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Generalized Wilson-Fisher critical points from conformal symmetry

Based on arXiv:1604.07310 (JHEP), 1611.10344 (PRL) and 1702.03938 (JHEP) with F. Gliozzi, A. Guerrieri & C. Wen

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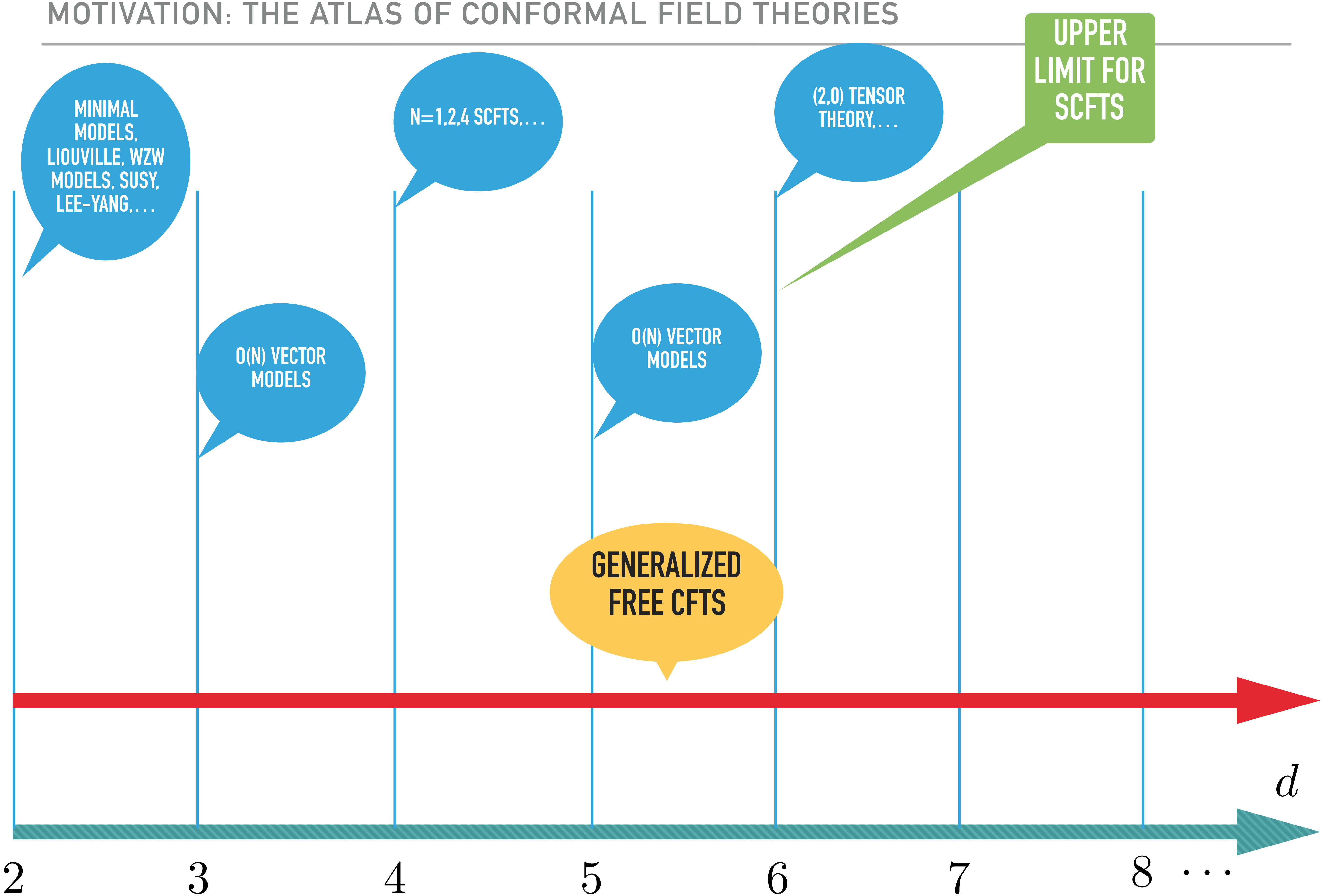
- Motivation: The Atlas of Conformal Field Theories.
- State-of-the-art evidence: Vector-models in general dimensions.
- A concrete framework: Generalized Free CFTs.
- Deformed GFCFTs: Generalized Wilson-Fisher like critical points.
- Examples: multi critical points, O(N) models, multiple deformations.
- Outlook

MOTIVATION: THE ATLAS OF CONFORMAL FIELD THEORIES



- Spectrum of **normal** free CFTs in any d:

Scalar $\phi : \phi^2, \phi^4, \ldots T_{\mu\nu} \ldots$

Fermions $\psi, \bar{\psi} : \bar{\psi}\psi, \ldots T_{\mu\nu} \ldots$

ALWAYS THERE

- Interacting CFTs (Large-N Vector Models)

e.g. scalars in d-dimensions - the mother theory

$$S = \frac{1}{2}\int \phi^a(-\partial^2)\phi^a + \int \sigma(\phi^a\phi^a - \frac{N}{g})\,, a = 1, 2, ..N$$

Lagrange multiplier

- The "gap equation" implies:

UV cutoff

$$\frac{:\phi^a\phi^a:}{N} \approx \frac{1}{g} = \frac{\Gamma(1-\frac{d}{2})}{2^d\pi^{d/2}}\sigma^{\frac{d}{2}-1} + \kappa_d\Lambda^{d-2}{}_2F_1\left(1,\frac{d}{2};\frac{d}{2}+1;-\frac{\sigma}{\Lambda^2}\right)$$

- A critical point is reached when $1/g$ can kill the UV divergence, allowing a regular $\Lambda\to\infty$ limit i.e. in $d=3$.

- An equivalent, and suggestive form of the gap eq. is

$$\Lambda^{6-d}\Phi(t)\frac{:\phi^a\phi^a:}{N} = \sigma^2$$

for some "RG" parameter $t=\Lambda^2/\sigma$.

For $d>6$ this has a finite number of poles as $t\to 0$.

- ϕ and $\sigma \equiv \sigma_2$ can be calibrated to coincide at $d = 6$ They have a relative "weight" in $d > 6$.

- A similar story holds for N fermions.

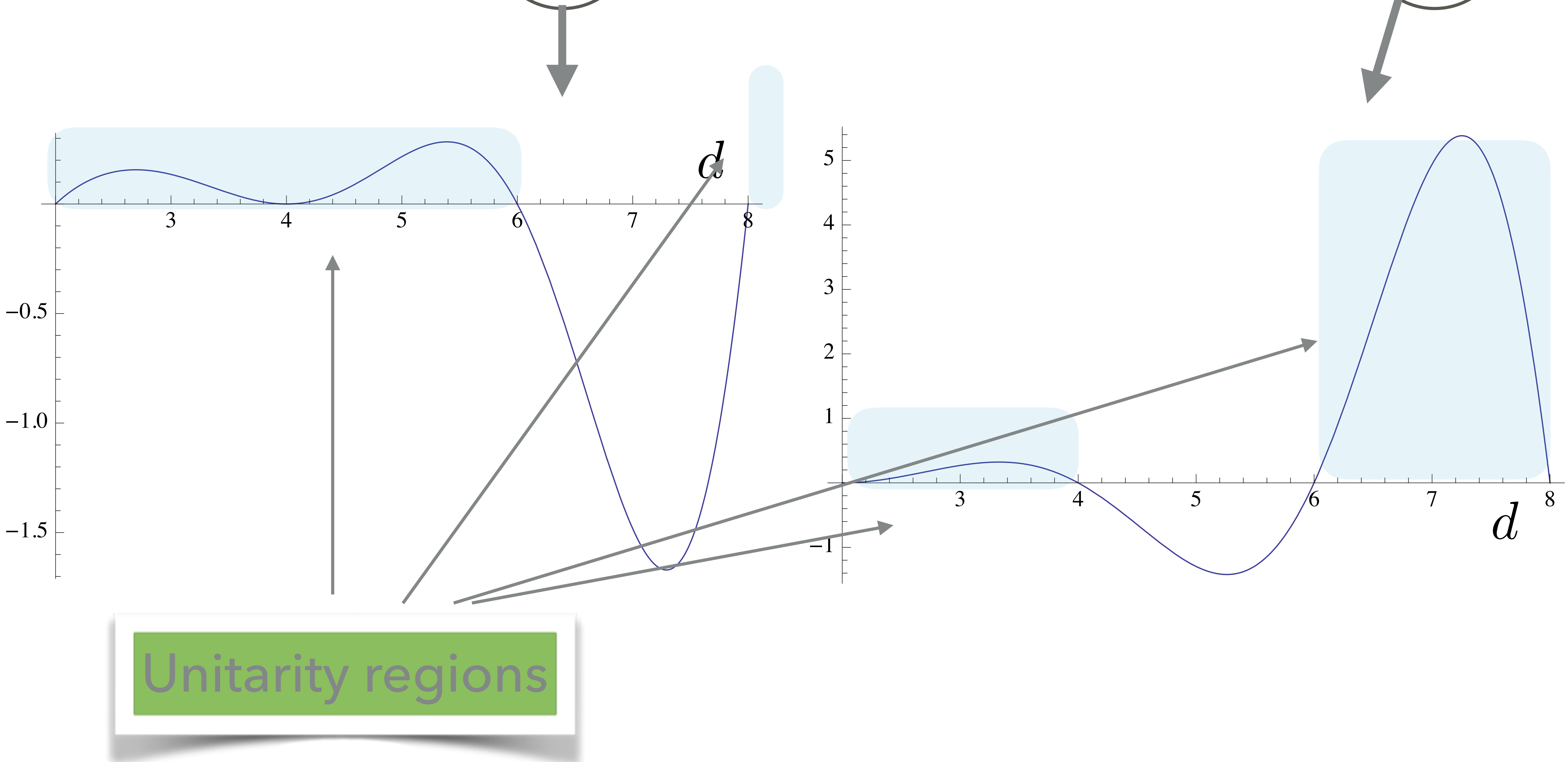
$$\Lambda^{2-d}\tilde{\Phi}(\tilde{t})\frac{\bar{\psi}^a\psi^a}{NTr\mathbb{I}} = \sigma_1\,, \quad \tilde{t} = \frac{\Lambda^2}{\sigma_1^2}$$

- The role of σ_2 and σ_1 was studied in an ancient version of the conformal bootstrap, that allowed a scan in d. ***[T.P. (93-94)]***

- Scalar and fermionic CFTs were studied for general dimensions, and results analytic in d were obtained.
  [See also related work by A. M. Vasiliev et. al (80s) and J. Gracey. (90-93)]

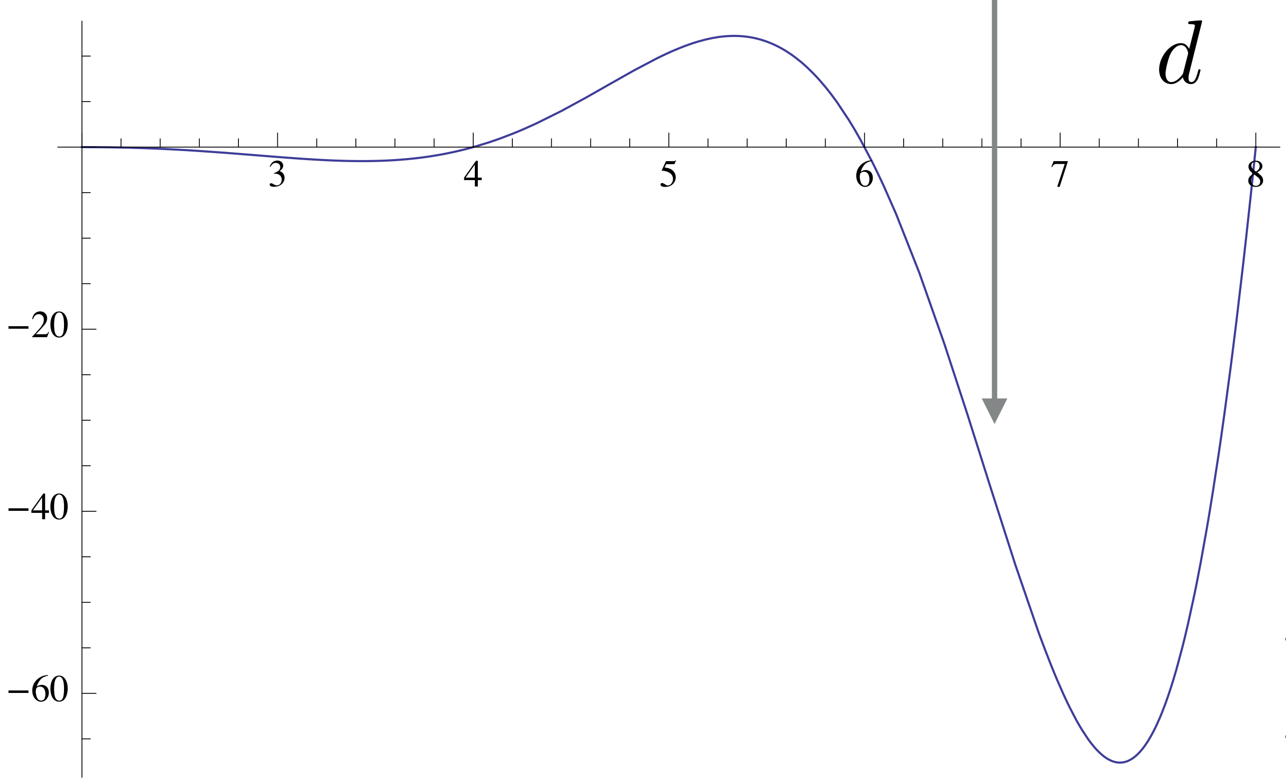
- In contrast to CFTs with gauge fields, all critical quantities are proportional to the anomalous dimensions of ϕ and ψ.

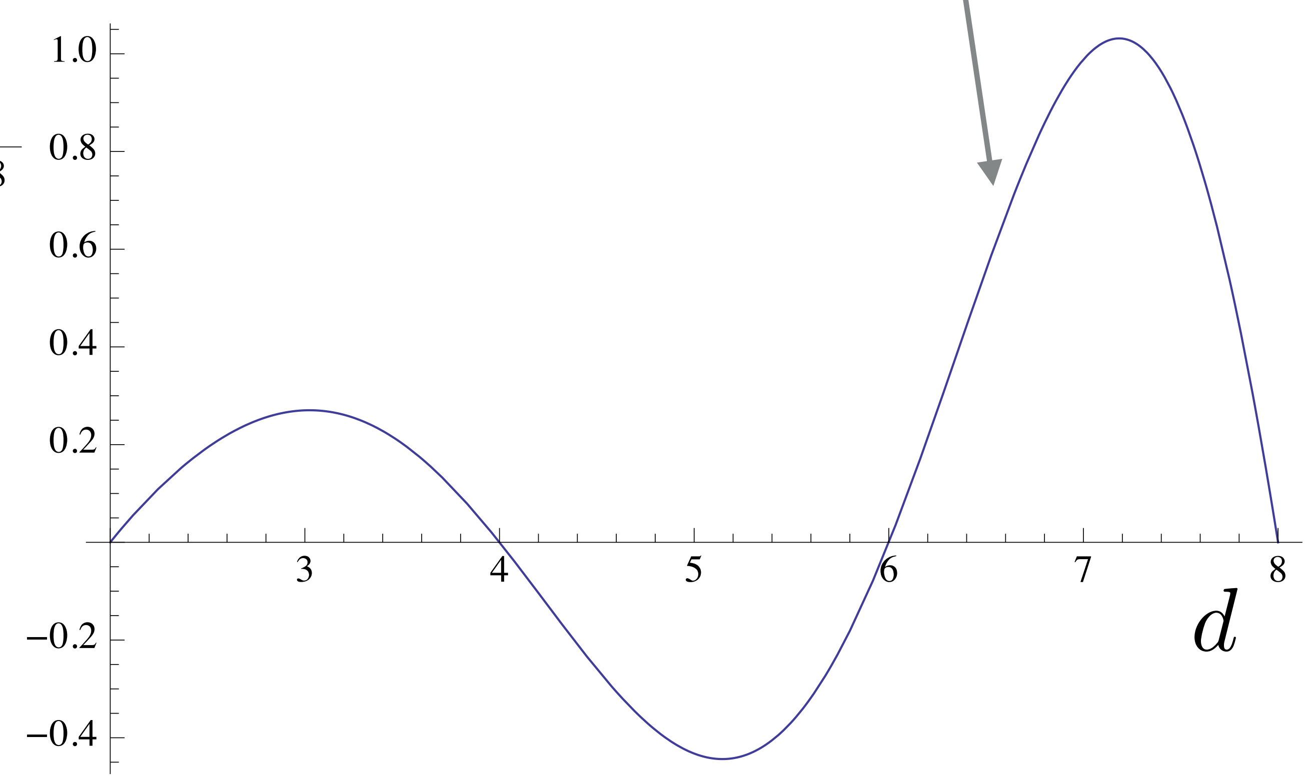
$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\gamma_\phi(d) \qquad \Delta_\psi = \frac{d}{2} - \frac{1}{2} + \frac{1}{N Tr\mathbb{I}}\gamma_\psi(d)$$



- The dimensions of the σ-fields are

$$\Delta_2 = 2 + \frac{1}{N}\frac{4(d-1)(d-2)}{d-4}\gamma_\phi(d) \qquad \Delta_1 = 1 + \frac{1}{N Tr\mathbb{I}}\frac{2}{d-2}\gamma_\psi(d)$$





- However, the "central charges" are given by

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$C_T^b(d) = C_T^{(0)}(d)\left[N + \frac{c_T(d)}{C_T^{(0)}(d)} + O(1/N)\right] \qquad C_T^{(0)}(d) = \frac{d}{(d-1)S_d^2}$$

$$C_T^f(d) = \mathcal{C}_T^{(0)}(d)\left[N + \frac{\tilde{c}_T(d)}{\mathcal{C}_T^{(0)}(d)} + O(1/N)\right] \qquad \mathcal{C}_T^{(0)}(d) = \frac{d}{2}\frac{1}{S_d^2}Tr\mathbb{I}$$

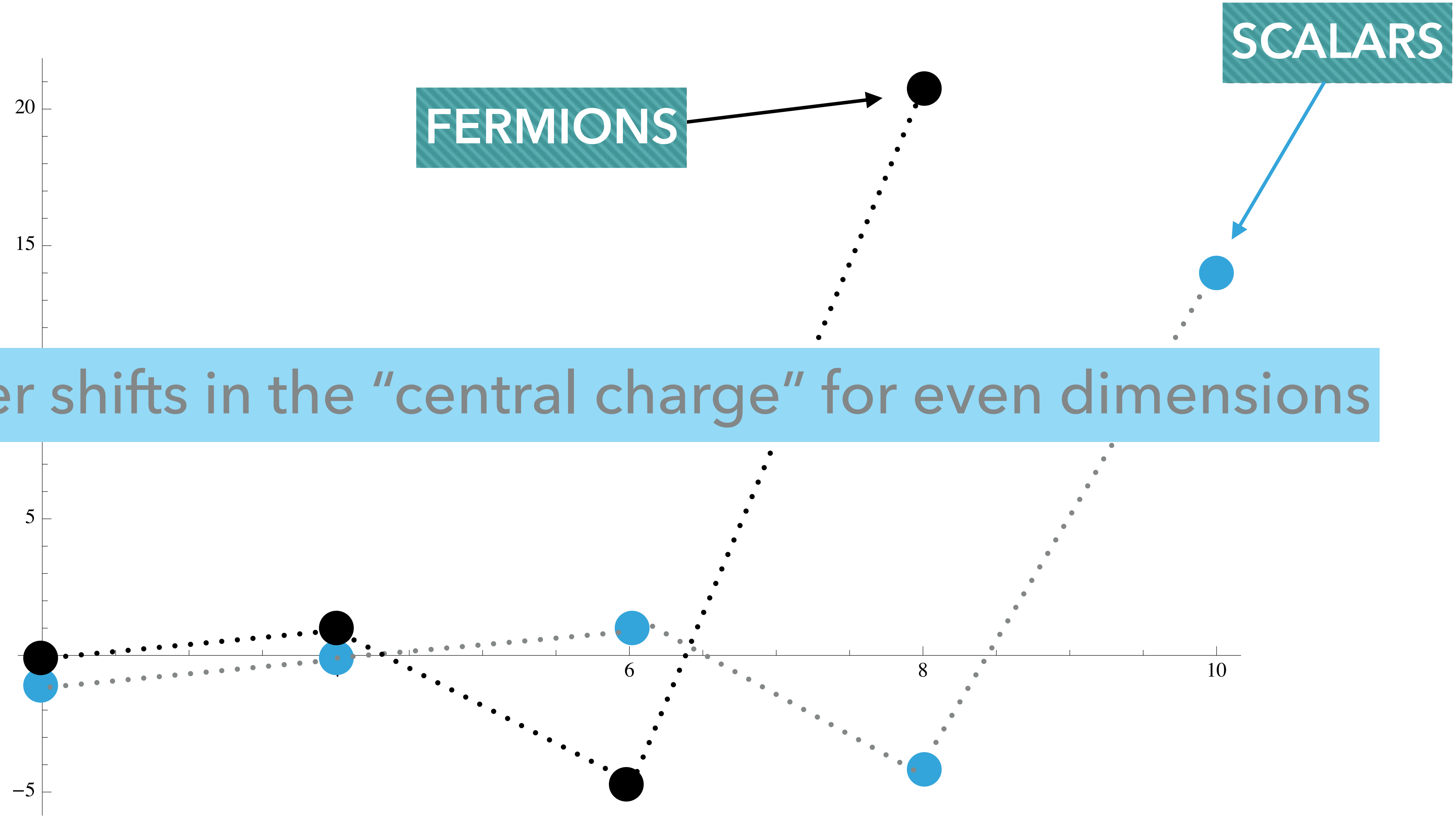
- Quite remarkably for even dimensions, they are non zero:

$$c_T(d) = \frac{(-1)^{\frac{d}{2}+1} d(d-4)(d-2)!}{(d-1)\left(\frac{d}{2}+1\right)!\left(\frac{d}{2}-1\right)!S_d^2}\,, \quad \tilde{c}_T(d) = \frac{(-1)^{\frac{d}{2}} d(d-2)(d-2)!}{2\left(\frac{d}{2}+1\right)!\left(\frac{d}{2}-1\right)!S_d^2}$$

$$d = 6, 8, 10, 12, \ldots \qquad\qquad d = 4, 6, 8, 10, \ldots$$

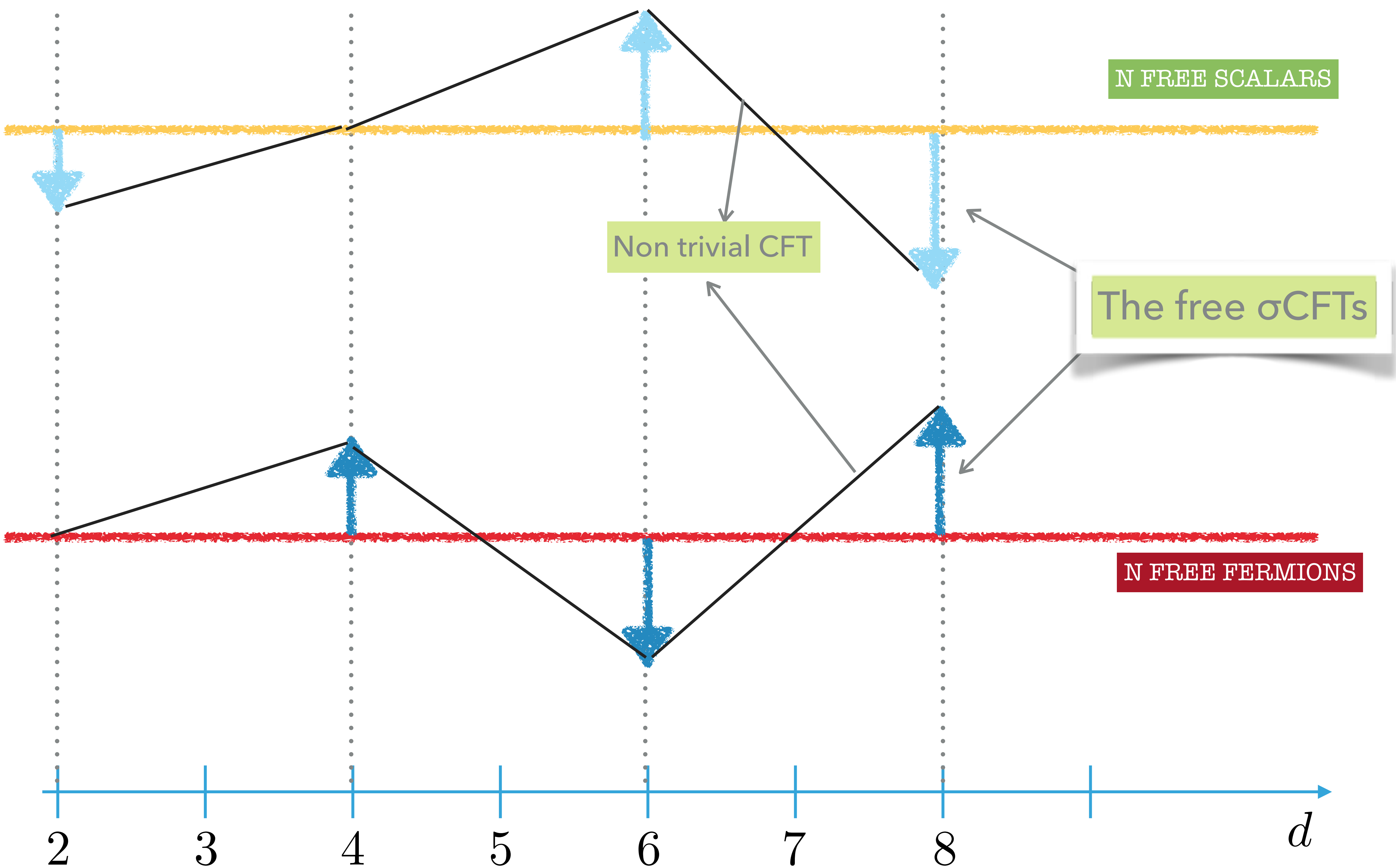
[The fermonic results first appeared in Diab et. al. arXiv:1601.07198]

- Moreover, the ratios $\dfrac{c_T(d)}{C_T^{(0)}(d)}$ and $\dfrac{\tilde{c}_T(d)}{\mathcal{C}_T^{(0)}(d)}$ are integers



Integer shifts in the "central charge" for even dimensions

- An atlas of Vector Model CFTs for all dimensions



- We suggest the existence of free CFTs - **the σCFTs** - for all even dimensions $d > 6$ and $d > 4$ that are **decoupled** from the **normal** free CFTs for special dimensions: **d=even**.
- They go beyond the older notion of Generalized Free CFTs ***[e.g Papadodimas & El Showk arXiv:1101.4163]*** that they are non-unitary since

$$\Delta_2, \Delta_1 < \frac{d}{2} - 1\,, \quad d > 6, d > 4$$

- The can be described by higher-derivative Lagrangians

$$L_2 = \frac{1}{2}\sigma_2(-\partial^2)^{\frac{d}{2}-2}\sigma_2\,, \quad L_1 = \frac{1}{2}\sigma_1(-\partial^2)^{\frac{d}{2}-1}\sigma_1$$

- They are intimately and universally related to the normal free CFTs: $\sigma_2\phi^2$ and $\sigma_1\bar{\psi}\psi$ are universal marginal couplings.

- We studied these CFTs with just the help of the OPE unveiling a rich and intriguing spectrum ***[T.P et. al. arXiv:1604:07310]***
- An *elementary* scalar conformal field has 2-pt function.

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}}$$

- It also satisfies the "elementariness" condition

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = g_\phi \frac{1}{(x_{12}^2 x_{23}^2 x_{13}^2)^{\Delta_\phi/3}} \equiv 0$$

- For σ_2 we had obtained ***[T.P. hep-th:9410093].***

$$g_{\sigma_2}(d) = 2(d-3)g_*(d)\,, \qquad g_*^2(d) = \frac{2\Gamma(d-2)}{\Gamma\left(3-\frac{d}{2}\right)\Gamma^3\left(\frac{d}{2}-1\right)}$$

which vanishes for all even $d > 6$.

- GFCFTs are fully determined by the 2-pt functions of their elementary fields. E.g. the 4-pt function is

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[1 + v^{\Delta_\phi}\left(1 + \frac{1}{(1-Y)^{\Delta_\phi}}\right)\right]$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\,, \quad Y = 1 - \frac{u}{v}$$

The CPWs (conf. blocks)

- The conformal OPE statement is

$$\langle\phi(x_1)...\phi(x_4)\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[1 + \frac{g_{\mathcal{O}_2}^2}{C_{\mathcal{O}_2}}\mathcal{H}_{\Delta_{\mathcal{O}_2}} + \sum_{k=1}^{\infty}\frac{g_{T_{2,2k}}^2}{C_{T_{2,2k}}}\mathcal{H}_{\Delta_{T_{2,2k}}}\right]$$

3-pt function coefficients

2-pt function coefficients

- We wish to be totally agnostic regarding the spectrum.
- The generic CPW for a scalar is

$$\mathcal{H}_\Delta = v^{\frac{\Delta}{2}} \left[F_\Delta^0(Y) + \sum_{n=1}^{\infty} \frac{v^n}{n!} \frac{\left(\frac{1}{2}\Delta\right)_n^4}{(\Delta)_{2n}\left(\Delta+1-\frac{d}{2}\right)_n} F_\Delta^n(Y) \right]$$

$$F_a^k(Y) \equiv {}_2F_1\left(\frac{a}{2}+k, \frac{a}{2}+k; a+2k; Y\right) , \; F_{a+2}^k(Y) = F_a^{k+1}(Y) , \; k = 0,1,2,3..$$

- We learn that if Δ is nor related to d, the CPW entails a natural $1/d$ expansion

$$\mathcal{H}_\Delta = v^{\frac{\Delta}{2}} \left[F_\Delta^0(Y) + \sum_{n=1}^{\infty} \left(\frac{v}{d}\right)^n H_\Delta^{(n)} \right]$$

- A similar results holds for all higher-spin CPWs

$$\mathcal{H}_{\Delta_s} = v^{\frac{\Delta_s - s}{2}} \left[Y^s F^0_{\Delta_s+s}(Y) + \sum_{n=0}^{\infty} \left(\frac{v}{d}\right)^n H^{(n)}_{\Delta_s} \right]$$

- The free 4-pt function involved just the single power v^{Δ_ϕ} but infinite powers of Y.

- In fact it holds

$$1 + \frac{1}{(1-Y)^{\Delta_\phi}} = \sum_{s=0,2,4,\dots}^{\infty} \alpha_s(\Delta_\phi) Y^s F^0_{2\Delta_\phi+2s}(Y)$$

$$\alpha_s(\Delta_\phi) \equiv \frac{g^2_{T_{2,s}}}{C_{T_{2,s}}} = \frac{(\Delta_\phi)_{s/2}(\Delta_\phi)_s}{2^{s-1} s! \left(\Delta_\phi + \frac{s-1}{2}\right)_{s/2}}$$

Contribution of leading twist fields with

$\Delta_s = 2\Delta_\phi + s\,,\ s = 0, 2, 4, ..$

- But the CPWs contain an infinity of additional terms. Where are they?

- Terms of order $v^{\Delta_\phi+1}$, without Y factors arise from the scalar and spin-2 CPWs. They yield

$$v^{\Delta_\phi+1}\left[\frac{g_2^2}{C_2}\frac{\Delta_\phi^3}{8(\Delta_\phi+1)(2\Delta_\phi+2-d)}F^1_{2\Delta_\phi}(Y)-\frac{g^2_{T_{2,2}}}{C_{T_{2,2}}}\frac{4}{d}F^0_{2\Delta_\phi+2}(Y)\right]$$

- This gives

$$v^{\Delta_\phi+1}\frac{2\Delta_\phi^2(d-2-2\Delta_\phi)}{d(4\Delta_\phi+2-d)}F^1_{2\Delta_\phi}(Y)$$

- To make it vanish **one option** is to require

$$\Delta_\phi=\frac{d}{2}-1$$

- This implies that the higher-spin fields have dimensions

$$\Delta_s = d - 2 + s$$ → HS conserved currents

We find the **normal** free CFT whose 4-pt function is constructed just from higher-spin conserved currents.

- When $\Delta_\phi \neq d/2 - 1$ we have some other options to "kill" the unwanted term from the OPE:

- An intriguing one is to fix Δ_ϕ and take the $d \to \infty$ limit. This produces the same OPE as the normal free CFT.

- This is reminiscent of a similar limit in spin-glass systems. It is also consistent with the perception that $d \to \infty$ is the mean-field theory limit.

- The other way to deal with $\Delta_\phi \neq d/2-1$ is to enhance the OPE. For example, the $v^{\Delta_\phi+1}$ term could be cancelled by the CPW of the scalar

$$\mathcal{O}_4(x) = \frac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x) - \frac{\Delta_\phi}{2+2\Delta_\phi-d}\phi(x)\partial^2\phi(x)$$

$$\langle\mathcal{O}_4(x_1)\mathcal{O}_4(x_2)\rangle = \frac{C_4}{(x_{12}^2)^{2\Delta_\phi+2}}\,,\ \ \langle\phi(x_1)\phi(x_2)\mathcal{O}_4(x_3)\rangle = g_4\frac{x_{12}^2}{(x_{13}^2x_{23}^2)^{\Delta_\phi+1}}$$

$$C_4 = \frac{2d\Delta_\phi^2(2+4\Delta_\phi-d)}{(2+2\Delta_\phi-d)}\,,\ \ g_4 = 2\Delta_\phi^2$$

- But then, the bag of Aeolus opens up and an infinite number of higher-twist higher-spin towers need to be included in the unitary case when $\Delta_\phi > d/2 - 1$.
- To actually evaluate Δ_ϕ one can used the modern bootstrap, or even the "old" skeleton expansion. In both cases one departs from free CFTs.
- However, when $\Delta_\phi = d/2 - 1 - \tau\,, \;\; \tau = 2, 4, \ldots$ the twist towers terminate for even d. But this requires that we enter into the realm of **non-unitary CFTs.**

- For the 4-pt functions of σ_2 and σ_1 we find:

$$C_4\Big|_{\sigma_2} = \frac{8d(10-d)}{6-d}\,,\; C_4\Big|_{\sigma_1} = \frac{2d(8-d)}{4-d}$$

This means that $\mathcal{O}_2(x)$ is a **ghost** for $d=8\,,\; d=6$.

- Morever, for $d=10\,,\; d=8$ it becomes **null**, despite the fact that its 3-pt function with the σ's is non-zero.

- What happens is that because the e.o.m. is high-derivative, one can construct a tower of higher-twist conformal primaries, until the HS current towers is reached.

A CONCRETE FRAMEWORK: GENERALIZED FREE CFTS

- A caricature of the OPE in free σCFTs

spin-4 CPW

$$\begin{aligned}\langle\sigma(x_1)..\sigma(x_4)\rangle \sim 1 &+ [2\Delta_\sigma]_0 + [2\Delta_\sigma+2]_2 + [2\Delta_\sigma+4]_4 + \cdots \\ &+ [2\Delta_\sigma+2]_0 + [2\Delta_\sigma+2+2]_2 + \cdots \\ &+ \cdots \\ &+ [d-2]_0 + [d]_2 + \cdots\end{aligned}$$

e.m. tensor

HS conserved currents

- For e.g. σ_2 all fields with spins $s = 2, 4, ..., d-4$ contribute to the e.m. tensor.

- Going over to non-unitary Generalized Free CFTs gives an unexpected bonus: the analytic structure of the **free OPE** becomes interesting.

- Up to an overall factor we have (I now switch to the "modern" notation..)

$$\langle\phi_1(x_1)...\phi_4(x_4)\rangle \sim g(v,u)$$

- This contains terms like u^δ that can be expanded as

$$u^\delta = \sum_{\tau=0}^{\infty}\sum_{\ell=0}^{\infty} c^{a,b}_{\tau,\ell}\, G^{a,b}_{2\delta+2\tau+\ell,\ell}(u,v)$$

Conformal block

- The OPE coefficients are calculated as

$$c^{a,b}_{\tau,\ell} = \frac{(-1)^{\ell}(2\nu)_{\ell}}{\tau!\ell!(\nu)_{\ell}(\nu+\ell+1)_{\tau}} \frac{(a+\delta)_{\ell+\tau}(b+\delta)_{\ell+\tau}(a+\delta-\nu)_{\tau}(b+\delta-\nu)_{\tau}}{(2\delta+\ell+2\tau-1)_{\ell}(2\delta-\nu+\ell+\tau-1)_{\tau}(2\delta-2\nu+\tau-1)_{\tau}}$$

- with

$$\nu = \tfrac{d-2}{2},\ a = -\tfrac{1}{2}\Delta_{12} \text{ and } b = \tfrac{1}{2}\Delta_{34} \text{ with } \Delta_{ij} = \Delta_i - \Delta_j .$$

- Setting $\Delta_\ell = 2\delta + \ell + 2\tau$ these have poles at

$$\begin{aligned} \Delta_\star &= 1 - \ell + k \quad (k = 1, 2, \ldots, \ell)\,, \\ \Delta_\star &= 1 + \nu + k \quad (k = 1, 2, \ldots, \tau)\,, \\ \Delta_\star &= 1 + \ell + 2\nu + k \quad (k = 1, 2, \ldots, \tau) \end{aligned}$$

- Since the expanded function is regular, the poles must cancel.
- This can be done by conformal blocks like $G^{a,b}_{2\delta+2\tau'+\ell',\ell'}(u,v)$ that must be present in the OPE.
- Indeed, generic conformal blocks have poles for certain values of dimensions and spin ***[Kos et. al. arXiv:1406.4858]***

$$G^{a,b}_{\Delta',\ell'}(u,v) \sim \frac{R}{\Delta' - \Delta_{\star'}} G^{a,b}_{\Delta_\star,\ell}(u,v)$$

- Notice that $\Delta_{*'}$ and Δ_* are different.

- The deep reason for this cancellation is the fact that the conformal blocks with with Δ_* and $\Delta_{*'}$ have the same eigenvalues wrt to the quadratic and quartic Casimirs.

$$C_2(\Delta_\star,\ell) = C_2(\Delta_{\star'},\ell')\,, \quad C_4(\Delta_\star,\ell) = C_4(\Delta_{\star'},\ell')$$

$$\begin{aligned} C_2(\Delta,\ell) &= \Delta(\Delta-d)+\ell(\ell+d-2)\,, \\ C_4(\Delta,\ell) &= \Delta^2(\Delta-d)^2+\frac{1}{2}d(d-1)\Delta(\Delta-d) \\ &+ \ell^2(\ell+d-2)^2+\frac{1}{2}(d-1)(d-4)\ell(\ell+d-2)\,. \end{aligned}$$

- Solving the Casimir equations, we get three series of poles

$\Delta_{\star'}$	$\Delta_\star$	ℓ	
$1-\ell'-k$	$1-\ell+k$	$\ell'+k$	$k=1,2,\ldots$
$\frac{d}{2}-k$	$\frac{d}{2}+k$	ℓ'	$k=1,2,\ldots$
$d+\ell'-k-1$	$d+\ell+k-1$	$\ell'-k$	$k=1,2,\ldots,\ell$

- $\Delta_{*'}$ are the poles found in ***[Kos et. al. arXiv:1406.4858]***
- Δ_* are the poles of our OPE coefficients.
- Then, schematically it holds

$$c(\Delta\to\Delta_*)=\frac{\tilde{R}}{\Delta-\Delta_*}\quad\Rightarrow\quad\tilde{R}+c(\Delta_{*'})R=0$$

- We learn that the spectrum of Generalized Free CFTs is actually very rich.
- Nevertheless, their 4pt function look very simple, due to some highly nontrivial cancellations.
- In CFT language (Zamolodchikov), a descendant becomes a null state, and it is cancelled out.
- But in an interacting CFT descendants may become quasi primaries. Then it is tempting to try to "guess" the deformed 4pt function by matching it smoothly to the known free 4pt function as
[Rychkov & Tan, 1505.00963, Nakayama unpublished notes]

$$\lim_{\Delta\to\Delta_*, d\to d_*} g_{int}(v,u) = g_{free}(v,u)$$

- Example at $d = 3 - \epsilon$. Consider the normal free CFT

$$\langle \phi^2(x_1)\phi^3(x_2)\phi^2(x_3)\phi^3(x_4)\rangle \sim g_{free}(v,u)$$

$$g_f(u,v) = 3\, G^{a_f,b_f}_{\Delta_{\phi_f},0}(u,v) + 10\, G^{a_f,b_f}_{\Delta_{\phi_f^5},0}(u,v) + \text{spinning blocks}$$

$$g_I(u,v) = (3 + \mathcal{O}(\epsilon))\, G^{a,b}_{\Delta_\phi,0}(u,v) + \dots$$

$$a = b = \tfrac{1}{2}(\Delta_{\phi^3} - \Delta_{\phi^2}), \text{ and } \Delta_{\phi^i} = \Delta_{\phi_f^i} + \gamma_i^{(1)}\epsilon + \gamma_i^{(2)}\epsilon^2 + \dots$$

- The conformal block of the interacting theory has a pole as

$$G^{a,b}_{\Delta_\phi,0}(u,v) = \frac{(\gamma_3^{(1)} - \gamma_2^{(1)})^2 \epsilon^2}{12(\Delta_\phi - \Delta_{\phi_f})} G^{a_f,b_f}_{\Delta_{\phi_f^5},0}(u,v) + \dots$$

- This should reproduce the free OPE, hence

$$3\frac{(\gamma_3^{(1)} - \gamma_2^{(1)})^2 \epsilon^2}{12(\Delta_\phi - \Delta_{\phi_f})} = 10$$

- This means that $\gamma_1^{(1)} = 0$ and we find

$$\frac{(\gamma_3^{(1)} - \gamma_2^{(1)})^2}{\gamma_1^{(2)}} = 40$$

- If we next consider

$$\langle\phi(x_1)\phi^2(x_2)\phi(x_3)\phi^2(x_4)\rangle$$

$$g_f(u,v) = 2\,G^{a_f,b_f}_{\Delta_{\phi_f},0}(u,v) + 3\,G^{a_f,b_f}_{\Delta_{\phi_f^3},0}(u,v) + \text{spinning blocks}$$

$$g_I(u,v) = (2+\mathcal{O}(\epsilon))\,G^{a,b}_{\Delta_\phi,0}(u,v) + \ldots$$

$$a = b = \frac{1}{2}(\Delta_{\phi^2} - \Delta_\phi)$$

- The interacting OPE has now the pole

$$G^{a,b}_{\Delta_\phi,0}(u,v) = \frac{(\gamma_2^{(1)})^2}{12\gamma_1^{(2)}}G^{a_f,b_f}_{\Delta_{\phi_f^5},0}(u,v) + \ldots$$

- But ϕ_f^5 is not in the free OPE, hence we must have $\gamma_2^{(1)} = 0$ leading to

$$(\gamma_3^{(1)})^2 = 40\,\gamma_1^{(2)}$$

- One can do a similar analysis for other correlation functions obtaining the known results

$$\gamma_1 = \frac{1}{1000}\epsilon^2 + \mathcal{O}(\epsilon)\,, \quad \gamma_2 = \frac{3}{100}\epsilon^2 + \mathcal{O}(\epsilon)$$
$$\gamma_n = \frac{1}{30}n(n-1)(n-2)\epsilon + \mathcal{O}(\epsilon^2)$$

- We can play this game for all non-unitary GFCFTs with

$$\Delta_{\phi_f} = \frac{d}{2} - k, \quad k = 1, 2, 3, ..$$

- We consider OPEs of the form

$$\phi^p \phi^{p+1} \sim \phi^{2n-1} + \cdots \, , \quad n = 1, 2, 3...$$

- In the interacting theory, the OPE coefficients are deformed by terms that vanish when $\epsilon \to 0$
- Moreover, the operator ϕ^{2n-1} becomes a descendant and it should drop out of the spectrum.
- Essentially, the above operator becomes the now regular part of the conformal block of ϕ

- The generic conformal block of ϕ has a singularity when

$$\Delta \to = \frac{d}{2} + k$$

- Hence, the above mechanism (multiplet recombination) happens when

$$d = \frac{2nk}{n-1}$$

- This is equivalent to having a marginal e.o.m. of the form

$$\partial^{2k}\phi \sim \phi^{2n-1}$$

- Our results for the anomalous dimensions are:

$$\gamma_p^{(1)} = \frac{(n-1)}{(n)_n}(p-n+1)_n$$

$$\gamma_1^{(2)} = (-1)^{k+1} 2 \frac{n\left(\frac{k}{n-1}\right)_k}{k\left(\frac{n\,k}{n-1}\right)_k}(n-1)^2\left[\frac{(n!)^2}{(2n)!}\right]^3$$

- and for the OPE coefficients

$$C_{1,p,p+2m+1} = \epsilon^2\left[\frac{(n!)^3}{(2n)!}\right]^2 \frac{(n-1)^2\Gamma(p+1)\Gamma(2m+p+2)}{\Gamma^2(n-m)\Gamma^2(n+m+1)\Gamma^2(m-n+p+2)}$$

$$\times \frac{\left(\frac{k}{n-1}\right)_k}{\left(\frac{n\,k}{n-1}\right)_k} \frac{\Gamma^2(k)\,(\frac{k}{n-1})_{2k}}{\left[(\frac{(m+1)k}{n-1})_k\,(\frac{-mk}{n-1})_k\right]^2}$$

- We can study O(N) models the same way….
- We can also study models with marginal deformations (like our σCFTs before in $d = 6 - \epsilon$) where the theory is deformed by σ^3 and $\phi^2\sigma$ (we also assume O(N) symmetry).
- Here $\mathcal{O}_d = g_1\sigma^2 + g_2\phi^2$ becomes a descendant of σ.
- However the following remains a primary.

$$\mathcal{O}_p = \frac{1}{\sqrt{2N}\sqrt{g_1^2 N + g_2^2}}(g_1 N\sigma^2 - g_2\phi^2)$$

- We obtain the following results that are consistent with the complicated loop calculations of ***[Giombi et. al. arXiv: 1404.1094]***

$$\frac{\gamma_\phi^{(1)}}{\gamma_\sigma^{(1)}} = \frac{2g_1^2}{g_1^2 N + g_2^2} \qquad C_{\sigma\sigma\sigma}^2 = \frac{12\, g_2^2}{g_1^2 N + g_2^2}\gamma_\sigma^{(1)}\epsilon$$

$$\frac{C_{\phi_i\phi_i\sigma}C_{\sigma\sigma\sigma}}{6\gamma_\sigma^{(1)}\epsilon} - \frac{2g_1 g_2}{g_1^2 N + g_2^2} = 0$$

OUTLOOK

- Generalized Free CFTs provide the backbone structure in the web of CFTs in any dimension.
- They have a rich analytic structure.
- They provide crucial information regarding the theory space near them.
- Using this information, we have confirmed all known results for the leading order anomalous dimensions at multicritical points in d=3,4,6 and and also 2<d<3 dimensions, including those for theories with O(N) symmetry.
- We have new results for non-unitary theories in d>6.
- The holographic duals?
- The d–> infinity limit?
- The relationship to real systems with long-range interactions?