ESQPT Session @ QPTN9 Padova

Introduction to Excited-State Quantum Phase Transitions

Pavel Cejnar

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Plan:

1) Example: ESQPTs in the Dicke model
2) Classification & some signatures of ESQPTs
3) Overview of the Session

Example: ESQPTs @ Dicke model

M. Kloc, P. Stránský, P. Cejnar, Ann. Phys. 382 (2017) 85
T. Brandes, Phys. Rev. E 88 (2013) 032133
M. Bastarrachea-Magnani et al., Phys. Rev. A, 89 (2014) 032101

Based on [R.H. Dicke, Phys. Rev. 93 (1954) 99], the **Extended Dicke Model** model describes an ensemble of *N* two-level atoms in a cavity interacting with a single-mode bosonic field

$$\hat{H} = \omega \hat{b}^\dagger \hat{b} + \omega_0 \hat{J}_z + \frac{\lambda}{\sqrt{N}} \left[\hat{b}^\dagger \hat{J}_- + \hat{b} \hat{J}_+ + \delta \left(\hat{b}^\dagger \hat{J}_+ + \hat{b} \hat{J}_- \right) \right]$$

free Hamiltonian
of field & atoms with excitation energies ω and ω_0

atom-field interaction
governed by control parameter
$\lambda \in [0, \infty)$

weight parameter of counter-rotating terms
$\delta \in [0,1]$

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free Hamiltonian of field & atoms with excitation energies ω and ω_0

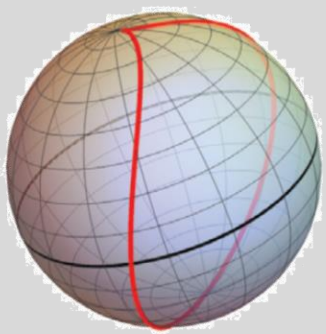
atom-field interaction governed by control parameter $\lambda \in [0, \infty)$

weight parameter of counter-rotating terms $\delta \in [0,1]$

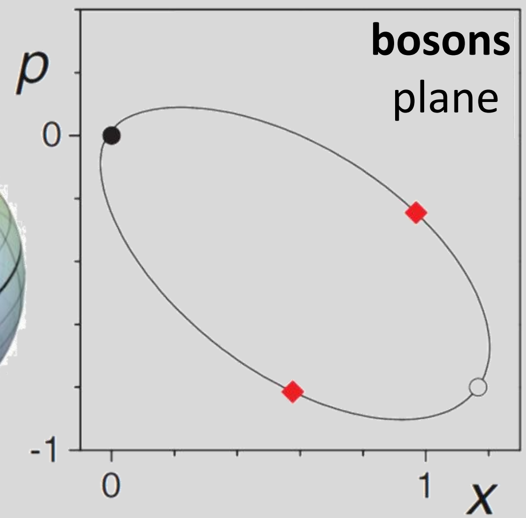
Classical limit: The model has $f = 2$ degrees of freedom, one associated with atoms, the other with bosons

phase space

atoms Bloch sphere



bosons plane



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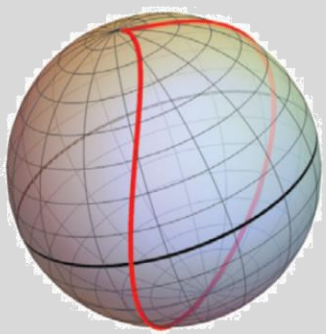
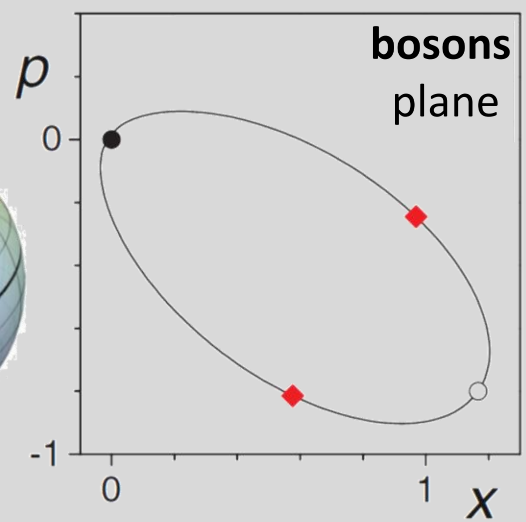




Integrable (Tavis-Cummings) regime: $\delta = 0$

Additional integral of motion

$$\hat{M} = \underbrace{\hat{b}^+ \hat{b}}_{\hat{n}} + \underbrace{\hat{J}_3 + j}_{\hat{n}^*}$$

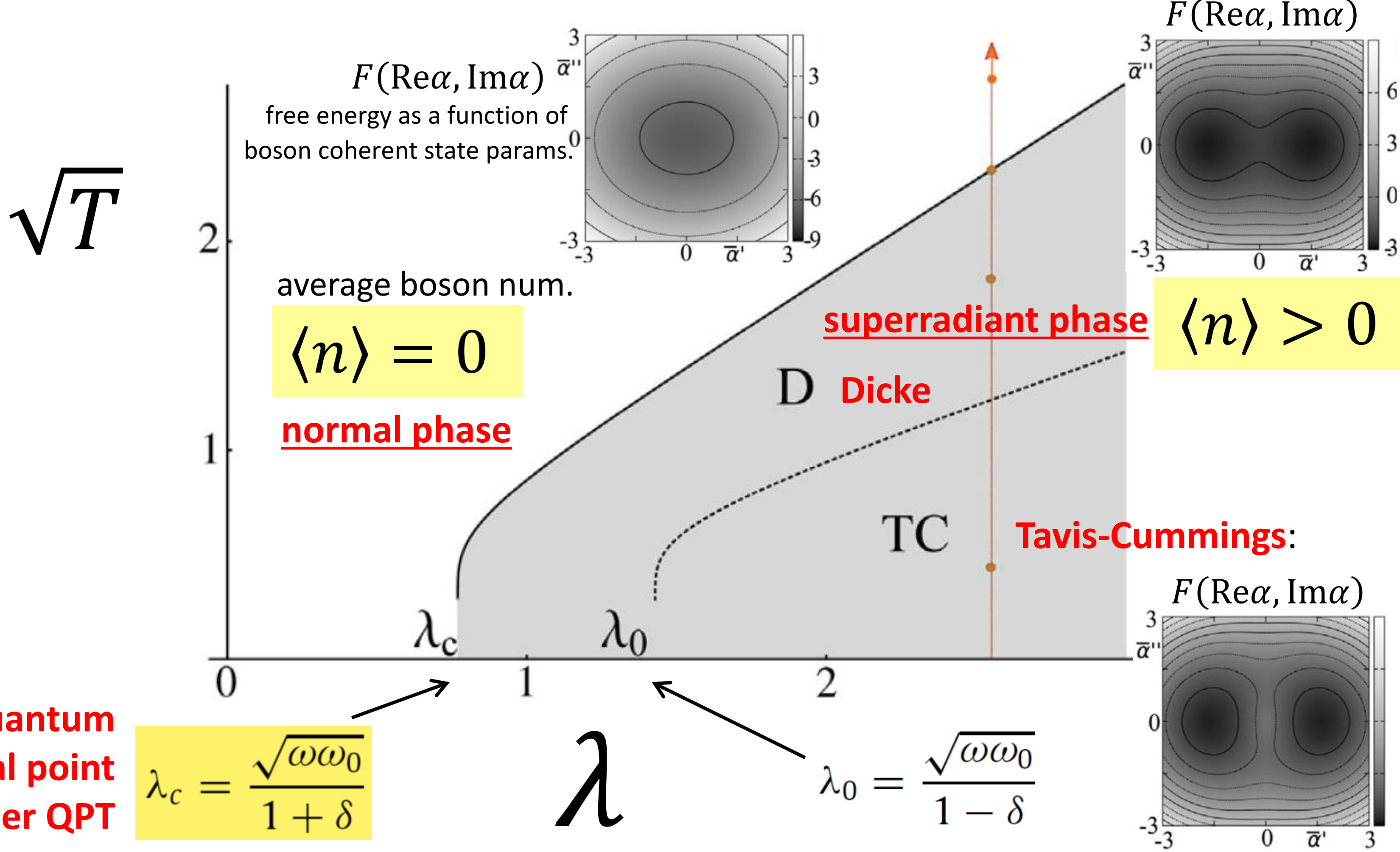
- Spectrum splits to noninteracting subsets: $M = 0,1 \ldots 2\mathrm{j}, \ldots$
- Number of effective degrees of freedom reduces: $f = 2 \rightarrow 1$

Example: ESQPTs @ Dicke model

C. Emary, T. Brandes, Phys. Rev. Lett. 90 (2003) 044101
M. Kloc, P. Stránský, P. Cejnar, Ann. Phys. 382 (2017) 85
M. Bastarrachea-Magnani et al., J. Stat. Mech. (2016) 093105

The model shows a **thermal phase transition** between normal and superradiant phases, first predicted in [Y.K. Wang, F.T. Hioe, Phys. Rev. A 7 (1973) 831, K. Hepp, E.H. Lieb, Phys. Rev. A 8 (1973) 2517]

At $T = 0$, this transition becomes a **quantum phase transition**



Quantum critical point 2nd order QPT

$$\lambda_c = \frac{\sqrt{\omega\omega_0}}{1+\delta}$$

$$\lambda_0 = \frac{\sqrt{\omega\omega_0}}{1-\delta}$$

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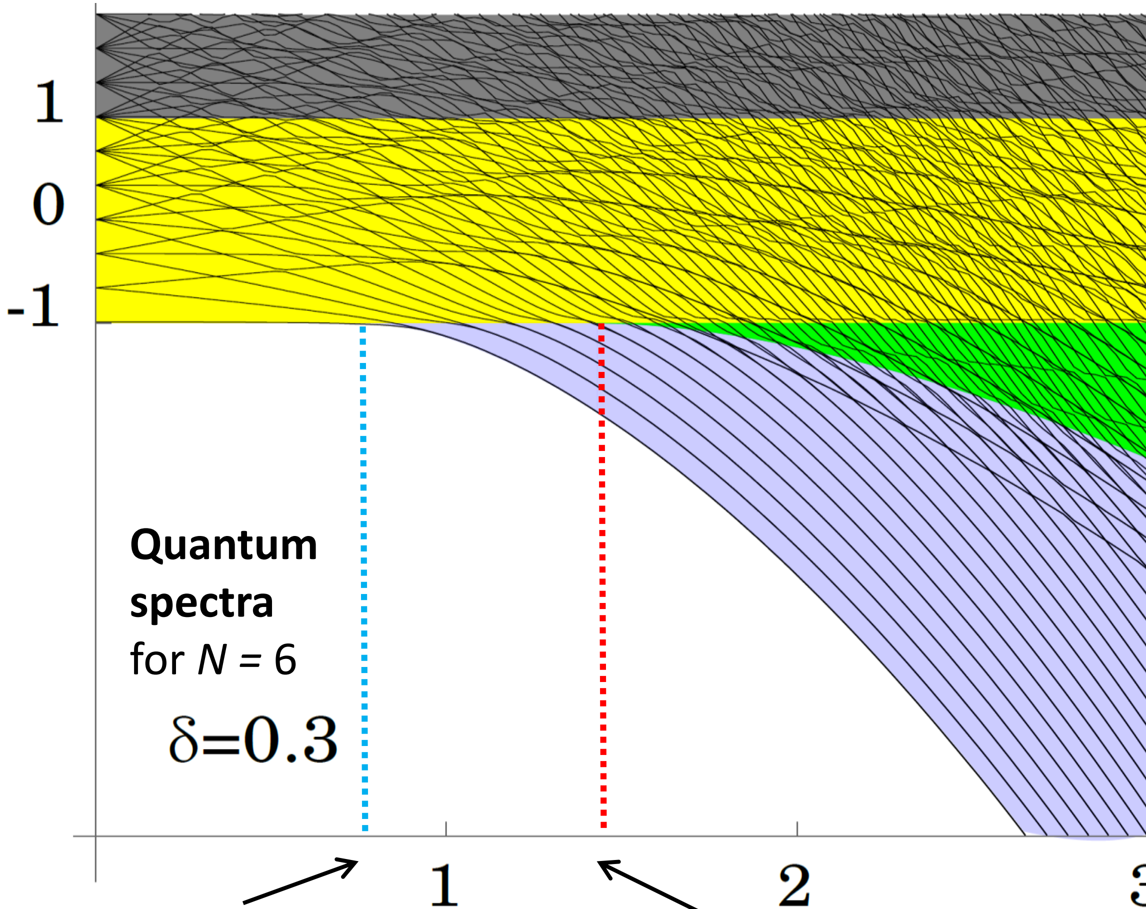
The model also shows several kinds of **Excited-State Quantum Phase Transition** and the corresponding types of **quantum phases**

$$\frac{1}{\omega_0 j}E$$



S: $\frac{E_{c3}}{\omega_0 j} = +1$ for $\lambda \in [0,\infty)$

N: $\frac{E_{c2}}{\omega_0 j} = -1$ for $\lambda \in [\lambda_0,\infty)$

TC: $\frac{E_{c1}}{\omega_0 j} = \begin{cases} -1 & \text{for } \lambda \in [\lambda_c,\lambda_0) \\ -\frac{1}{2}\left(\frac{\lambda_0^2}{\lambda^2}+\frac{\lambda^2}{\lambda_0^2}\right) & \text{for } \lambda \in [\lambda_0,\infty) \end{cases}$

D: $\frac{E_{\text{g.s.}}}{\omega_0 j} = \begin{cases} -1 & \text{for } \lambda \in [0,\lambda_c) \\ -\frac{1}{2}\left(\frac{\lambda_c^2}{\lambda^2}+\frac{\lambda^2}{\lambda_c^2}\right) & \text{for } \lambda \in [\lambda_c,\infty) \end{cases}$

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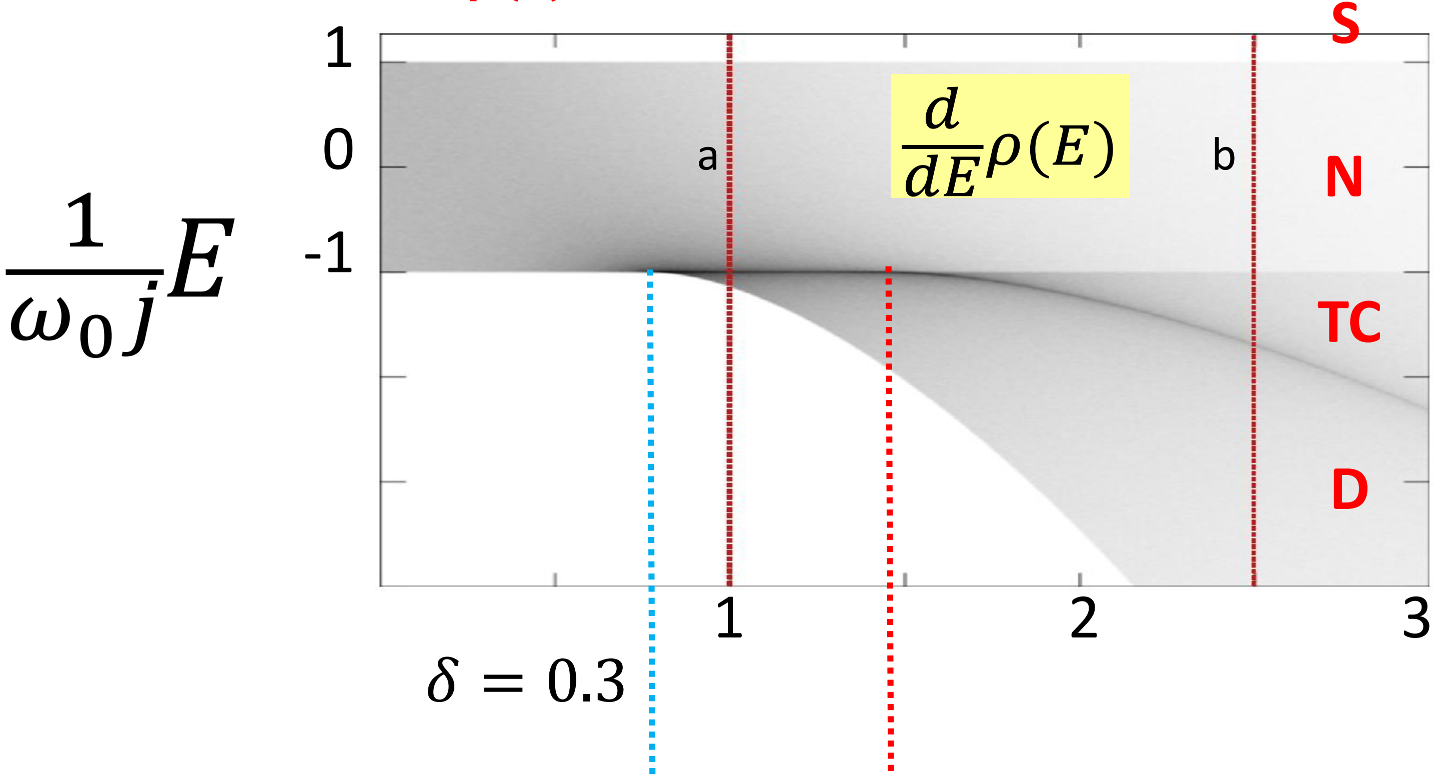
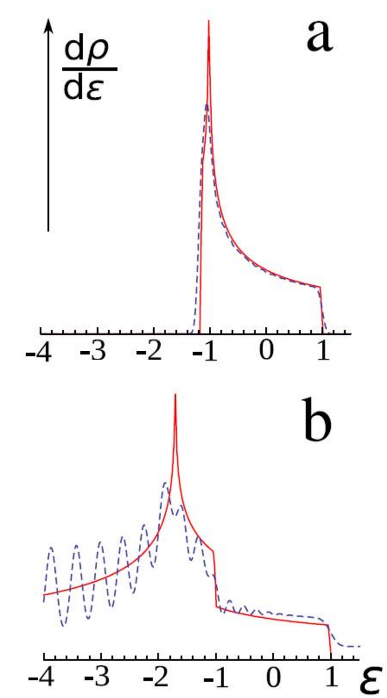
The model also shows several kinds of **Excited-State Quantum Phase Transition** and the corresponding types of **quantum phases**. ESQPTs in general show up as **singularities of the quantum state density $\rho(E)$**, in the present case its **first derivative**





λ

Quantum critical point 2nd order QPT

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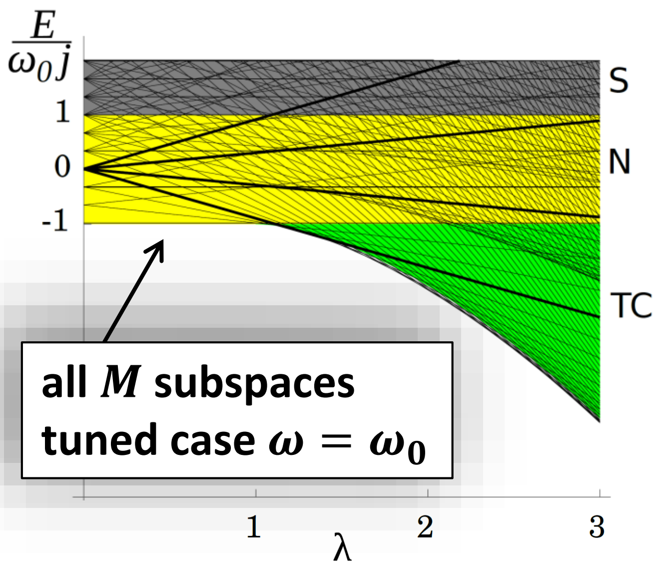
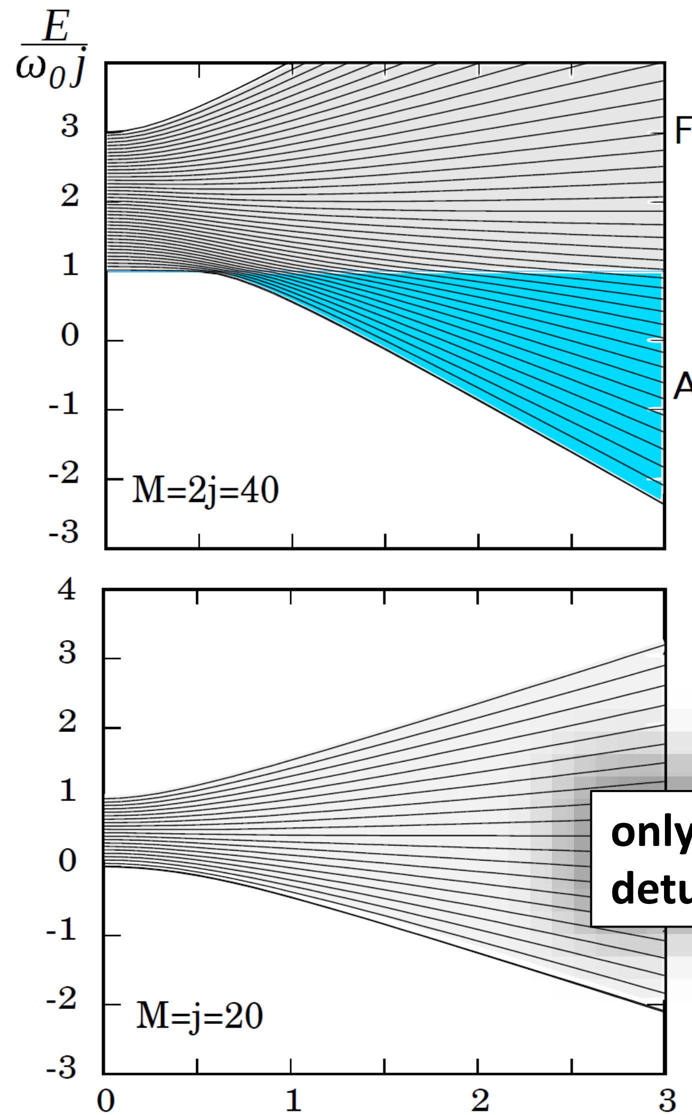
P. Pérez-Fernández, P. Cejnar et al., Phys. Rev. A 83 (2011) 033802

In the **Tavis-Cummings regime**, subspaces with different values of the conserved quantity M can be treated separately. The subspace $M = 2j$ shows both **QPT & ESQPT** properties



all M subspaces
tuned case $\omega = \omega_0$



only single M subspaces
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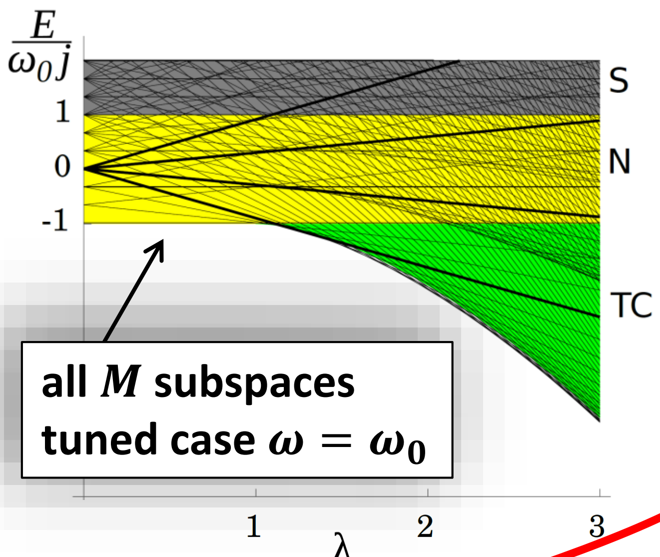

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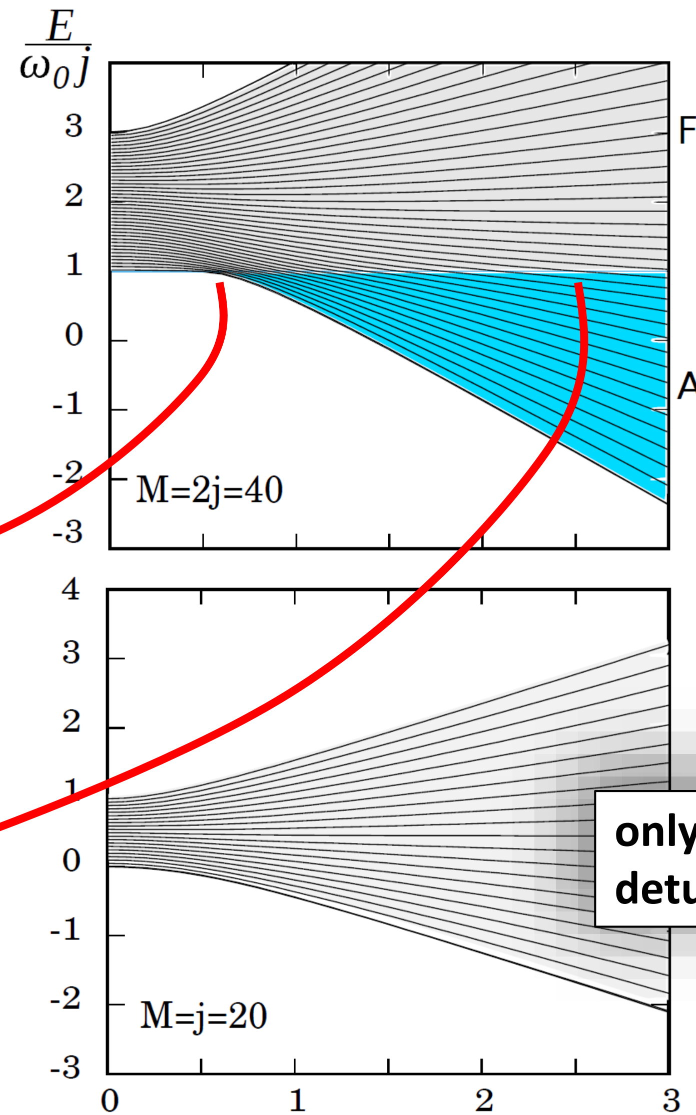

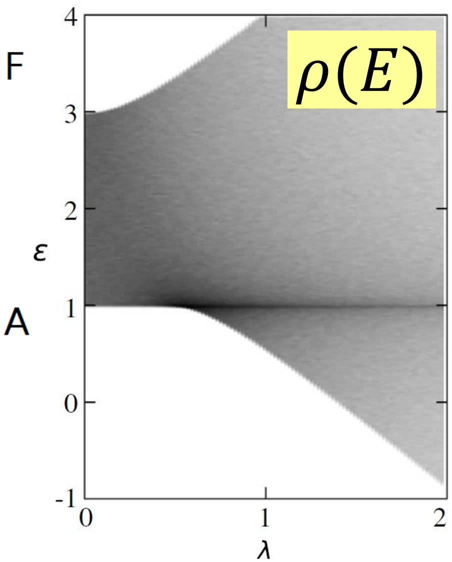

only single M subspaces
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QPT 2[nd] order

$$\bar{\lambda}_c = \frac{\omega - \omega_0}{2}$$

ESQPT with $\rho \to \infty$

$$\frac{E_{c4}}{\omega_0 j} = +1 \quad \text{for } \lambda \in [\bar{\lambda}_c, \infty)$$

ESQPT classification

P. Stránský, P. Cejnar, Phys. Lett. A 380 (2016) 2637
P. Stránský, M. Macek, P. Cejnar, Ann. Phys. 345 (2014) 73
M. Kastner, Rev. Mod. Phys. 80 (2008) 167

The classical Hamiltonian function $H(\underbrace{q,p}_{x})$ of the system determines the smooth part of the **quantum density of states**

$$\sum_l \delta(E-E_l) \equiv \rho(E) = \overline{\rho}(E) + \tilde{\rho}(E)$$

smooth & **oscillatory** components

not relevant for $N \to \infty$
but sometimes very relevant for finite N, see P. Stránský et al., Ann. Phys. 356 (2015) 57

$$\bar{\rho}(E) = \frac{1}{(2\pi\hbar)^f} \iint d^{2f}x\ \delta[E-H(x)] = \frac{1}{(2\pi\hbar)^f} \int\limits_{H(x)=E} d^{2f-1}\sigma \left|\nabla_{2f}H(x)\right|^{-1}$$

Integral over (2f –1)-dim energy hypersurface in the 2f-dim phase space. For analytic Hamiltonians it is **nonanalytic only at stationary points**.

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Nondegenerate (quadratic) stationary point [gradient $\boldsymbol{\partial_i H = 0}$, Hessian $\mathbf{det}\, \boldsymbol{\partial_i \partial_j H \neq 0}$]

For such points the type of nonanalyticity in $\bar{\rho}$ can be determined explicitly.
It depends on: **(f,r)**

f ... number of degrees of freedom
r ... index of stationary point = number of negative eigenvalues of the Hessian matrix

$$\frac{\partial^{f-1}\bar{\rho}}{\partial E^{f-1}} \propto \begin{cases} (-)^{\frac{r+1}{2}} \ln|\delta E| & \text{... for } r \text{ odd} \\ (-)^{\frac{r}{2}} \quad \Theta(\delta E) & \text{... for } r \text{ even} \end{cases}$$

step function

(f,r) provide a **classification of ESQPT** connected with any nondegenerate stationary point of a general Hamiltonian

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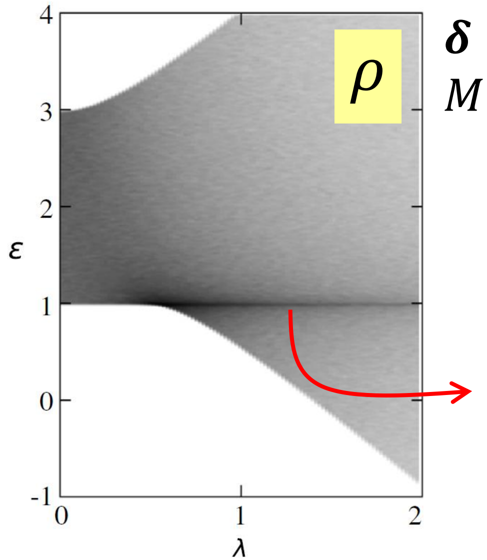


$\delta = 0$ model
$M = 2j$ subspace

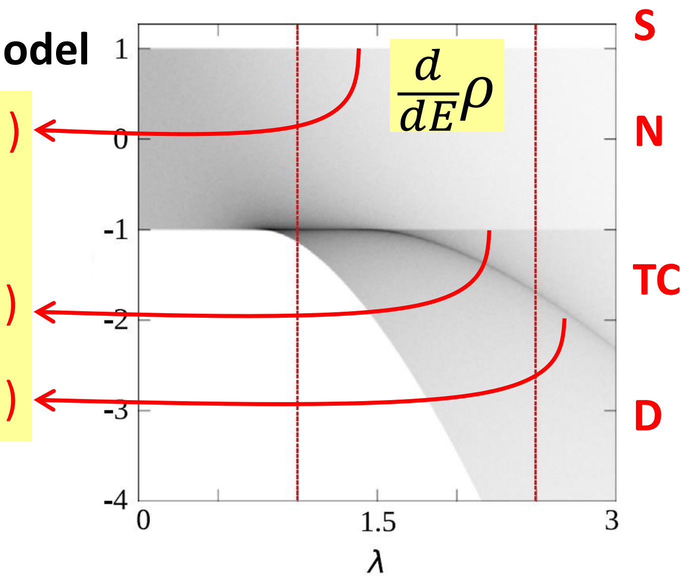
$(f, r) \sim (1,1)$

$\delta = 0.3$ model

$(f, r) = (2,2)$

$(2,2)$

$(2,1)$



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The “flow” (slope, velocity) of the spectrum with running control parameter

$$\bar{\rho}(\lambda, E) = \sum_l F(E - E_l(\lambda))$$

smoothed **level density** (*F* = smoothing function)

$$\bar{\phi}(\lambda, E) = \frac{1}{\bar{\rho}(\lambda, E)} \sum_l F(E - E_l(\lambda)) \frac{dE_l(\lambda)}{d\lambda}$$

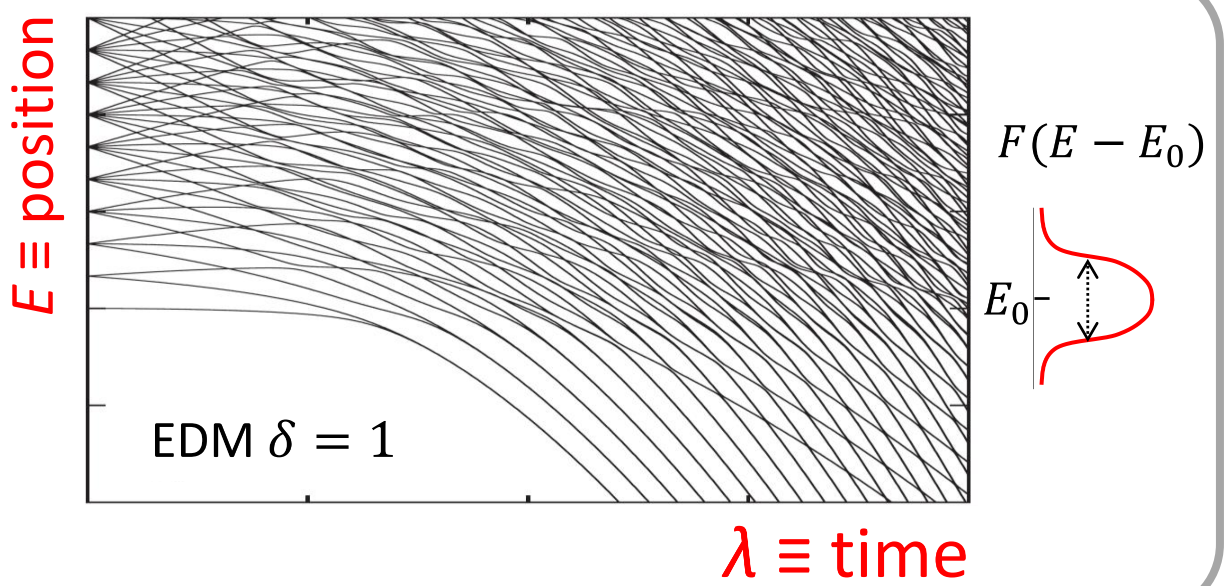
smoothed **flow rate** = average slope of the spectrum at given energy & parameter

$$\frac{\partial}{\partial\lambda}\bar{\rho}(\lambda, E) + \frac{\partial}{\partial E}\left[\bar{\rho}(\lambda, E)\bar{\phi}(\lambda, E)\right] = 0$$

continuity equation

$\Downarrow$

At the ESQPT energy, the smoothed flow rate generically exhibits **the same type of nonanalyticity** as the smoothed level density



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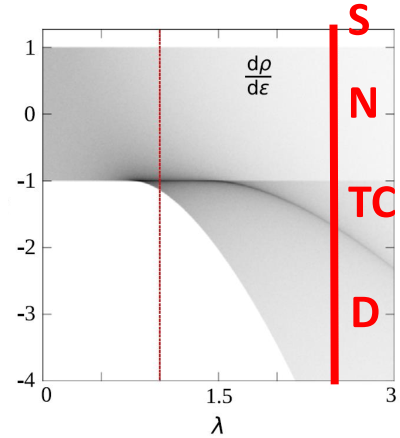
For Hamiltonians of the form $\hat{H} = \hat{H}_0 + \lambda\hat{V}$ the slope of energy levels is given by the **Hellman-Feynman formula**:

$$\frac{dE_l}{d\lambda} = \langle\psi_l|\hat{V}|\psi_l\rangle$$

$\Rightarrow$ The smoothed flow rate is related to "**order parameters**" of quantum phases below and above the ESQPT

$$\frac{\partial^{f-1}}{\partial E^{f-1}}\overline{\langle\psi_l|\hat{V}|\psi_l\rangle} = \frac{\partial^{f-1}}{\partial E^{f-1}}\bar{\phi} \sim \frac{\partial^{f-1}}{\partial E^{f-1}}\rho$$



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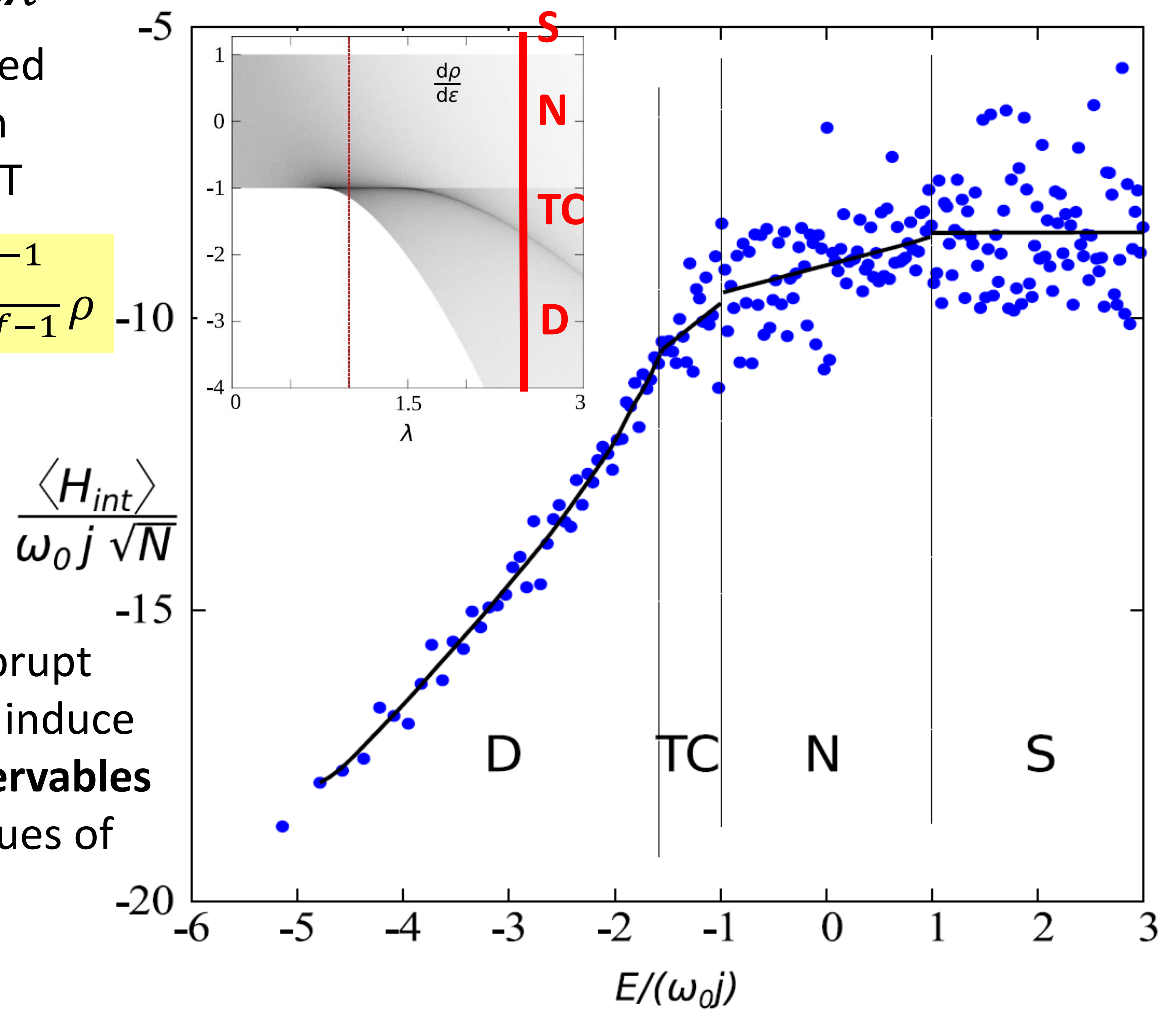
$\Downarrow$

ESQPTs do not in general cause abrupt changes of order parameters, but induce "**changes of trends**" **of some observables** that may be considered as analogues of order parameters

EDM $\delta = 0.3$



Quantum phases

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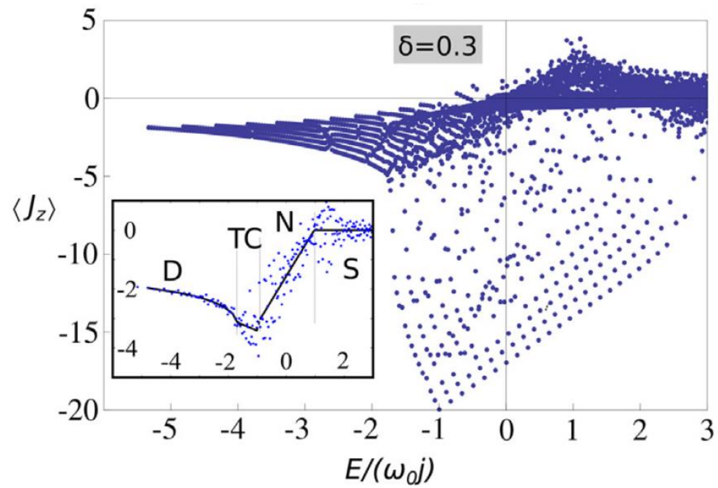
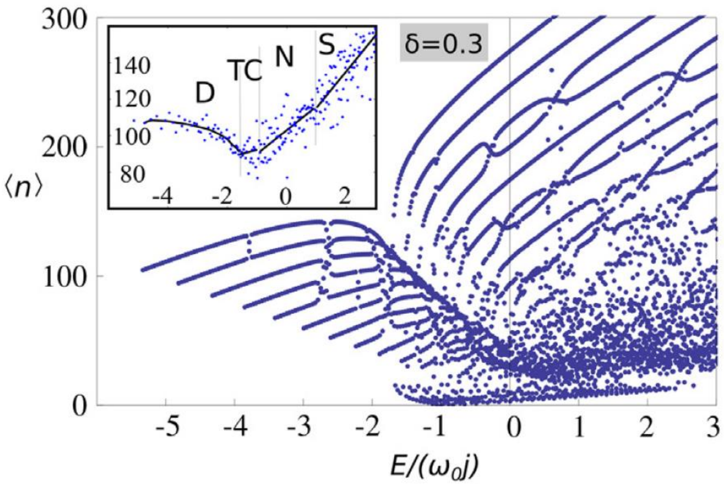
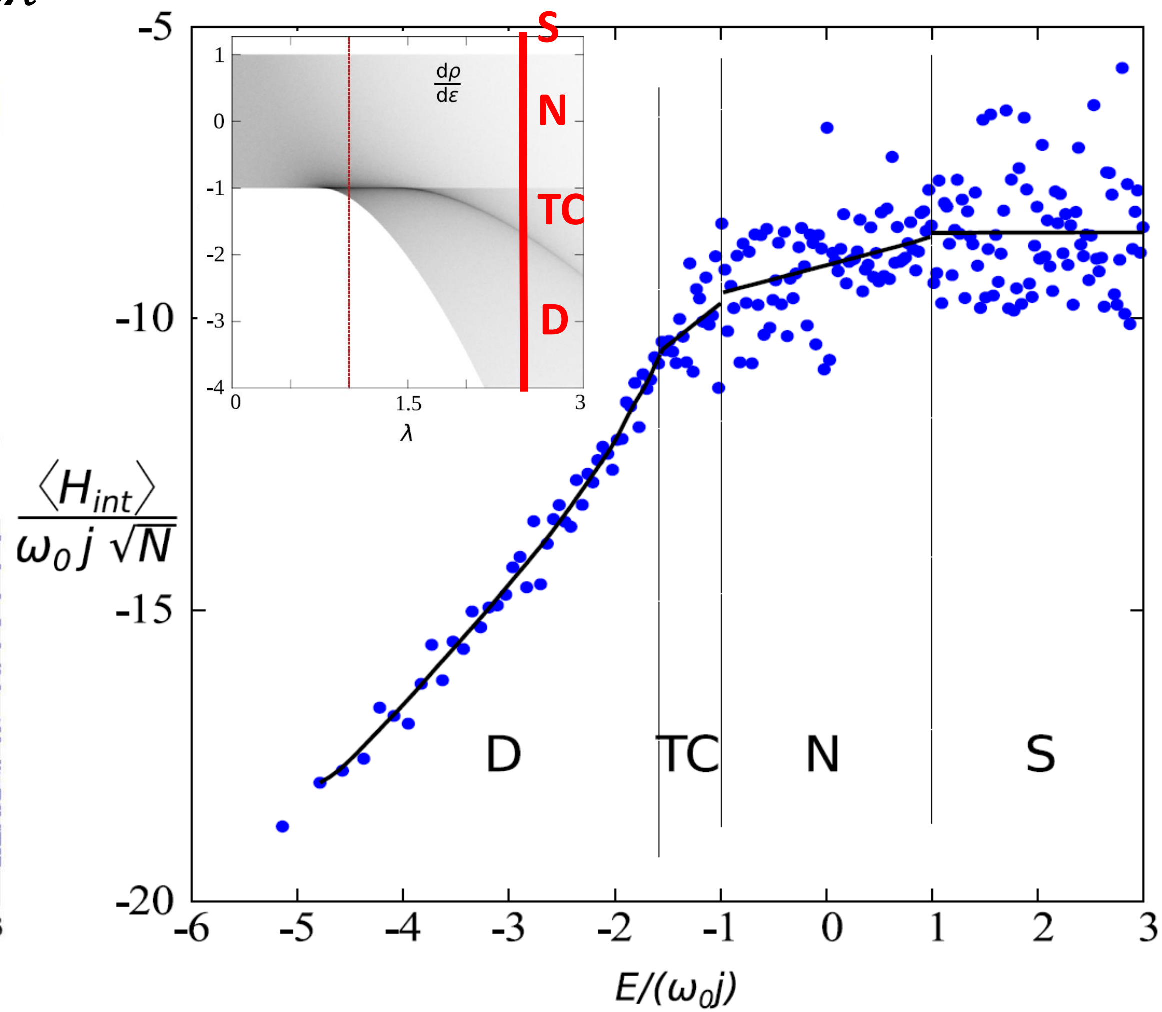
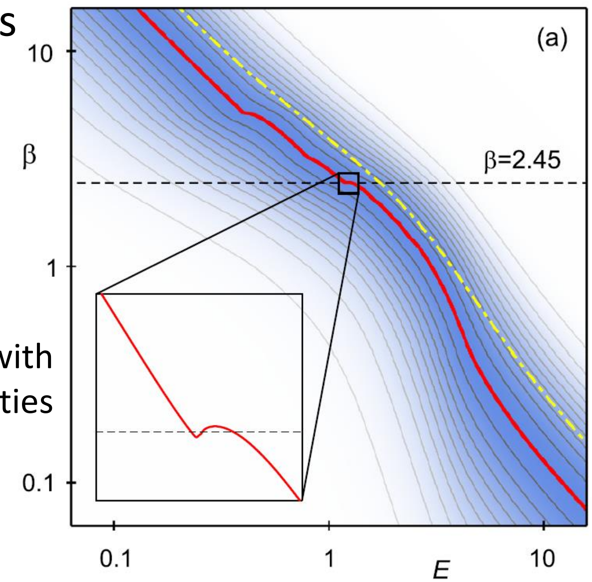






(Thermo)dynamical consequences of ESQPT

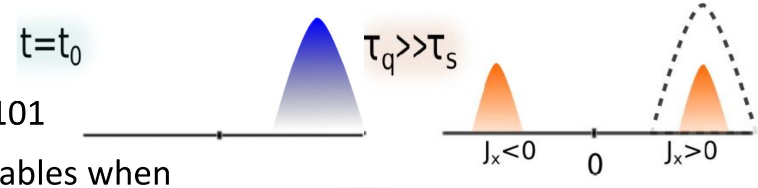
ESQPT affect various thermodynamical & dynamical properties



- **Canonical & microcanonical thermodynamics**
  - P. Cejnar, P. Stránský, Phys. Lett. A 381 (2017) 984
  - P. Pérez-Fernández, A. Relaño, Phys. Rev. (2017) E 96, 012121
  - M. Bastarrachea-Magnani et al., J. Stat. Mech. (2016) 093105
  - P. Stránský, M. Macek, P. Cejnar, Ann. Phys. 345 (2014) 73

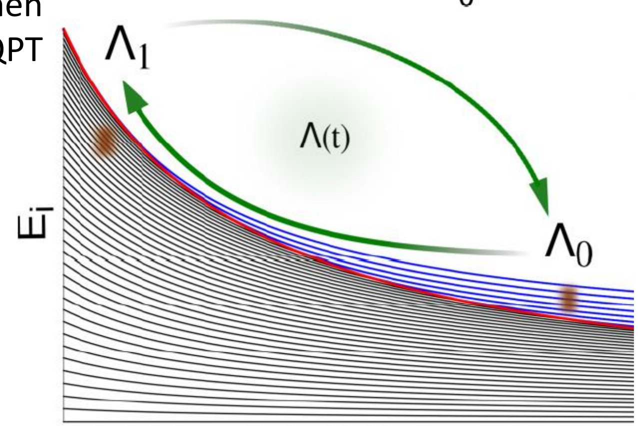
Example: canonical and microcanonical caloric curves in a system with multiple ESQPTs. The microcanonical curve (red) shows irregularities

- **Open systems**
  - W. Kopylov, T. Brandes, New J.Phys. 17 (2015) 103031
  - A. Relaño et al., Phys. Rev. A 78 (2008) 060102



- **Driven systems**
  - **adiabatic** R. Puebla, A. Relaño, Phys. Rev. E 92 (2015) 012101

Example: irreversibility observed in some observables when performing an adiabatic cycle across the ESQPT



  - **periodic** V.M. Bastidas et al., Phys. Rev. Lett. 112 (2014) 140408
    Phys. Rev. A 90 (2014) 063628

- **Quantum Quench Dynamics**
  - L. Santos, M. Távora, F. Pérez-Bernal, Phys. Rev. A 94 (2016) 012113
  - L. Santos, F. Pérez-Bernal, Phys. Rev. A 92 (2015) 050101
  - P. Pérez-Fernández et al., Phys. Rev. A 83 (2011) 033802

Program of the Session

Driven and dissipative dynamics

- **Wassilij Kopylov**: ESQPT in the LMG model and its influence on the non-adiabatic dynamics
- **Wassilij Kopylov**: Smearing out of the ESQPT properties in the dissipative LMG model and their restore by delayed feedback control

Quantum quenches

- **Lea Santos**: Nonequilibrium quantum dynamics: from full random matrices to real systems
- **Michal Kloc**: Quantum quench dynamics in an extended Dicke model

Exceptional points

- **Milan Šindelka**: ESQPTs studied from a non-Hermitian perspective
- **Pavel Stránský**: Exceptional points for randomly perturbed critical Hamiltonians

ESQPT in nuclear structure

- **Rostislav V. Jolos**: Analytical description of the ESQPT to octupole deformed shape in alternating parity bands

Level dynamics

- **Akhi Qureshi**: Landau-Zener transitions in the Pechukas-Yukawa formalism under the influence of Brownian noise