9th Quantum Phase Transitions in Nuclei and Many-Body Systems

Breaking and restoration of rotational symmetry in the spectrum of $\alpha-$conjugate nuclei on the lattice

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24th May 2018

Motivation

We investigate **rotational symmetry breaking** in the low-energy spectra of

$$\boxed{\text{light } \alpha\text{-conjugate nuclei: } {}^{8}\text{Be},\ {}^{12}\text{C},\ {}^{16}\text{O}, \ldots}$$

on a cubic lattice. In particular, we aim at

- identifying lattice eigenstates in terms of SO(3) irreps

  $\Longrightarrow$ Phys. Lett. B 114, 147-151 (1982), PRL 103, 261001 (2009)
- exploring the dependence of physical observables on spacing and size

  $\Longrightarrow$ PR D 90, 034507 (2014), PR D 92, 014506 (2015)
- testing techniques for the suppression of discretization artifacts

  $\Longrightarrow$ Symanzik improvement scheme Lect. Notes in Phys. 788 (2010)

Applications

Nuclear Lattice EFT: ab initio nuclear structure PRL 104, 142501 (2010), PRL 106, 192501 (2011), PRL 112, 102501 (2014), PRL 117, 132501 (2016)
and scattering Nature 528, 111-114 (2015)

The Hamiltonian of the system

The macroscopic α-cluster model[1] of B. Lu et al. PR D 90, 034507 (2014) is adopted
$\Longrightarrow$ nuclei are decomposed into M structureless α-particles

$$H = -\frac{\hbar^2}{2m_\alpha}\sum_{i=1}^{M}\nabla_i^2 + \sum_{i>j=1}^{M}\left[\boxed{V_C(\mathbf{r}_{ij})} + \boxed{V_{AB}(\mathbf{r}_{ij})}\right] + \sum_{i>j>k=1}^{M}\boxed{V_T(\mathbf{r}_{ij}, \mathbf{r}_{ik}, \mathbf{r}_{jk})}$$

with $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. The potentials are of the type

Erf-Coulomb[2]

$$\frac{4e^2}{4\pi\epsilon_0}\frac{1}{r_{ij}}\mathrm{erf}\left(\frac{\sqrt{3}r_{ij}}{2R_\alpha}\right)$$

with $R_\alpha = 1.44$ fm
rms radius of the ^{4}He
NB: Erf *adsorbs* the
singularity at $r = 0$

Ali-Bodmer[2]

$$V_a\,\mathrm{f}\,e^{-\eta_a^2 r_{ij}^2} + V_r e^{-\eta_r^2 r_{ij}^2}$$

with $\eta_r^{-1} = 1.89036$ fm,
$V_r = 353.508$ MeV
and $\eta_a^{-1} = 2.29358$ fm,
$V_a = -216.346$MeV,
auxiliary param. f $= 1$

Gaussian

$$V_0 e^{-\lambda(r_{ij}^2 + r_{ik}^2 + r_{jk}^2)}$$

with $\lambda = 0.00506$ fm^{-2},
$V_0 = -4.41$ MeV for ^{12}C [3]
s.t. $E_{g.s.} = -\Delta E_{Hoyle}$
and $V_0 = -11.91$ MeV for ^{16}O [4]
s.t. $E_{g.s.} = -\Delta E_{4\alpha}$

[1] G.S. et al. JP G 43, 8 (2016), [2] NP 80, 99-112 (1966). , [3] Z. Physik A 290, 93-105 (1979). , [4] G.S. (2017)

The lattice environment

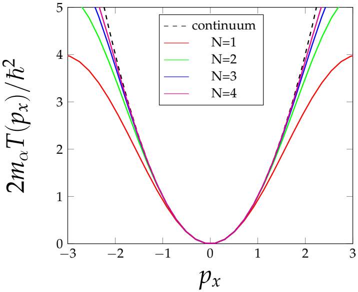
The configuration space in relative d.o.f. of an $M - body$ physical system into a cubic lattice reduces to

$$\boxed{\mathbb{R}^{3M-3} \longrightarrow L^{3M-3}}$$

where: $L \Longrightarrow$ number of points per dimension ($\equiv$ **lattice size**)
$a \Longrightarrow$ **lattice spacing**

Consequences: **discretization effects**

1. the action of differential operators is represented via finite differences: $\Longrightarrow$ Symanzik improvement scheme
2. breaking of Galiean invariance
3. breaking of continuous translational invariance (free-particle case)



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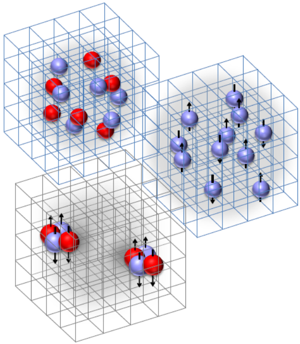
and **finite-volume effects** on physical observables



With periodic boundary conditions:

1. configuration space becomes isomorphic to a torus in $3M - 3$-dimensions
2. lattice momenta become $\mathbf{p} = \hbar \frac{2\pi \mathbf{n}}{La}$ where $\mathbf{n}$ is a vector of integers

Symmetries

On the lattice SO(3) symmetry reduces to the invariance under the cubic group $\mathcal{O}$.

Accordingly

«Only eight [five: A_1, A_2, E, T_1, T_2] different possibilities exist for rotational classification of states on a cubic lattice. So, the question arises: how do these correspond to the angular momentum states in the continuum? [...] To be sure of higher spin assigments and mass predictions it seems necessary to follow all the relevant irreps simultaneously to the continuum limit. »

R.C. Johnson, Phys. Lett. B 114, 147-151, (1982).

SO(3) to $\mathcal{O}$ irreps decomposition

$$D^0 = A_1, D^1 = T_1$$
$$D^2 = E \oplus T_2$$
$$D^3 = A_2 \oplus T_1 \oplus T_2$$
$$D^4 = A_1 \oplus E \oplus T_1 \oplus T_2$$
$$D^5 = E \oplus T_1 \oplus T_1 \oplus T_2$$
$$D^6 = A_1 \oplus A_2 \oplus E \oplus T_1 \oplus T_2 \oplus T_2$$

Symmetries

Degenerate states belonging to the same $\mathscr{O}$ irrep can be labeled with the irreps I_z of the cyclic group $\mathscr{C}_4$, generated by an order-three element of $\mathscr{O}$ (e.g. $\mathcal{R}_z^{\pi/2}$):

$$\begin{array}{ccc} \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow \\ l & & m, \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{O} & \supset & \mathcal{C}_4 \\ \downarrow & & \downarrow \\ \Gamma & & I_z, \end{array}$$

Conversely, the discrete symmetries of the Hamiltonian are preserved:

time reversal, parity, exchange symmetry

Applications

Within an iterative approach for the diagonalization of $\mathcal{H}$, the states belonging to an irrep Γ of a point group $\mathcal{G}$ can be extracted applying the projector

$$P_\Gamma = \sum_{g \in \mathcal{G}} \chi_\Gamma(g) D(g)$$

where $D(g)$ is a representation of dimension $3M - 3$ for the operation $g \in \mathcal{G}$

Finite-volume energy corrections

LO finite-volume energy corrections for relative two-body states with angular momentum ℓ and belonging to the Γ irrep of $\mathcal{O}$ PRL 107, 112011 (2011)

$$\Delta E_B^{(\ell,\Gamma)} = \alpha\left(\frac{1}{\kappa L}\right)|\gamma|^2\frac{e^{-\kappa L}}{\mu L} + \mathcal{O}\left(e^{-\sqrt{2}\kappa L}\right)$$

with γ the asymptotic normalization, κ the binding momentum and $\alpha(x)$ a polynomial:

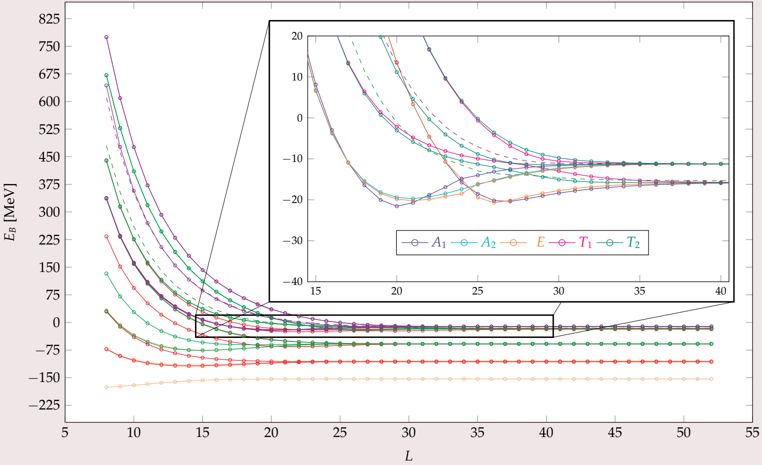
ℓ	Γ	$\alpha(x)$
0	A_1^+	-3
1	T_1^-	$+3$
	T_2^+	$30x + 135x^2 + 315x^3 + 315x^4$
2	E^+	$-\frac{1}{2}(15 + 90x + 405x^2 + 945x^3 + 945x^4)$
	A_2^-	$315x^2 + 2835x^3 + 122285x^4 + 28350x^5 + 28350x^6$
3	T_2^-	$-\frac{1}{2}(105x + 945x^2 + 5355x^3 + 19530x^4 + 42525x^5 + 42525x^6)$
	T_1^-	$-\frac{1}{2}(14 + 105x + 735x^2 + 3465x^3 + 11340x^4 + 23625x^5 + 23625x^6)$

Although no analytic LO FVEC formula for the three-body case exists, results for zero-range potentials PRL 114, 091602 (2015) and the asymptotic ($\equiv$ large L) behaviour are available Phys. Lett. B 779, 9-15 (2018).

The low-energy ^{8}Be spectrum

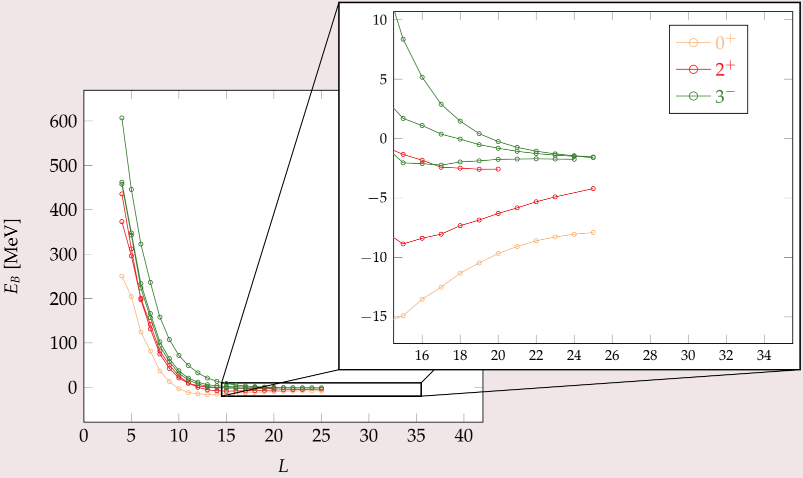
Increasing the parameter V_a of V_{AB}, a set of bosonic bound states can be analyzed.

f=2.5, $a = 0.25$ fm



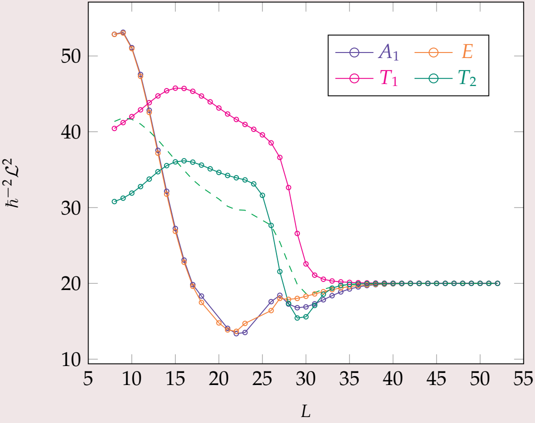
The low-energy ^{12}C spectrum

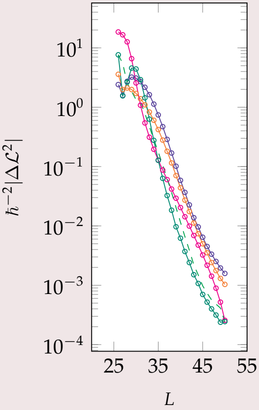
f=1.0, $a = 0.50$ fm (in progress)



The low-energy ^{8}Be spectrum: the 4_2^+ multiplet

f = 2.5, $a = 0.25$ fm

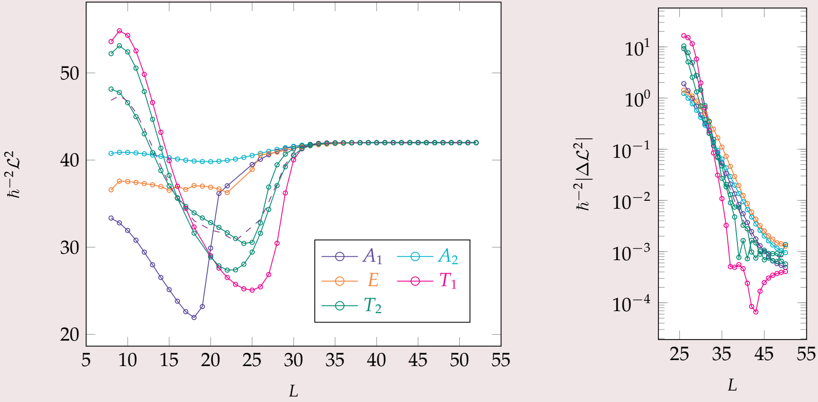




Remark: for $L \gtrsim 25$ $\quad |\Delta\mathcal{L}^2| \propto \exp(m_\kappa L)$ with $m_\kappa < 0$

The low-energy ^{8}Be spectrum: the 6_1^+ multiplet

f $= 2.5$, $a = 0.25$ fm

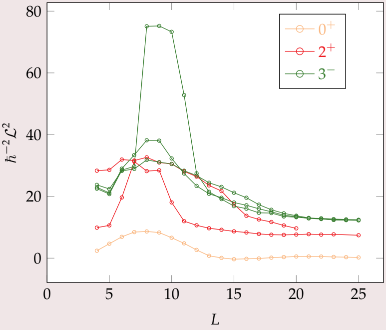


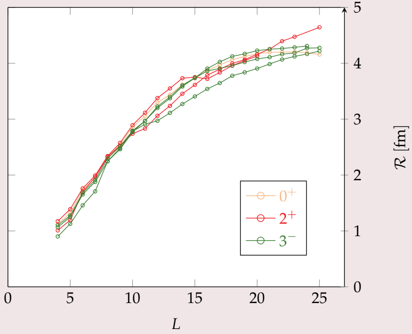
Remark: for $L \gtrsim 25$ $\quad |\Delta\mathcal{L}^2| \propto \exp(m_\kappa L)$ with $m_\kappa < 0$

The low-energy ^{12}C spectrum

As a consequence of the isotropy of the potentials, the nucleus has an equilateral triangular equilibrium configuration, i.e. $\langle r_{12} \rangle = \langle r_{23} \rangle = \langle r_{13} \rangle \equiv \mathcal{R}$.

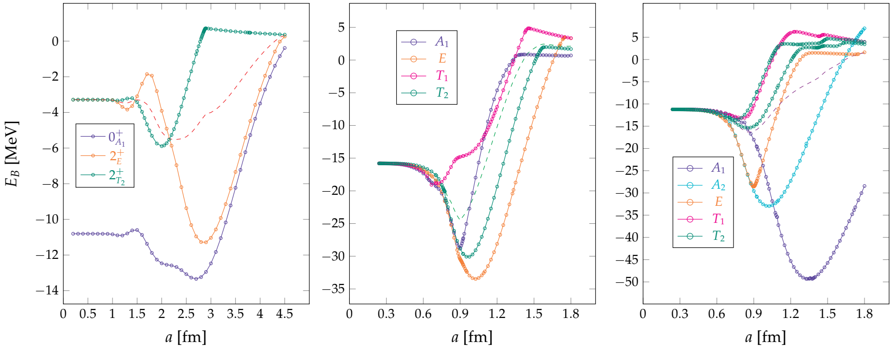
f = 1.0, a = 0.50 fm (in progress)





Discretization effects on energy

Unlike finite-volume effects, the dominant behaviour of dicretization corrections on energy, $\Delta E_B(a)$, is unknown.



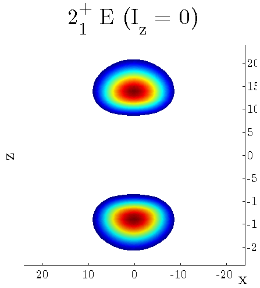
Nevertheless: some extrema of $E_B(a)$ can be associated to the maxima of the **p**robability **d**ensity **f**unction corresponding to the given energy eigenstate.

NB: If the primary maxima of the pdf lie at distance d^* w.r.t. the origin, the most probable $\alpha - \alpha$ separation $\mathcal{R}^*$ is given by d^*

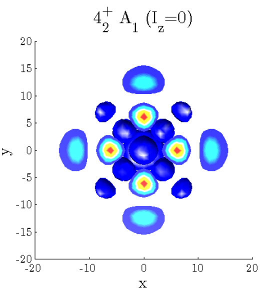
Discretization effects on energy

If the all pdf maxima are absolute and lie along the coordinate axes, ∃ a value of a s.t. all the maxima of the pdf are included in the cubic lattice.

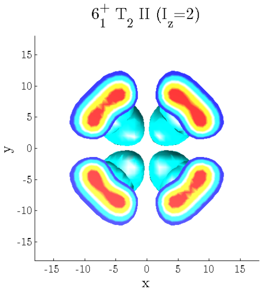
> In particular: for $a = d^* \implies E_B(a)$ is minimized and if $|\Psi_B^{\mathrm{Max}}|^2 \gg |\Psi_B(\mathbf{r})|^2$ where $|\mathbf{r}| = nd^*$ and $n \geq 2 \implies \langle \mathcal{R} \rangle \approx d^*$ and $\langle V \rangle$ is approximately minimized



✓ conditions fulfilled



✗ secondary maxima



✗ maxima off the axes

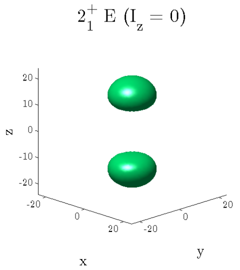
Discretization on ^{8}Be: the 2_1^+ E states

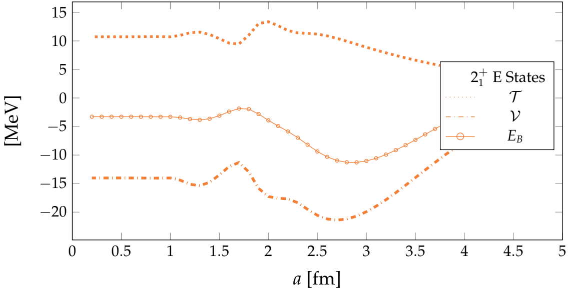
$\boxed{I_z = 0 \text{ Pdf}}$: two principal maxima along the z axis, located at a distance $d^* = 2.83$ fm from the origin.

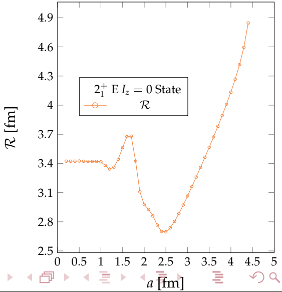
$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$\boxed{a = \frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.83, 1.42, 0.94, \ldots}$$

In practice: two E_B minima at $a \approx 1.36$ and 2.85 fm are observed







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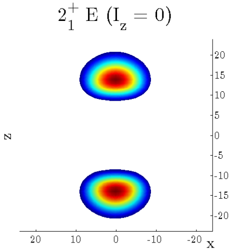

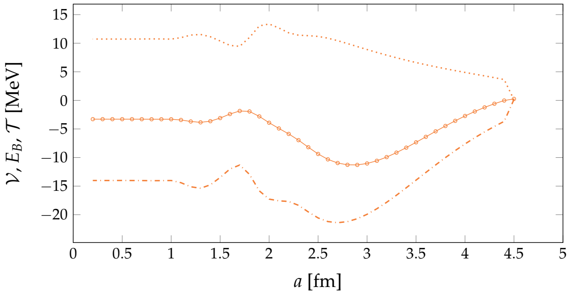

In addition:

$\mathcal{V} \approx -21.21$ MeV @ $a = d^*$
$\mathcal{V}^{min} \approx -21.40$ MeV @ $a \approx 2.70$ fm

and

$\mathcal{R} \approx 2.88$ fm @ $a = d^*$
$\mathcal{R}^{min} \approx 2.70$ fm @ $a \approx 2.50$ fm

Discretization on ^{8}Be: the 2_1^+ E states

$\boxed{I_z = 2 \text{ Pdf}}$: 4 principal maxima on the x and y axes, located at a distance $d^* = 2.83$ fm from the origin.

$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$\boxed{a = \frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.83, 1.42, 0.94, \ldots}$$

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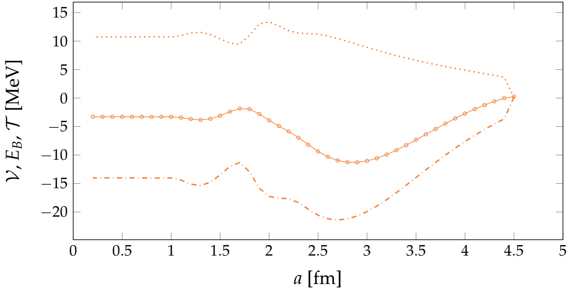
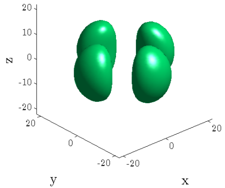


2_1^+ E ($I_z = 2$)



Still:

$$\mathcal{V} \approx -21.21 \text{ MeV} @ a = d^*$$
$$\mathcal{V}^{min} \approx -21.40 \text{ MeV} @ a \approx 2.70 \text{ fm}$$

Discretization on ^{8}Be: the 2_1^+ E states

$\boxed{I_z = 2 \text{ Pdf}}$: 4 principal maxima on the x and y axes, located at a distance $d^* = 2.83$ fm from the origin.

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$$\boxed{a = \frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.83, 1.42, 0.94, \dots}$$

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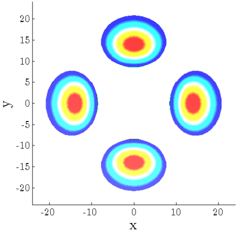

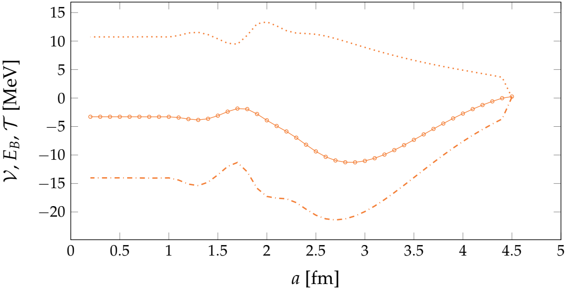

Still:

$\mathcal{V} \approx -21.21$ MeV @ $a = d^*$

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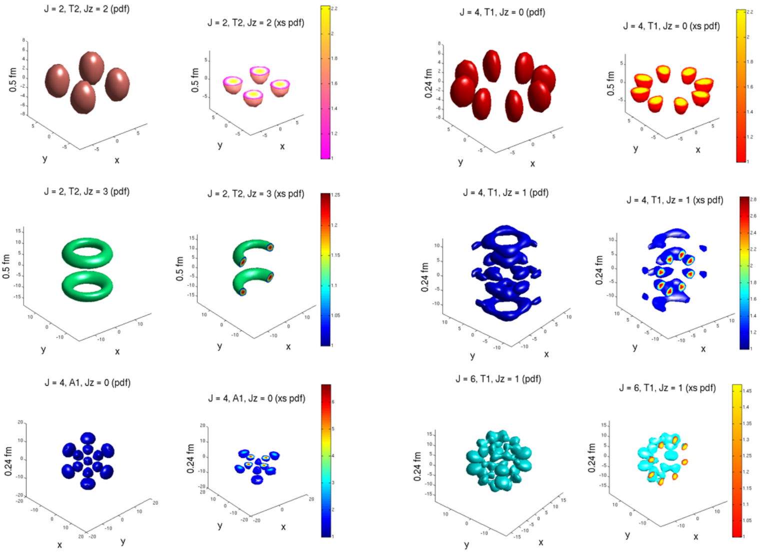
Other low-energy ^{8}Be wavefunctions



Conclusions & Outlook

The macroscopic α-cluster model in PR D 90, 034507 (2014) has been applied to the ^{8}Be and ^{12}C on the lattice. A fully-parallel method based on the power iteration has been adopted for the diagonalization of the Hamiltonian, allowing for

1. the reconstruction of a larger part of the low-energy spectrum of the two nuclei, made possible by the application of projectors;

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Perspectives and hints

- ♠ Extension of the analysis to the ^{16}O $\Rightarrow$ adaption of the present MPI diagonalization code to GPU calculations;
- ♠ Derivation of an analytical formula for the leading order FV energy corrections for bound states in presence of a Coulomb-type potential.

Thanks for your attention!

Grazie per l'attenzione!

Rotational Symmetry

On the lattice 3-dim rotational symmetry reduces to a subgroup of SO(3), the cubic group $\mathcal{O}$. A process of descent in symmetry takes place: α = x; y; z

$$\text{continuum, } \infty - \text{volume}: \;\; SO(3) \Longrightarrow [H, L^2] = 0, [H, L_\alpha] = 0$$

$$\Downarrow$$

$$\text{continuum, finite volume}: \;\; \mathcal{O} \subset SO(3) \Longrightarrow [H, L^2] = 0, [H, L_\alpha] \neq 0$$

$$\Downarrow$$

$$\text{discrete, finite volume}: \;\; \mathcal{O} \subset SO(3) \Longrightarrow [\mathcal{H}, \mathcal{L}^2] \neq 0, [\mathcal{H}, \mathcal{L}_\alpha] \neq 0$$

Accordingly

«Only eight [five: A_1, A_2, E, T_1, T_2] different possibilities exist for rotational classification of states on a cubic lattice. So, the question arises: how do these correspond to the angular momentum states in the continuum? [...] To be sure of higher spin assigments and mass predictions it seems necessary to follow all the relevant irreps simultaneously to the continuum limit. »

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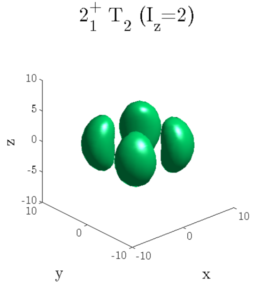
Discretization on ^{8}Be: the 2_1^+ T_2 states

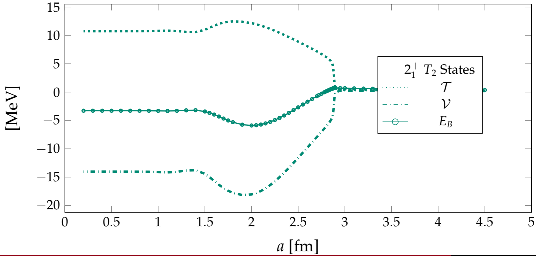
$\boxed{I_z = 2 \text{ Pdf}}$: four principal maxima in the intersection betw. the $z = 0$ plane and the $x = \pm y$ planes, s.t. $d^* = 2.83$ fm.

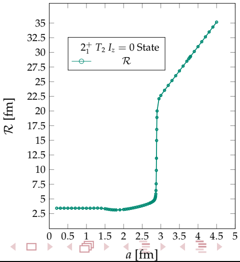
$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$\boxed{a = \frac{\sqrt{2}}{2}\frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.02, 1.01, 0.67, \ldots}$$

In practice: two E_B minima at $a \approx 1.05$ and 2.02 fm are observed







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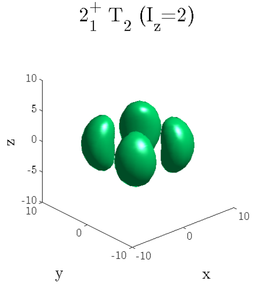
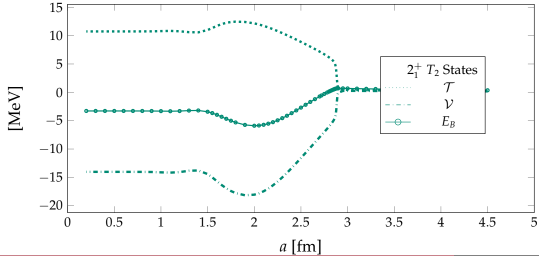
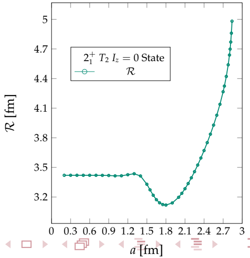
In practice: two E_B minima at $a \approx 1.05$ and 2.02 fm are observed







Discretization on ^{8}Be: the 2_1^+ T_2 states

$\boxed{I_z = 2 \text{ Pdf}}$: four principal maxima in the intersection betw. the $z = 0$ plane and the $x = \pm y$ planes, s.t. $d^* = 2.83$ fm.

$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$\boxed{a = \frac{\sqrt{2}}{2}\frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.02, 1.01, 0.67, \ldots}$$

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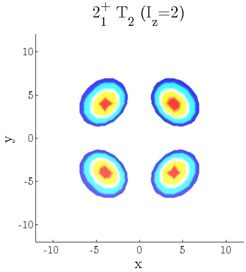
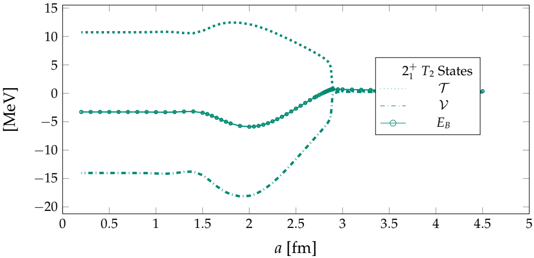




In addition:

$\mathcal{V} \approx -5.43$ MeV @ $a = d^*$
$\mathcal{V}^{min} \approx -18.05$ MeV @ $a \approx 1.15$ fm
and
$\mathcal{R} \approx 4.86$ fm @ $a = d^*$
$\mathcal{R}^{min} \approx 3.11$ fm @ $a \approx 1.78$ fm

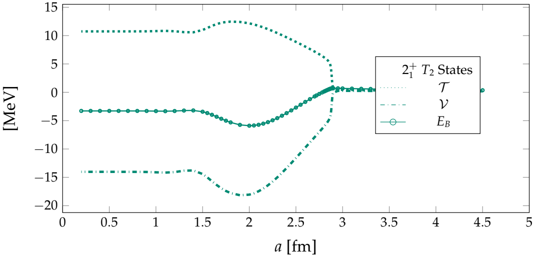
Discretization on ^{8}Be: the 2_1^+ T_2 states

$\boxed{I_z = 1, 3 \text{ Pdf}}$: 2 circles of principal maxima about the z axis,
located at a distance $d^* = 2.83$ fm from the origin.

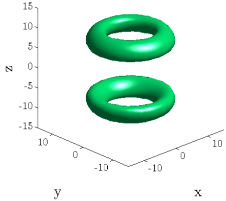
$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$\boxed{a = \frac{\sqrt{2}}{2}\frac{d^*}{n}} \text{ with } n \geq 1, \text{ i.e. } \boxed{a \approx 2.02, 1.01, 0.67, \ldots}$$

In practice: two E_B minima at $a \approx 1.05$ and 2.02 fm are observed



2_1^+ T_2 (I_z=1,3)



Still:

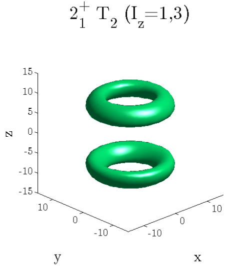
$$\mathcal{V} \approx -5.43 \text{ MeV} @ a = d^*$$
$\mathcal{V}^{min} \approx -18.05$ MeV @ $a \approx 1.15$ fm

Discretization on ^{8}Be: the 2_1^+ T_2 states

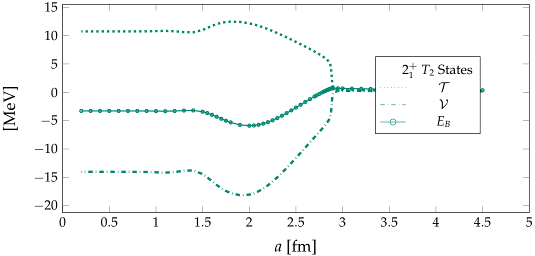
$\boxed{I_z = 1, 3 \text{ Pdf}}$: 2 circles of principal maxima about the z axis,
located at a distance $d^* = 2.83$ fm from the origin.

$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$\boxed{a = \frac{\sqrt{2}}{2}\frac{d^*}{n}}$ with $n \geq 1$, i.e. $\boxed{a \approx 2.02, 1.01, 0.67, \ldots}$



In practice: two E_B minima at $a \approx 1.05$ and 2.02 fm are observed



Still:

$\mathcal{V} \approx -5.43$ MeV @ $a = d^*$

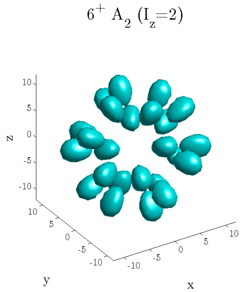
$\mathcal{V}^{min} \approx -18.05$ MeV @ $a \approx 1.15$ fm

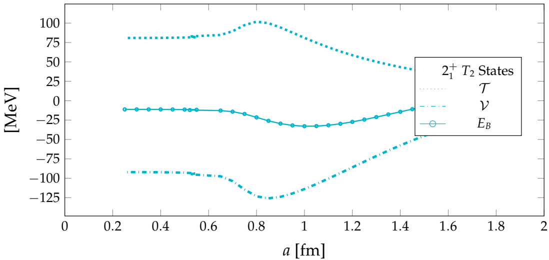
Discretization on ^{8}Be: the 6_1^+ A_2 state

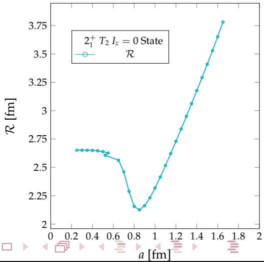
$\boxed{I_z = 2 \text{ Pdf}}$: four equidistant couples of principal maxima separated by an angle $\gamma \approx 34.2°$ and located at a distance $d^* \approx 2.31$ fm from the origin in the x, y and $z = 0$ planes.

$\Downarrow$

$\boxed{\text{The 24 maxima cannot be included on the lattice}}$







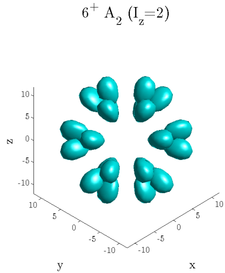
Discretization on ^{8}Be: the 6_1^+ A_2 state

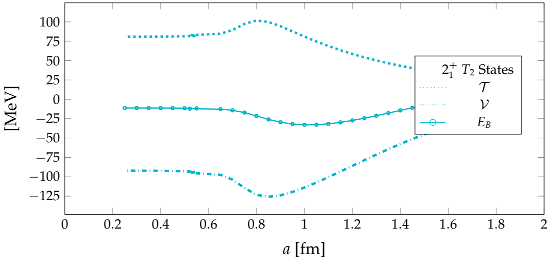
Considering the inclusion conditions of a couple of maxima in the 1st quadrant of the xy plane ($n \geq 1$):

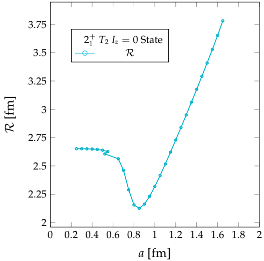
$$\boxed{a_x = \frac{d^*}{n}\cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}\ , \text{ i.e } \boxed{a_y \approx 2.04, 1.02, 0.68\ldots}$$

$$\boxed{a_y = \frac{d^*}{n}\sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}\ , \text{ i.e } \boxed{a_y \approx 1.08, 0.54, 0.36\ldots}$$

In practice: an E_B minimum at $a \approx 1.03$ fm is observed !







Discretization on ^{8}Be: the 6_1^+ A_2 state

Considering the inclusion conditions of a couple of maxima in the 1st quadrant of the xy plane ($n \geq 1$):

$$\boxed{a_x = \frac{d^*}{n}\cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)}, \text{ i.e } \boxed{a_y \approx 2.04, 1.02, 0.68...}$$

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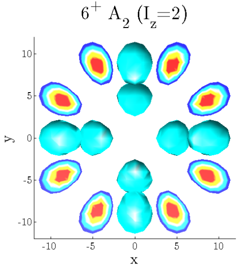
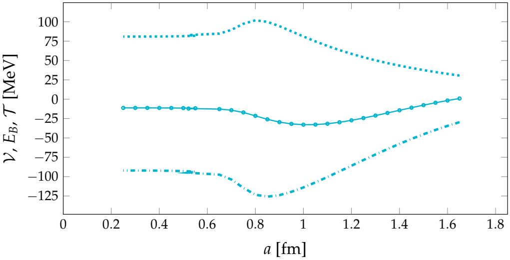
In practice: an E_B minimum at $a \approx 1.03$ fm is observed !





In addition:

$\mathcal{V} \approx 0.0$ MeV @ $a = d^*$ (unbound)

$\mathcal{V}^{min} \approx -125.85$ MeV @ $a \approx 0.85$ fm

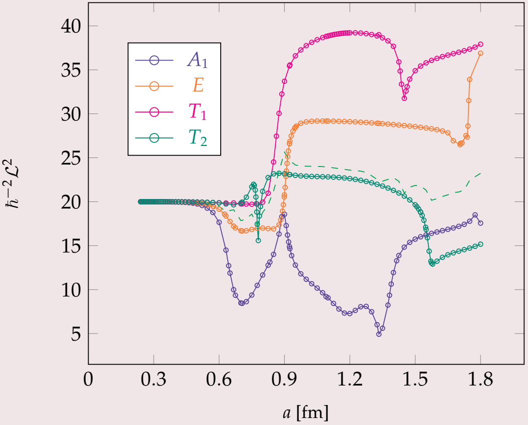
and

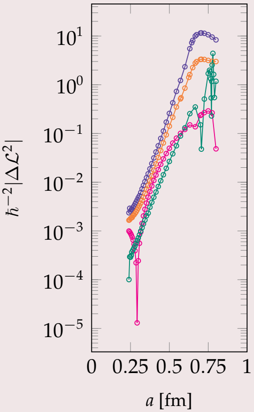
$\mathcal{R} \gg \mathcal{R}^{min}$ @ $a = d^*$

$\mathcal{R}^{min} \approx 2.13$ fm @ $a \approx 0.85$ fm

The low-energy ^{8}Be spectrum: the 4_2^+ multiplet

f = 2.5, $a = 0.25$ fm

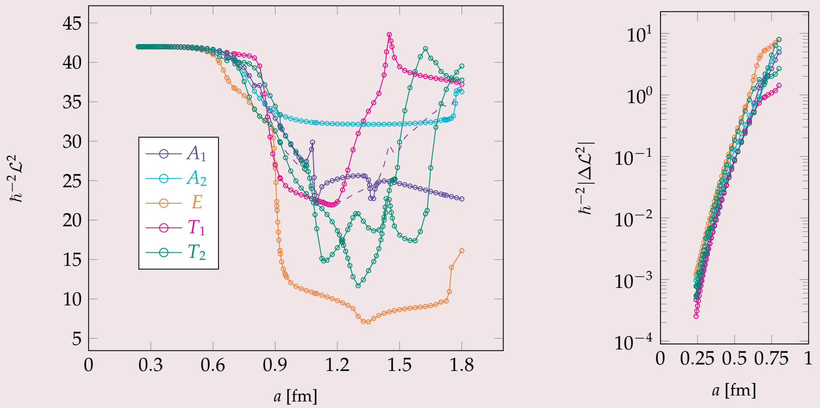




Remark: for $a \lesssim 0.80$ fm $\quad |\Delta\mathcal{L}^2| \propto \exp(c_\kappa a) \quad$ with $c_\kappa > 0$

The low-energy ^{8}Be spectrum: the 6_1^+ multiplet

f = 2.5, $a = 0.25$ fm



Remark: for $a \lesssim 0.80$ fm $\quad |\Delta\mathcal{L}^2| \propto \exp(c_\kappa a)$ with $c_\kappa > 0$