac{10^{12}\mathrm{GeV}}{f_a} \right)$$

Predictions about θ-dependence - III

Large-N_c for low T $SU(N_c)$ gauge theories (Witten, 1980)

$$L_{QCD}(\theta) = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

$g^2 N_c = \lambda$ **is kept fixed as** $N_c \to \infty$ $\Longrightarrow$ **if any non-trivial dependence on** θ **exist in the large-**N_c **limit, the dependence must be on** $\bar\theta = \theta / N_c$**.**

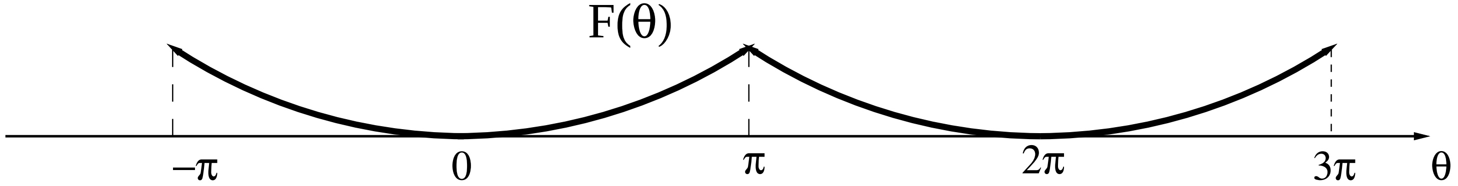
$$F(\theta, T) - F(0, T) = N_c^2 \bar F(\bar\theta, T)$$

$$\bar F(\bar\theta, T) = \frac{1}{2} \bar\chi \bar\theta^2 \left[1 + \bar b_2 \bar\theta^2 + \bar b_4 \bar\theta^4 + \cdots \right]$$

Matching powers of $\bar\theta$ **and** θ **we obtain**

$$\chi \sim N_c^0 \ ; \quad b_2 \sim N_c^{-2} \ ; \quad b_{2n} \sim N_c^{-2n}$$

$P(Q)$ **is Gaussian in the large** N_c**, as the toy model. Periodicity in** θ **enforces a multibranched structure with phase transitions at** $\theta = (2k+1)\pi$**.**



Numerical Results from Lattice QCD

main technical and numerical issues

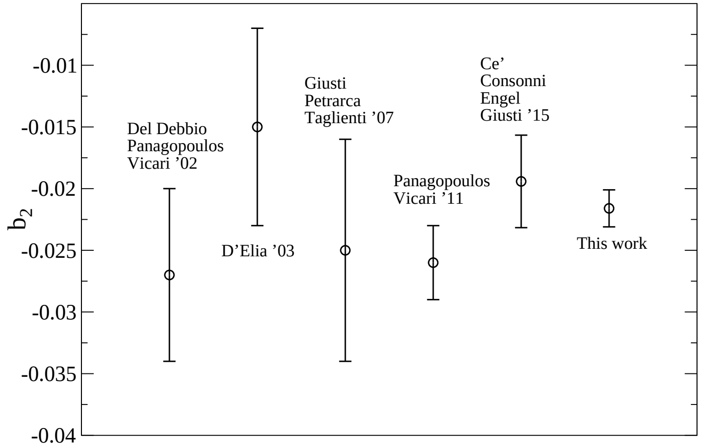
- topological charge renormalizes, naive lattice discretizations are non-integer valued. Various methods devised leading to consistent results
  - field theoretic compute renormalization constants and subtract
  - fermionic definitions use the index theorem to deduce Q from fermionic zero modes
  - smoothing methods use various techniques to smooth gauge fields and recover integer Q

- Determination of higher cumulants is numerically challenging: need to detect deviations from a Gaussian, but as $V_4 \to \infty$ Gaussian modes dominate.

- Freezing of topological modes in the continuum

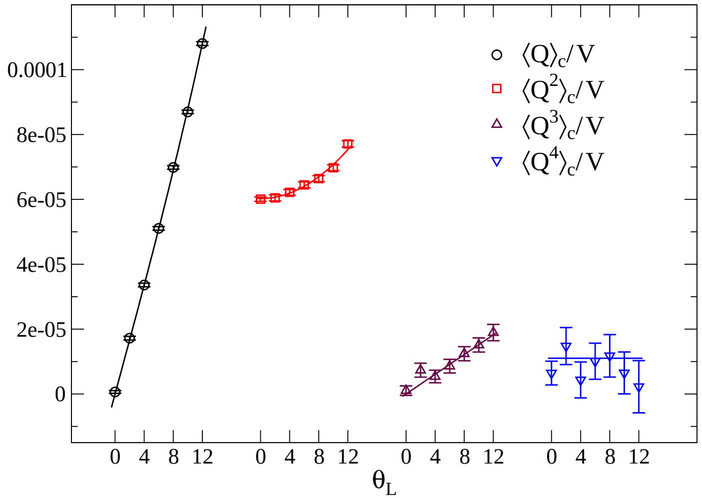
Pure gauge results: $T = 0$ (Yang-Mills vacuum)

Topological susceptibility well known, with increasing refinement, since 20 years, and compatible with the Witten-Veneziano mechanism for $m_{\eta'}$, $\chi^{1/4} \sim 180$ MeV



Determination of b_2 more difficult. Most recent determination for $SU(3)$ (Bonati, D'Elia, Scapellato, 1512.01544) obtained by introducing an external imaginary θ source to improve signal/noise.

Introduce an imaginary $\theta = i\theta_L$ in the lattice action, then perform a global fit to the first four cumulants:

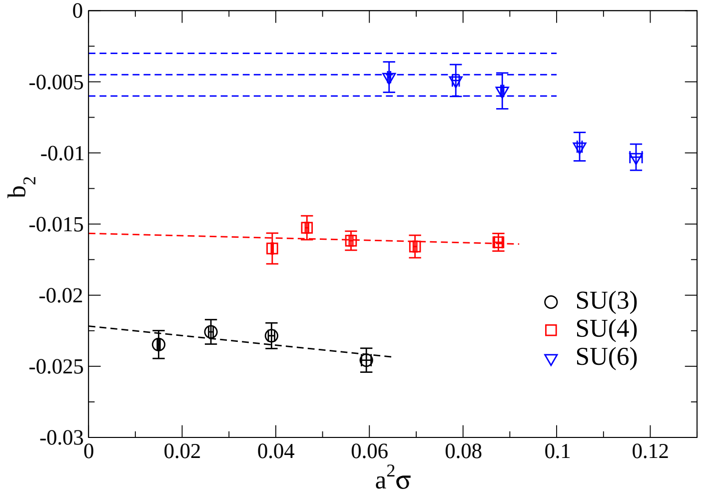


$$\frac{\langle Q\rangle}{\mathcal{V}} = \chi Z\theta_L(1 - 2b_2 Z^2\theta_L^2 + 3b_4 Z^4\theta_L^4 + \dots)\,,$$

$$\frac{\langle Q^2\rangle_c}{\mathcal{V}} = \chi(1 - 6b_2 Z^2\theta_L^2 + 15b_4 Z^4\theta_L^4 + \dots)\,,$$

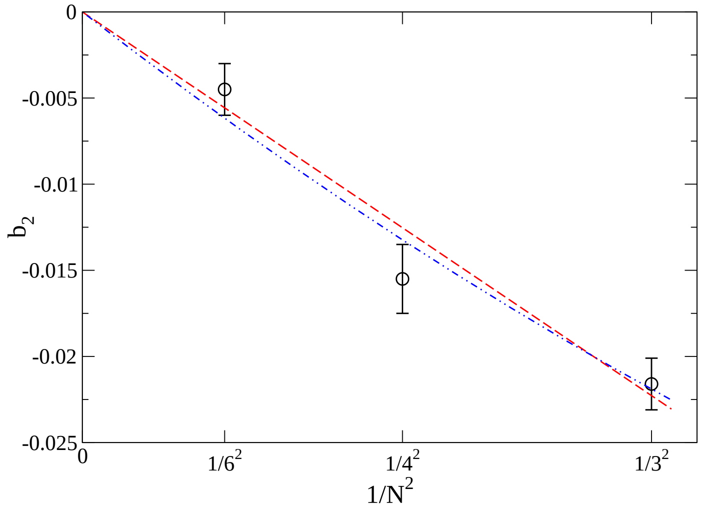
$$\frac{\langle Q^3\rangle_c}{\mathcal{V}} = \chi(-12b_2 Z\theta_L + 60b_4 Z^3\theta_L^3 + \dots)\,,$$

$$\frac{\langle Q^4\rangle_c}{\mathcal{V}} = \chi(-12b_2 + 180b_4 Z^2\theta_L^2 + \dots)\,.$$



By these means, we have recently been able to determine the scaling of b_2 to the large N limit.

(Bonati, D'Elia, Rossi, Vicari, 1607.06360)



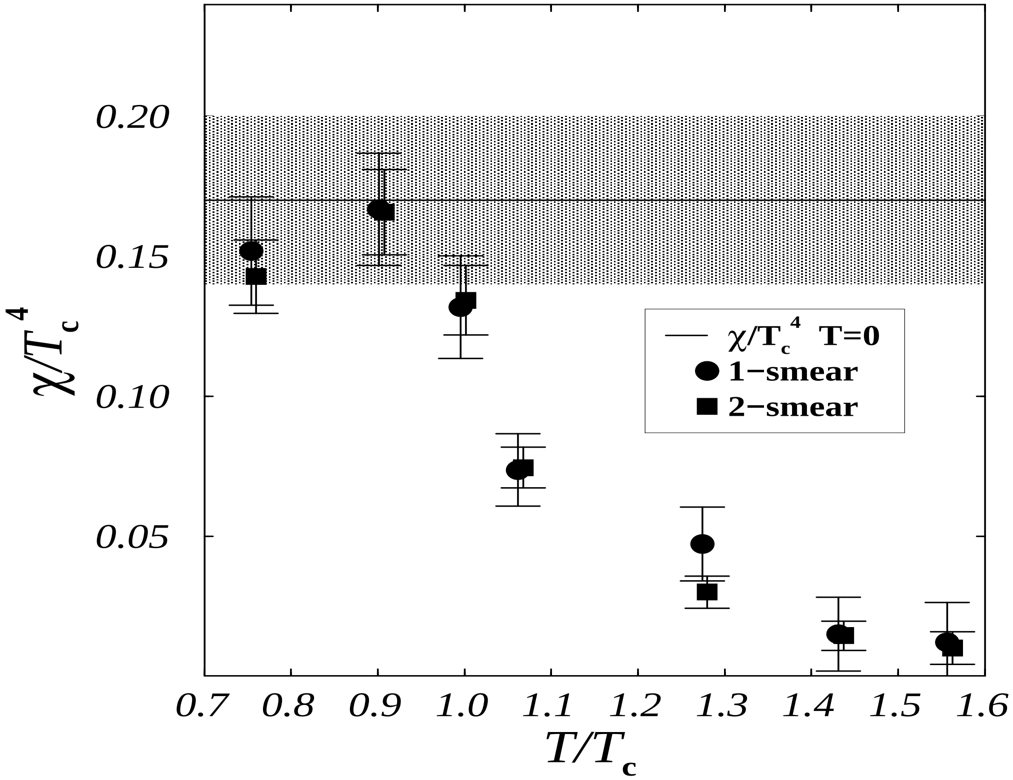
Clear evidence for the predicted large-N_c scaling of b_2:

$$b_2 \simeq \frac{\bar{b}_2}{N^2}$$

with $\bar{b}_2 = -0.20(2)$

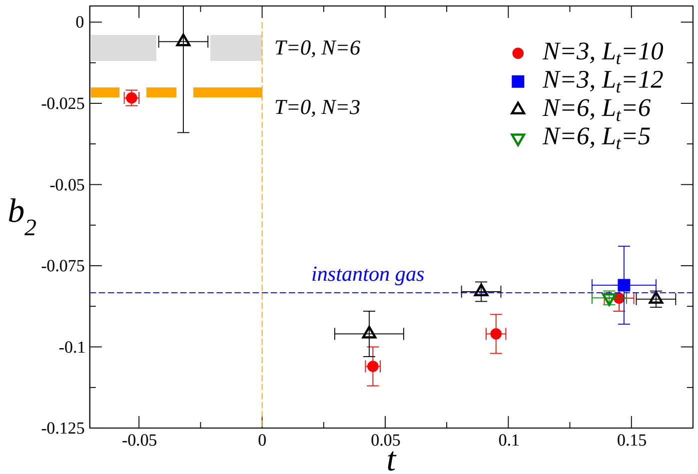
Pure gauge results: Finite T, across and above T_c

Topological activity stays almost unchanged till T_c and then χ drops suddenly: known since 20 years



B. Alles, MD, Di Giacomo, hep-lat/9605013 and my PhD Thesis

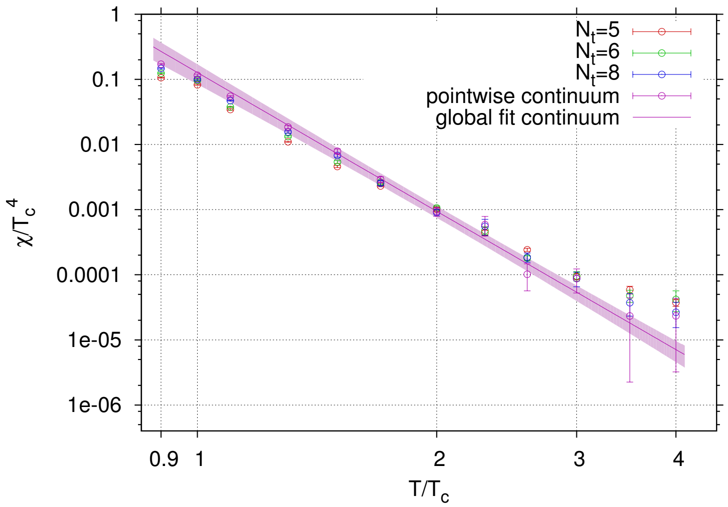
we had some recent progress:



from Bonati, D'Elia, Panagopoulos, Vicari 1301.7640

DIGA values for higher cumulants reached quite soon, already for $T \gtrsim 1.1\,T_c$.

Small deviations compatible with repulsive instanton-instanton interactions



from S. Borsanyi et al. 1508.06917

The perturbative power law behavior predicted for χ at high T has been verified

$\chi(T) \propto 1/T^b$, where $b = 7.1(4)(2)$ (perturbative prediction $b = 7$), but absolute value a factor 10 larger

Emerging picture:

- shortly after T_c, topological excitations behave as a dilute non-interacting gas, $F(\theta) \propto (1 - \cos(\theta))$. Residual interactions around T_c are repulsive. Agreement with perturbative DIGA, at least for the power law.

- the scenario changes completely crossing the confinement transition, large N predictions sets in and $F = F(\theta/N)$.

- Sometimes this is interpreted in terms of decomposition into topological objects with charge $1/N$ (instanton quarks). However our results show that the picture could be naïve, or at least such objects are not weakly interacting.

  Non interacting gas of $1/N$ charged objects would give

$$F \propto (1 - \cos(\theta/N)) \implies b_2 = -\frac{0.08333}{N^2}$$

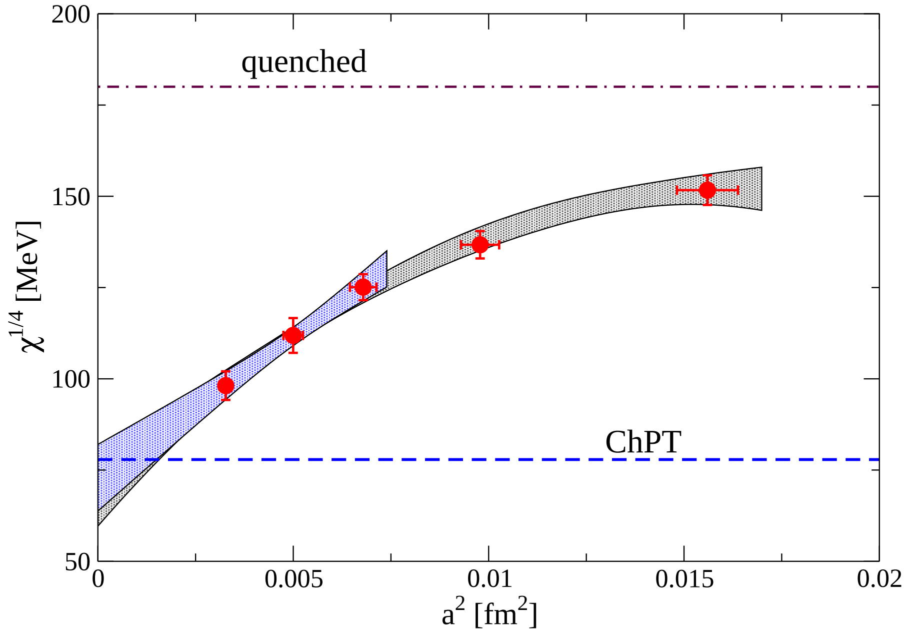
  we obtain instead $b_2 = -0.20(2)/N^2$, hence corrections must be significant.

Full QCD results

I will show some results obtained for $N_f = 2+1$ QCD with physical quark masses

C. Bonati et al., JHEP 1603 (2016) 155 [`arXiv:1512.06746`]

stout improved staggered fermions, a tree-level Symanzik gauge action

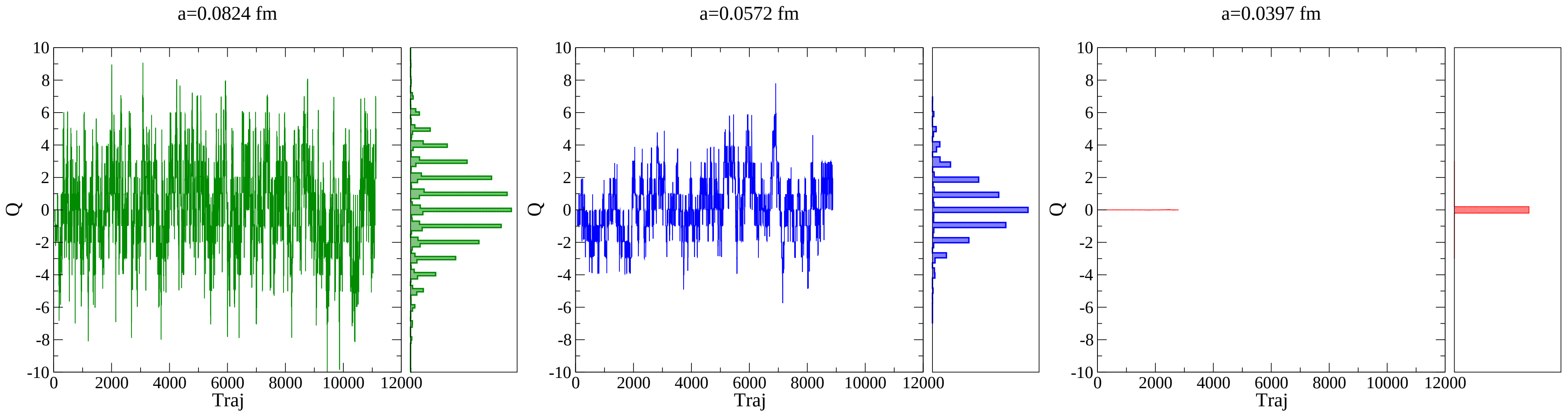


The approach to the continuum limit is quite slow and lattice spacing well below 0.1 fm are needed

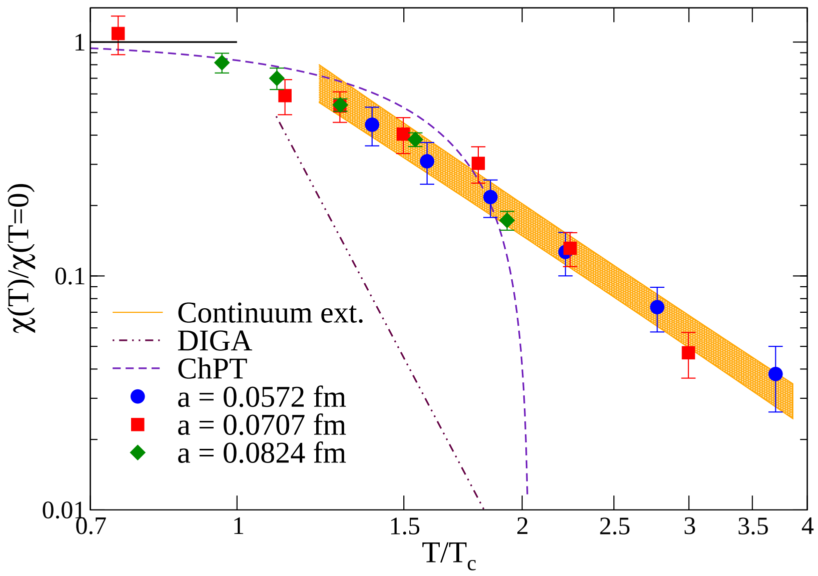
continuum limit compatible with ChPT ($73(9)\mathrm{MeV}$ against $77.8(4)\mathrm{MeV}$)

slow convergence to the continuum is strictly related to the slow approach to the correct chiral properties of fermion fields

The need for quite small lattice spacings, in order to correctly extrapolate to the continuum limit, has brought us to the frontier of frozen topology

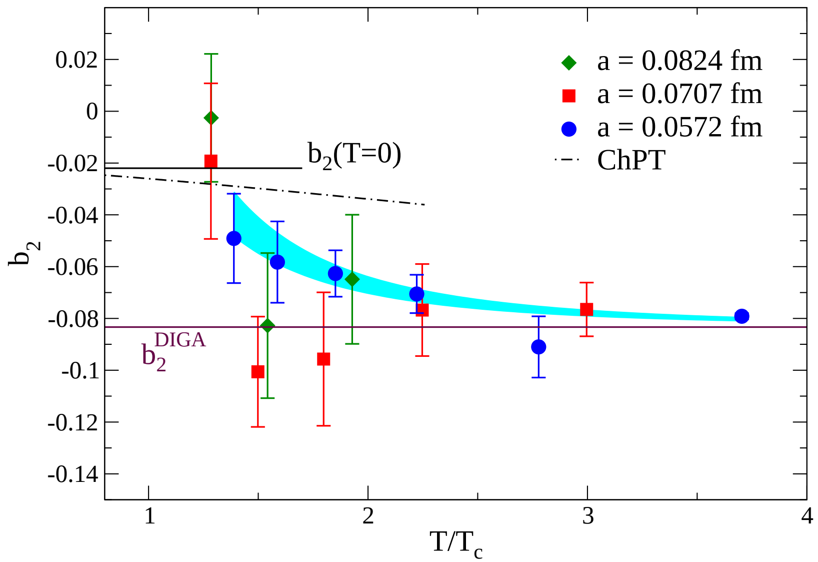


Finite T results



Cut-off effects strongly reduced in the ratio $\chi(T)/\chi(T=0)$

drop of the chiral susceptibility much smoother than perturbative estimate:
$\chi(T) \propto 1/T^b$ **with** $b = 2.90(65)$ **(DIGA prediction:** $b = 7.66 \div 8$**)**



Values for b_2 converge faster to DIGA prediction, however deviations seem of opposite sign with respect to the quenched case:

quark mediated attractive instanton-instanton interaction?

What are the consequences of our results for axion physics?

Main source of axion relics: misalignment. Field not at the minimum after PQ symmetry breaking. Further evolution (zero mode approximation, $H =$ Hubble constant):

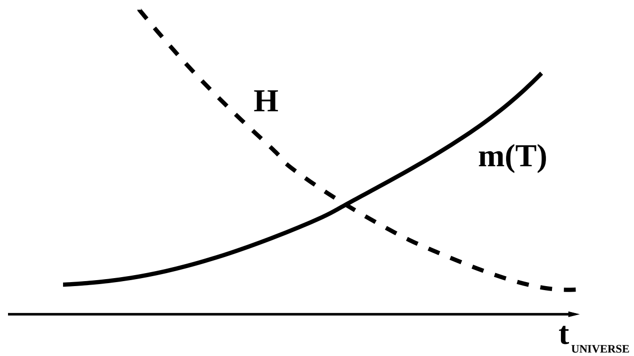
$$\ddot{a}(t) + 3H(t)\dot{a}(t) + m_a^2(T)a(t) = 0 \ ; \quad m_a^2 = \chi(T)/f_a^2$$

$T \gg \Lambda_{QCD}$ **2^{nd} term dominates** $\Longrightarrow \quad a(t) \sim \mathrm{const}$

$m_a \gtrsim H$ **oscillations start** $\Longrightarrow$ **adiabatic invariant**

$N_a = m_a A^2 R^3 \sim$ **number of axions ($\sim$ cold DM)**

$A =$ **oscill. amplitude;** $R =$ **Universe radius**



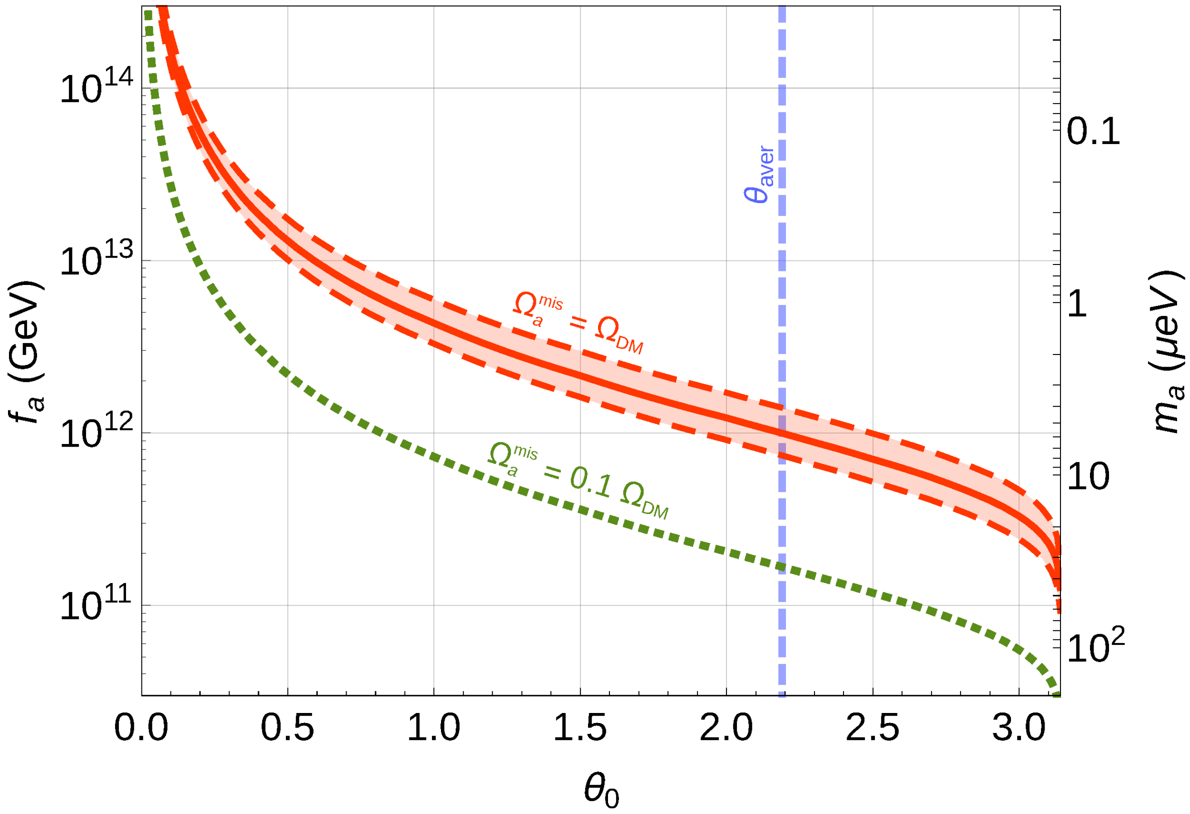
A larger $\chi(T)$ implies larger m_a and moves the oscillation time earlier (higher T, smaller Universe radius R)

Requiring a fixed N_a ($\Omega_{axion} \sim \Omega_{DM}$)

$\chi(T)$ **grows** $\Longrightarrow$ **oscill. time anticipated** $\Longrightarrow$ **less axions** $\Longrightarrow$ **require larger f_a to maintain N_a**

On the other hand, larger f_a means smaller m_a today

Our results translated in predictions for f_a, hence m_a at our times, depending on the required amount of axion dark matter. f_a factor 10 larger (m_a smaller) wrt perturbative DIGA predictions



An unknown variable is the initial misalignment θ_0. Moreover, if PQ symmetry breaks before inflation the initial value is constant, otherwise an average over the initial value has to be performed. **order of magnitude prediction for present $m_a \sim 10 - 100\ \mu\mathrm{eV}$**

BEST WISHES KEN!