Strongly-Coupled Infrared Fixed Points, Confinement and Chiral Symmetry Breaking
Confinement and RG flow

Free YM & matter

UV f.p.

X

IR f.p.

empty

Duality Meissner effect

Abelianization

monopoles

weakly-coupled

local, non-Abelian

symmetry breaking

CFT

strongly-coupled nonlocal, non-Abelian monopoles, dyons, quarks

coupling

symmetry breaking

red curves = deformations by some relevant operators
QCD

• Abelian dual superconductor (dynamical Abelianization) ?

\[ SU(3) \rightarrow U(1)^2 \rightarrow 1 \]

\[ \langle M \rangle \neq 0 \]

\( \Longrightarrow \) Doubling of the meson spectrum (*)

\[ \Pi_1(U(1)^2) = \mathbb{Z} \times \mathbb{Z} \]

‘t Hooft, Nambu, Mandelstam ’80

• Non-Abelian monopole condensation ?

\[ SU(3) \rightarrow SU(2) \times U(1) \rightarrow 1 \]

\( \Longrightarrow \) Problems (*) avoided

\[ \Pi_1(SU(2) \times U(1)) = \mathbb{Z} \]

But non-Abelian monopoles expected to be strongly coupled (no sign flip of \( b_0 \))

• Both the electric (quarks and gluons) and magnetic (monopoles and dyons) d.o.f. become strongly coupled in the Infrared!
Can $\mathcal{N}=2$ SQCD teach us anything useful?

- Or more humbly: any analogy between the phenomena occurring in $\mathcal{N}=2$ SQCD and in the real-world QCD?

- A recent observation: the most singular ("Argyres-Douglas") SCFT, in $\mathcal{N}=2$ SU(N) QCD with $N_F$ flavors, under an $\mathcal{N}=1$ perturbation,

$$\mu \Phi^2|_F = \mu \psi \psi + \ldots$$

flows down (RG) towards an infrared-fixed-point theory described by massless mesons $M$ in the adjoint representation of $G_F$

- Further relevant deformations (shift of bare mass parameters)

\[\text{Confinement and flavor symmetry breaking:}\]

$M$ (or a part of it) make metamorphosis to massless Nambu-Goldstone particles
Blown-up RG flow

$\Lambda_{N=1}$

$\Lambda_{N=2}$

$\mathcal{N}=2$ SCFT

$\mathcal{N}=1$ SCFT

Confinement

Flavor Symmetry Breaking

$\mu \Phi^2 + \delta m \bar{Q}Q$

(local theory of massless mesons)

(strongly-coupled monopoles, dyons, quarks)
Why remarkable:

- $\mathcal{N}=2$ SCFT is a complicated, nonlocal theory of strongly interacting massless monopoles, dyons and quarks *

- $\mathcal{N}=1$ SCFT is a theory of weak coupled local theory of mesons M

- In the nearby $\mathcal{N}=1$ confining vacuum, $M \sim \text{NG bosons of symmetry breaking}$ **

- Analogous to the real-world QCD * **
Massless mesons in the adjoint representation of $G_F$ in the IR

Tools:

- **Seiberg-Witten curves** for $\mathcal{N}=2$ gauge theories
  - SCFT by appropriate tuning of the vacuum parameters (VEVs) and m’s

- **Trace anomalies** (any theory)
  \[
  \langle T^{\mu}_\mu \rangle = \frac{1}{16\pi^2} \left[ c \left( R^{2}_{\mu\nu\rho\sigma} - 2R^{2}_{\mu\nu} + \frac{R^2}{3} \right) - a \left( R^{2}_{\mu\nu\rho\sigma} - 4R^{2}_{\mu\nu} + R^2 \right) \right] \\
  \text{(Weyl)^2 Euler density}
  \]

- **For any $\mathcal{N}=1$ susy theory** ( $R = U_R$ (1) charge )
  \[
  a = \frac{3}{32}(3\text{Tr} R^3 - \text{Tr} R) ; \quad c = \frac{1}{32}(9\text{Tr} R^3 - 5\text{Tr} R) ,
  \]

- **For $\mathcal{N}=2$ susy fields**
  \[
  \text{Tr} R^3_{\mathcal{N}=2} = \text{Tr} R_{\mathcal{N}=2} = 48(a - c) ; \quad \text{Tr} R_{\mathcal{N}=2} I_a I_b = \delta_{ab}(4a - 2c) .
  \]

Any $\mathcal{N}=2$ theory has global R symmetries

\[
SU_R(2) \times U_R(1) ; \quad \mathcal{R}_{\mathcal{N}=2} \equiv U_R(1) \text{ charge}, \quad I_3 \subset SU_R(2)
\]
Flowing down from $\mathcal{N}=2$ SCFT to $\mathcal{N}=1$ SCFT

$$\mu \Phi^2|_F = \mu \psi \psi + \ldots$$

- $\mathcal{N}=1$ curves ($\mathcal{N}=2$ SW curves + factorization condition)
- relations

$$\{R_{\mathcal{N}=2}, I_3\} \leftrightarrow R_{\mathcal{N}=1}$$

e.g., for $SU(2)$, $N_F = 1$, 
$$R_{\mathcal{N}=1} = \frac{5}{6} R_{\mathcal{N}=2} + \frac{1}{3} I_3$$

for $SU(N)$, $N_F = 2N-1$, 
$$R_{\mathcal{N}=1} = \frac{2}{3} R_{\mathcal{N}=2} + \frac{2}{3} I_3$$

- Known $\{R_{\mathcal{N}=2}, I_3\}$ of $\mathcal{N}=2$ SCFT $\Rightarrow$ $R_{\mathcal{N}=1}$

- 't Hooft anomaly matching conditions

  $Tr R^3, Tr R, Tr R(G_F)^2$

- Known $Tr R^3, Tr R$ of the IR theory $\Rightarrow$ $a, c$ of the IR theory

e.g., for $SU(2)$, $N_F = 1$, 
$$a = \frac{1}{48}, \quad c = \frac{1}{24}$$

for $SU(2)$, $N_F = 3$, 
$$a = \frac{1}{6}, \quad c = \frac{1}{3}$$

- Works for $\mathcal{N}=2$ SCFT with $SU(N)$ color, any $N_F$

Very nontrivial check / powerful info on the massless d.o.f.

This is $3^2 - 1 = 8$ massless meson chiral fields!!
The result has been checked by following different RG paths.

Figure 1: RG flow for various values of $\mu$.

BTW: our mesons $M \sim$ Meson $M$ in the Seiberg's $\mathcal{N}=1$ duality

$\mathcal{N}=1$ L_eff of SQCD

(M, B, q, \ldots)

SU(N), $N_F$ Q's

\[ < \quad \rightarrow \quad > \]

SU($N_F - N$), $N_F$ q's, $M$
A puzzle?

Weak GST duals for \( SU(N) \) \( N_F = 2n \) Infrared theory

- \( \mathcal{N}=1 \) perturbation \( \rightarrow \) confinement and XSB

in simplest cases: \( \text{USp}(2N), N_F = 4; \) \( \text{SU}(3), N_F = 4; \) \( \text{SU}(4), N_F = 4; \)

\( \text{SO}(2N), N_F = 2. \)

- In all cases the GST dual description correctly realize XSB;
- In simplest cases, d.o.f \( \sim \) monopoles carrying flavor q.n.'s

- \( \text{SU}(N), N_F = 2n, \) GST dual looked more difficult

\[
\begin{array}{c}
S_{N-n+1} - SU(2) - R_n, \\
AD_{N_f=2}(SU(N-n+1))
\end{array}
\]

- Actually \( \mathcal{N}=1 \) deformation can be worked out also in this case

- Free mesons in adjoint representation of \( SU(2n) \)

(Tools; Conformal anomalies, 't Hooft anomaly matching conditions)
To conclude:

**Real-world QCD**

\( \mathcal{N}=0 \) SCFT

\[
a_{UV} = \frac{11N_fN_c}{360} + \frac{31}{180}(N_c^2 - 1)
\]

\( a_{UV} > a_{IR} \) if \( N_f < \frac{11}{2}N_c \)

**UV**

**Confinement**

\( \mathcal{N}=2 \) SCFT

\[
a_{UV} = \frac{7N^2 - N - 5}{24}
\]

\( a_{N=2SCFT} = \frac{7N(N - 1)}{24} \)

**Flavor Symmetry Breaking**

\( \mathcal{N}=1 \) SCFT

\[
a_{IR} = \frac{(2N - 1)^2 - 1}{48}
\]

\( a_{IR} = \frac{N_f^2 - 1}{360} \)

**IR**

Free pions (Infrared freedom!)

\( \mathcal{N}=2 \) SCFT

\( SU(N), N_f = 2N - 1 \)

\( \Lambda_{N=2} \)

\( \Lambda_{N=1} \)
Thank you, all
Grazie a tutti
ありがとうございました
(arigatou)
”古稀“
ありがとうございます (ARIGATOU)

= ありがとうたく (ARIGATAKU)

= Rare, difficult to have, precious （luck, etc）

“ひとの生をうくるはかたく、死すべきものの、生命あるもありがとうございます”

“That a man is given life in this universe, where death is normal, is an incredibly rare and precious thing ”

(from "Dhammapada", "法句経")
Thank you, all
Grazie a tutti
ありがとう
’t Hooft anomaly matching conditions between $\mathcal{N}=2$ SCFT and $\mathcal{N}=1$ SCFT

$SU(N), N_F = 2N-1$ AD vacuum

$\mathcal{N}=2$ SCFT ("UV") Input:

$$a' = \frac{7}{24} N(N - 1), \quad c' = \frac{1}{3} N(N - 1), \quad k_{SU(2N-1)} = 2N - 1$$

$$R_{N=1} = \frac{2}{3} R_{N=2} + \frac{2}{3} I_3$$

$$\Rightarrow \quad Tr R_{N=1} = \frac{2}{3} Tr R_{N=2} = -\frac{4}{3} N(N - 1)$$

$$Tr R_{N=1}^3 = \frac{8}{27} [Tr R_{N=2}^3 + 3 Tr R_{N=2} I_3^2] = -\frac{4}{27} N(N - 1)$$

$$Tr R_{N=1} SU(2N - 1)^2 = \frac{2}{3} Tr R_{N=2} SU(2N - 1)^2 = \frac{1 - 2N}{3}$$

$\mathcal{N}=1$ SCFT ("IR")

$$a = \frac{(2N - 1)^2 - 1}{48}, \quad c = \frac{(2N - 1)^2 - 1}{24}$$

$$\Rightarrow \quad Tr R_{N=1} = 16(a - c) = -\frac{4}{3} N(N - 1), \quad Tr R_{N=1}^3 = \frac{16}{9} (5a - 3c) = -\frac{4}{27} N(N - 1)$$

$$\Rightarrow \quad Tr R_{N=1} SU(2N - 1)^2 = (R_{N=1}(M) - 1)(2N - 1) = \frac{1 - 2N}{3}$$

Massless mesons in the adjoint representation of $SU(N_F)$ !!!
Complex Structure of Susy 4D Gauge Theories

- Chiral superfields

\[ \Phi(x, \theta, \bar{\theta}) = \phi(y) + \sqrt{2} \theta \psi(x) + \theta \theta F(y), \quad y = x + i \theta \sigma \bar{\theta} \]

- Vector superfields \( V^\dagger = V \),

\[ W_\alpha = -\frac{1}{4} D^2 e^{-V} D_\alpha e^V = -i \lambda + \frac{\mu}{2} (\sigma^\mu \bar{\sigma}^\nu)_{\alpha} F_{\mu \nu} \theta_\beta + \ldots \]

- Supersymmetric Lagrangian (\( \int d\theta_1 \theta_1 = 1 \), etc)

\[ \mathcal{L} = \frac{1}{8\pi} \text{Im} \tau_{cl} \left[ \int d^4 \theta \Phi^\dagger e^V \Phi + \int d^2 \theta \frac{1}{2} W_\alpha W^\alpha \right] + \int d^2 \theta W(\Phi) \]

- \( W(\Phi) = \) superpotential; \( \tau_{cl} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \)

- Potential (F term and D term)

\[ V_{sc} = \sum_{\text{mat}} \left| \frac{\partial W}{\partial \phi} \right|^2 + \frac{1}{2} \sum_a \left( \sum_{\text{mat}} \phi^* t^a \phi \right)^2 \]

- Vacuum Degeneracy (Space of Vacua)

- Non-renormalization Theorem (perturbative) (cfr. Anomaly)

- Q: Superpotential Dynamically Generated?

\[ \mathcal{N} = 2 \quad \text{if} \quad \Phi \sim \text{adj} \quad \text{and} \]
\[ W(\Phi) = 0 \quad (W = M \Phi \tilde{M} \quad \text{in SQCD}) \]
Seiberg-Witten solution in $\mathbf{N}=2, \mathbf{SU}(2)$ susy gauge theories

- **Fields:**
  $$W = (A_\mu, \lambda), \quad \Phi = (\phi, \psi)$$

- **Moduli of vacua (degeneracy):**
  $$\langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad u = Tr \langle \phi^2 \rangle$$

- **$L_{\text{eff}}$:**
  $$L_{\text{eff}} = \text{Im} \left[ \int d^4 \theta \bar{A} \frac{\partial F(A)}{\partial A} + \int d^2 \theta \frac{\partial^2 F(A)}{\partial A^2} W^\alpha W_\alpha \right] = A_D$$

- **Duality:** $L=L_{\text{eff}}$ formally inv under $\text{SL}(2,\mathbb{Z})$ ($ad-bc=1$) $\supset \text{EM duality}$
  $$\begin{pmatrix} A_D \\ A \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix} \quad \begin{pmatrix} \delta L / \delta F_{\mu\nu}^+ \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta L / \delta F_{\mu\nu}^+ \end{pmatrix}$$

- **Which description?** Depends on $u$!

- **Assume:**
  $$\text{massless monopoles at } u = \pm \Lambda^2 \quad \rightarrow \quad \text{F(A)}!$$
Seiberg-Witten curve (SU(2) YM)

\[ y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2) \]

- solves the theory

\[
\frac{dA_D}{du} = \int_\alpha \frac{dx}{y}, \quad \frac{dA}{du} = \int_\beta \frac{dx}{y}, \quad \rightarrow \quad F(A)
\]

\[ \text{Im} \int_\alpha \frac{dx}{y} \cdot \frac{dy}{dx} > 0 \]

- Perturbative and nonperturbative quantum effects (instantons) fully encoded in ★

- Effective theory near \( u = \Lambda^2 \)

\[
L_{\text{eff}}(A_D, F_D^{\mu\nu}, \ldots) + \int d^4 \theta \bar{\mathcal{M}} e^{V_D} \mathcal{M} + (\mathcal{M} \rightarrow \bar{\mathcal{M}}) + \sqrt{2} \int d^2 \theta M A_D \bar{\mathcal{M}}
\]

Magnetic monopole coupled minimally to the dual gauge field
Seiberg-Witten curves for general gauge groups

• SU(N) with $N_F$ quarks

$$y^2 = \prod_{i=1}^{N} (x - \phi_i)^2 - \Lambda^{2N-N_F} \prod_{a=1}^{N_F} (x + m_a)$$

• SO(N) with $N_F$ quarks in vector representation

$$y^2 = x \prod_{i=1}^{\lfloor N/2 \rfloor} (x - \phi_i^2)^2 - 4\Lambda^{2(N-N_F-N_F)} x^{2+\epsilon} \prod_{a=1}^{N_F} (x + m_a^2)$$

• etc.
Argyres-Seiberg’s S duality

• SU(3) with $N_F = 6$ hypermultiplets ($Q_i$, $\tilde{Q}_i$’s) at infinite coupling

\[
SU(3) \text{ w/ } (6 \cdot 3 \oplus \bar{3}) \quad = \quad SU(2) \text{ w/ } (2 \cdot 2 \oplus \text{SCFT}_{E_6})
\]

\[g = \infty\quad g = 0\]

Flavor symmetry $\sim SU(6) \times U(1)$

$\mathcal{N}=2$ dualities

Explosive developments in studies of $\mathcal{N}=2$ SCFT
- \( U(1)^{N-n} \) gauge multiplets
- SU(2) gauge field coupled to the SU(2) flavor symmetry of the SCFT A & B
- The A sector: the SCFT entering in the Argyres-Seiberg dual of SU\((n)\), \(N_F = 2n\) with \(G_F = SU(2) \times SU(2n)\), known as \(R_n\)
- The B sector: the singular SCFT of the SU\((N-n+1)\) theory with two flavors

\[
b_0 = \frac{N - n}{N - n + 2}
\]

GST solved the problem of wrong dimensions for the masses in EHIY
GST eliminated a counter example (Shapere-Tachikawa) to the “a” theorem
GST dualities generalized to SO\((N)\), USp\((2N)\) theories
Gaiotto-Seiberg-Tachikawa (GST) 2011

• Apply the basic idea of Argyres-Seiberg duality to the IR f.p. SCFT

• SU(N) with $N_F = 2n$:

$$y^2 = (x^N + u_1 x^{N-1} + u_2 x^{N-2} + \cdots + u_N)^2 - \Lambda^{2N-2n} \prod_{i=1}^{2n} (x + m_i)$$

At $u = m = 0,^* \quad y^2 \sim x^{N+n}$ (EHIY point)

relatively non-local massless monopoles and dyons

• Straightforward treatment of fluctuations
  ⇒ wrong scaling laws for the masses

★ To keep the correct dimensions of masses, introduce two different scalings:

$$u_{N-n+2} \sim O(\epsilon_A^2), \quad u_{N-n+3} \sim O(\epsilon_A^3), \quad \ldots, \quad u_N \sim O(\epsilon_A^N).$$

$$u_1 \sim O(\epsilon_B), \quad u_2 \sim O(\epsilon_B^2), \quad \ldots, \quad u_{N-n+2} \sim O(\epsilon_B^{N-n+2}).$$

$$\epsilon_A^2 = \epsilon_B^{N-n+2} \quad \Rightarrow$$

$$a_i = \int_{\alpha_i} \lambda, \quad a_D \lambda = \int_{\beta_i} \lambda$$

$$\lambda \sim dx \frac{y}{x^n}$$

$$m(n_m, n_e, n_i) = \sqrt{2} |n_m a_D + n_e a + n_i m_i|$$

Note: * except for one $u$

Eguchi-Hori-Ito-Yang '96
GST dual for the singular point of USp(2N) (also SO(N))

$$y^2 \sim x^{2n} \cong \begin{array}{c}
\text{B} \\
\text{SU(2)} \\
\text{A}
\end{array}$$

- $U(1)^{N-n}$ gauge multiplets

- The A sector: a (in general) non-Lagrangian SCFT having SU(2)$\times$SO(4n) flavor symmetry

- The B sector: a free doublet (coupled to U(1) gauge boson)

For $N_F = 2n = 4$, A sector $\sim$ 4 free doublets
Figure 3: Zero loci of the discriminant of the curve of $\mathcal{N} = 2$, $SU(3)$, $n_f = 4$ theory at small $m$.

$$U = \text{Tr} \Phi^2 \simeq 3m^2; \quad V = \text{Tr} \Phi^3 \simeq 2m^3$$
QMS of $N=2$ SQCD ($SU(n)$ with $n_f$ quarks)

- $N=1$ Confining vacua (with $\Phi^2$ perturbation)
- $N=1$ vacua (with $\Phi^2$ perturbation) in free magnetic phi

$m = m^{cr}$

Di Pietro, Giacomelli '11
Quantum space of vacua in $\mathcal{N}=2$ SQCD with critical mass

QMS of $\mathcal{N}=2$ USp(2n) Theory with $n_f$ Quarks 
$m = 0$

Higgs Branches

Special Higgs Branch

Non Abelian monopoles

Dual Quarks

Coulomb Branch

- N=1 Confining vacua (with $\Phi^2$ perturbation)
- N=1 vacua (with $\Phi^2$ perturbation) in free magnetic pha
a and c coefficients for free particles

\[ c = \frac{1}{120} (N_S + 6 N_F + 12 N_V) ; \quad a = \frac{1}{360} (N_S + 11 N_F + 62 N_V) \]

For N=2 hypermultiplet (pair of chiral multiplets Q and Q bar):

\[ a = \frac{1}{360} (4 + 11) = \frac{1}{24} ; \quad c = \frac{1}{120} (4 + 6) = \frac{1}{12} . \]

For N=2 vector multiplet:

\[ a = \frac{1}{360} (2 + 11 + 62) = \frac{5}{24} ; \quad c = \frac{1}{120} (2 + 6 + 12) = \frac{1}{6} . \]

For real-world QCD:

\[ c_{UV} = \frac{1}{20} N_f N_c + \frac{N_c^2 - 1}{10} ; \quad a_{UV} = \frac{11 N_f N_c}{360} + \frac{31}{180} (N_c^2 - 1) , \]

\[ c_{IR} = \frac{N_f^2 - 1}{120} ; \quad a_{IR} = \frac{N_f^2 - 1}{360} . \]