The phase diagram of the (extended) Agassi model

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Why the Agassi model?

- It is a solvable many-body model that allows to mimic the main characteristics of the pairing-plus-quadrupole model.
- It can be exactly solved even in the case of large systems.
- Nowadays, it is used to benchmark many-body approximations because of its great flexibility and simplicity to be solved for large systems.
- The model (and in particular its extension) owns a very rich phase diagram.
- The model is, somehow, an extension of the two-level Lipkin-Meshkov-Glick model that incorporates pairing interaction.

Agassi model

Dan Agassi

"Validity of the BCS and RPA approximations in the pairing-plus-monopole solvable model", Dan Agassi, Nuclear Physics A **116**, 49 (1968).

The original Hamiltonian

$$H = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a^\dagger_{m\sigma} a_{m\sigma} + \frac{1}{2} V \sum_{mm'\sigma} a^\dagger_{m\sigma} a^\dagger_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}$$
$$-\frac{1}{4} G \sum_{mm'\sigma\sigma'} a^\dagger_{m\sigma} a^\dagger_{-m\sigma} a_{-m'-\sigma'} a_{m'\sigma'}$$

$\sigma = +1, -1$ and $m = -j, ..., -2, -1, 1, 2, ..., j$. Degeneracy $\Omega = 2j$

Agassi model

The O(5) as spectrum generator algebra

$$J^{+} = \sum_{m=-j}^{j} c^{\dagger}_{1,m} c_{-1,m} = \left(J^{-}\right)^{\dagger}, \quad J^{0} = \frac{1}{2} \sum_{m=-j}^{j} \left(c^{\dagger}_{1,m} c_{1,m} - c^{\dagger}_{-1,m} c_{-1,m}\right),$$

$$A^{\dagger}_{1} = \sum_{m=1}^{j} c^{\dagger}_{1,m} c^{\dagger}_{1,-m}, \; A^{\dagger}_{-1} = \sum_{m=1}^{j} c^{\dagger}_{-1,m} c^{\dagger}_{-1,-m}, \; A^{\dagger}_{0} = \sum_{m=1}^{j} \left(c^{\dagger}_{-1,m} c^{\dagger}_{1,-m} - c^{\dagger}_{-1-m} c^{\dagger}_{1,m}\right)$$

$$A_{1} = \sum_{m=1}^{j} c_{1,-m} c_{1,m}, \; A_{-1} = \sum_{m=1}^{j} c_{-1,-m} c_{-1,m}, \; A_{0} = \sum_{m=1}^{j} \left(c_{1,-m} c_{-1,m} - c_{1,m} c_{-1,-m}\right),$$

$$N_{\sigma} = \sum_{m=-j}^{j} c^{\dagger}_{\sigma,m} c_{\sigma,m}, \qquad N = N_{1} + N_{-1}.$$

A pictorial view

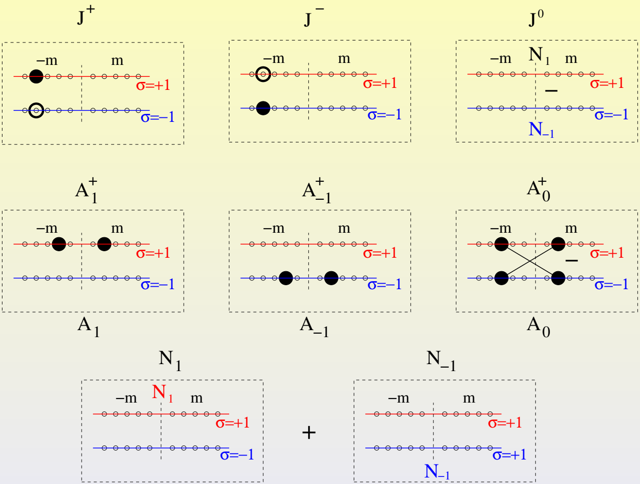

Figure: A pictorial view of the ten O(5) generators in the Agassi model Hilbert space

The Hamiltonian

$$H = \varepsilon J^0 - g \sum_{\sigma\sigma'} A^\dagger_\sigma A_{\sigma'} - \frac{V}{2}\left[\left(J^+\right)^2 + \left(J^-\right)^2\right] - 2hA^\dagger_0 A_0$$

For convenience

$$V = \frac{\varepsilon\chi}{2j-1}, \quad g = \frac{\varepsilon\Sigma}{2j-1}, \quad h = \frac{\varepsilon\Lambda}{2j-1}$$

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For convenience

$$V = \frac{\varepsilon\chi}{2j-1}, \quad g = \frac{\varepsilon\Sigma}{2j-1}, \quad h = \frac{\varepsilon\Lambda}{2j-1}$$

$$H = \varepsilon\left[J^0 - \frac{\Sigma}{2j-1}\sum_{\sigma\sigma'} A^\dagger_\sigma A_{\sigma'} - \frac{\chi}{2(2j-1)}\left[\left(J^+\right)^2 + \left(J^-\right)^2\right] - 2\frac{\Lambda}{2j-1} A^\dagger_0 A_0\right].$$

Hartree-Fock-Bogoliubov transformation (I)

The transformation

Hartree-Fock transformation

$$a^\dagger_{\eta,m} = \sum_\sigma D_{\eta\sigma} c^\dagger_{\sigma,m}$$

Bogoliubov transformation

$$\begin{aligned} \alpha^\dagger_{\eta,m} &= u_\eta a^\dagger_{\eta,m} - sig(m)\, v_\eta a_{\eta,-m}, \\ \alpha^\dagger_{\eta,-m} &= u_\eta a^\dagger_{\eta,-m} + sig(m)\, v_\eta a_{\eta,m}, \end{aligned}$$

with the constraint (at half filling)

$$u^2_{-1} = v^2_1,\ u^2_1 = v^2_{-1},\ v^2_\eta + u^2_\eta = 1$$

(E.D. Davis and W.D. Weiss, J. Phys. G **12**, 805 (1986).)

Hartree-Fock-Bogoliubov transformation (II)

A convenient parametrization

$$D_{1,1} = D_{-1,-1} = \cos\frac{\varphi}{2}\,,\; D_{-1,1} = -D_{1,-1} = \sin\frac{\varphi}{2}$$

$$v_1 = \sin\frac{\beta}{2},\; v_{-1} = \cos\frac{\beta}{2}.$$

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Bogoliubov phase selection

Phase selection	Surface
$u_{-1} = v_1 = \sin\frac{\beta}{2}$ $u_1 = v_{-1} = \cos\frac{\beta}{2}$	A
$u_{-1} = v_1 = \sin\frac{\beta}{2}$ $u_1 = -v_{-1} = -\cos\frac{\beta}{2}$	B

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The energy

$$E(\varphi,\beta) = \frac{\langle HFB(\varphi,\beta)|H|HFB(\varphi,\beta)\rangle}{\langle HFB(\varphi,\beta)|HFB(\varphi,\beta)\rangle}.$$

The energy surfaces

φ Hartree-Fock variational parameter. β Bogoliubov variational parameter

The energy surface A

$$E_A = -\varepsilon j \cos\varphi \cos\beta - g j^2 \sin^2\beta - V j^2 \sin^2\varphi \cos^2\beta$$

$$\frac{E_A}{j\varepsilon} = -\cos\varphi\cos\beta - \frac{\Sigma}{2}\sin^2\beta - \frac{\chi}{2}\sin^2\varphi\cos^2\beta$$

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The energy surface B

$$E_B = -\varepsilon j \cos\varphi \cos\beta - 2h j^2 \sin^2\beta \sin^2\varphi - V j^2 \sin^2\varphi \cos^2\beta$$

$$\frac{E_B}{j\varepsilon} = -\cos\varphi \cos\beta - \Lambda \sin^2\beta \sin^2\varphi - \frac{\chi}{2} \sin^2\varphi \cos^2\beta$$

The phases of the system

- The spherical phase: $\varphi = 0$ and $\beta = 0$.
- The Hartree-Fock deformed phase: $\varphi \neq 0$ and $\beta = 0$.
- The BCS deformed phase: $\varphi = 0$ and $\beta \neq 0$.
- The Hartree-Fock plus BCS deformed phase: $\varphi = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$.

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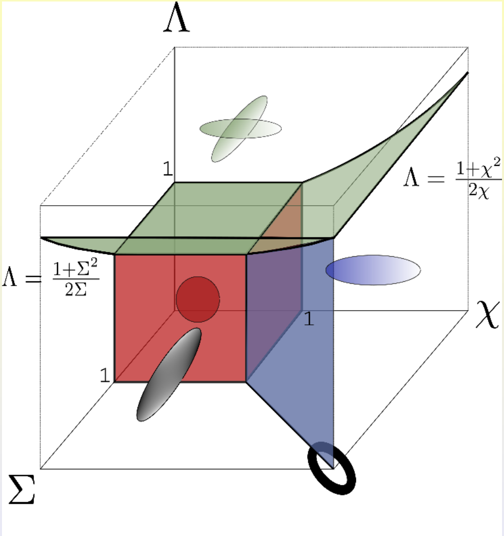
In the original formulation of the Agassi model only the three first phases were present, but in the extended version of the model the four basis can be found and, moreover, there is coexistence of some of the phases.

The phase diagram (I)



JEGR, J. Dukelsky, P. Pérez-Fernández, and J. M. Arias, PRC **97**, 054303 (2018)

The phase diagram (II)

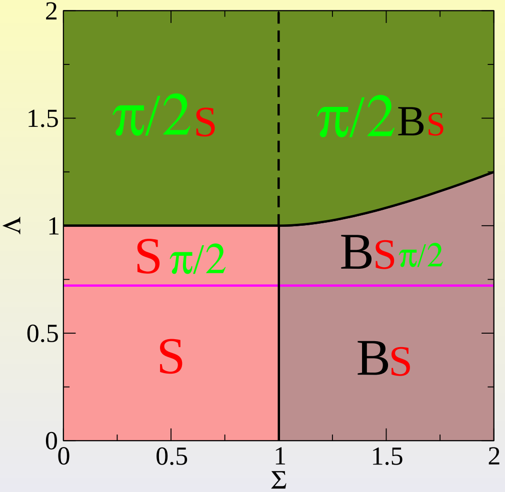


Figure: Phase diagram for the plane $\chi = 0.75$

Shape coexistence

Competition of phases

Shape coexistence

Competition of phases

- Spherical ($\varphi = 0, \beta = 0$), HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$) and BCS deformed one ($\varphi = 0, \beta \arccos(1/\Sigma)$).
- HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$), HF deformed minimum in ($\varphi = \arccos(1/\chi), \beta = 0$) and BCS deformed one in ($\varphi = 0, \beta = \arccos(1/\Sigma)$).
- **Closed valley minimum** (for $\boldsymbol{\Sigma} = \chi$) and the combined HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$).

Shape coexistence

Competition of phases

- Spherical ($\varphi = 0, \beta = 0$), HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$) and BCS deformed one ($\varphi = 0, \beta \, \mathrm{arccos}(1/\Sigma)$).
- HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$), HF deformed minimum in ($\varphi = \mathrm{arccos}(1/\chi), \beta = 0$) and BCS deformed one in ($\varphi = 0, \beta = \mathrm{arccos}(1/\Sigma)$).
- **Closed valley minimum** (for $\Sigma = \chi$) and the combined HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$).
- HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$), HF deformed minimum in ($\varphi = \mathrm{arccos}(1/\chi), \beta = 0$), BCS deformed minimum and the **closed valley minimum** along the line $\Lambda = \frac{1+\chi^2}{2\chi}$ with $\chi = \Sigma$. All the minima are degenerated.
- Spherical ($\varphi = 0, \beta = 0$), HF-BCS deformed minimum ($\varphi = \pi/2, \beta = \pi/2$), BCS deformed one ($\varphi = 0, \beta \, \mathrm{arccos}(1/\Sigma)$), HF deformed minimum in ($\varphi = \mathrm{arccos}(1/\chi), \beta = 0$), and the **closed valley minimum** at the point $\chi = \Sigma = \Lambda = 1$. All the minima are degenerated.

Numerical calculations

Order parameters in the laboratory frame

$$OP_{J^2} = \frac{< J_+^2 > + < J_-^2 >}{2j^2} \rightarrow \sin^2\varphi\cos^2\beta \; (\sin^2\varphi\cos^2\beta)$$

$$OP_{A_0^2} = \frac{< A_0^+ A_0 >}{j^2} \rightarrow 0 \; (\sin^2\beta\sin^2\varphi)$$

$$OP_{A_1^2} = \frac{< A_1^+ A_1 > + < A_{-1}^+ A_{-1} >}{2j^2} \rightarrow \frac{1}{2}\sin^2\beta \; (0)$$

Numerical calculations

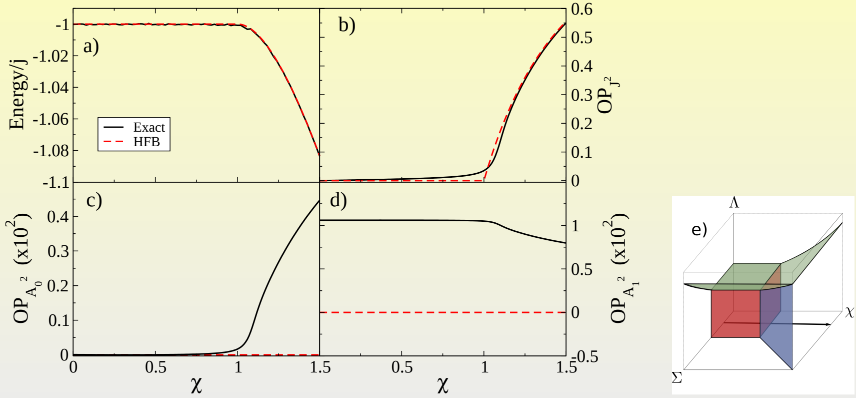


Figure: Comparison of HFB and exact results. $j = 100$ and Hamiltonian parameters $\Sigma = 0.5, \Lambda = 0$.

Numerical calculations

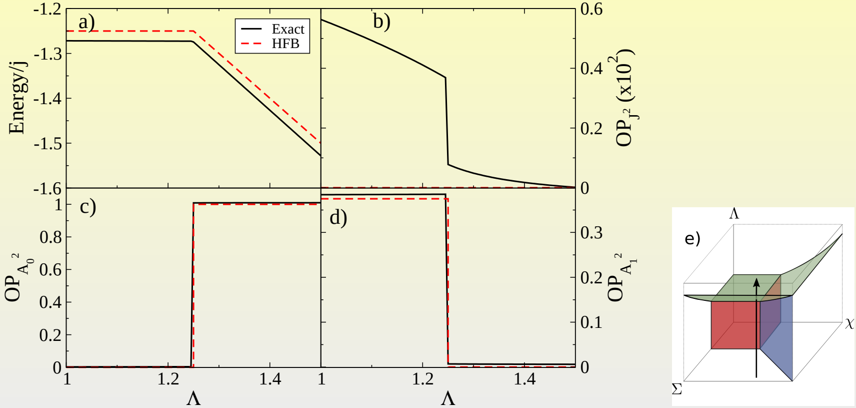


Figure: Comparison of HFB and exact results. $j = 100$ and Hamiltonian parameters $\chi = 1.5$, $\Sigma = 2$.

Conclusions

- We have presented an extended version of the Agassi model.
- We have obtained its phase diagram.
- The phase diagram of the present extended Agassi model shows a rich variety of phases.
- Phase coexistence is present in extended areas of the parameter space.
- The existence of coexisting phases is expected to have a strong influence on Excited-state Quantum Phase Transitions.

Thank you for your attention

Critical points of energy surface A

I-A: $\varphi = \beta = 0$ $(E_A/(j\varepsilon) = -1)$. Regardeless the values of Σ and χ.

- $\chi < 1$ and $\Sigma < 1$: it's a minimum
- $\chi > 1$ and $\Sigma > 1$: it's a Maximum
- $\chi > 1$ and $\Sigma < 1$ or $\chi < 1$ and $\Sigma > 1$: it's a saddle point.

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II-A: $|\varphi| = |\beta| = \frac{\pi}{2}$ ($E_A = -\frac{\Sigma}{2}$)

It is a saddle point

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III-A: $\beta = 0$, $\cos\varphi = \frac{1}{\chi}$ ($E_A/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$). Valid for $\chi > 1$

- $\chi > \Sigma$: it's a minimum.
- $\chi < \Sigma$: it's a saddle point.

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- $\chi > 1$ and $\Sigma < 1$ or $\chi < 1$ and $\Sigma > 1$: it's a saddle point.

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- $\chi > \Sigma$: it's a minimum.
- $\chi < \Sigma$: it's a saddle point.

IV-A: $\varphi = 0$, $\cos\beta = \frac{1}{\Sigma}$ $(E_A/(j\varepsilon) = -\frac{\Sigma^2+1}{2\Sigma})$ for $\Sigma > 1$

- $\chi < \Sigma$: it's a minimum.
- $\chi > \Sigma$: it's a saddle point.

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I-A: $\varphi = \beta = 0$ ($E_A/(j\varepsilon) = -1$). Regardeless the values of Σ and χ.

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IV-A: $\varphi = 0$, $\cos\beta = \frac{1}{\Sigma}$ ($E_A/(j\varepsilon) = -\frac{\Sigma^2+1}{2\Sigma}$) for $\Sigma > 1$

- $\chi < \Sigma$: it's a minimum.
- $\chi > \Sigma$: it's a saddle point.

V-A: Particular case $\chi = \Sigma$. $\cos\beta\cos\varphi = \frac{1}{\chi}$

This solution corresponds to a kind of closed valley in the $\varphi - \beta$ plane.

Critical points of energy surface B

I-B: $\varphi = \beta = 0$ $(E_B/(j\varepsilon) = -1)$. Regardeless the values of Λ and χ.

- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

Critical points of energy surface B

I-B: $\varphi = \beta = 0$ ($E_B/(j\varepsilon) = -1$). Regardeless the values of Λ and χ.

- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

II-B: $\beta = 0, \cos\varphi = \frac{1}{\chi}$ ($E_B/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$). Valid for $\chi > 1$

- $\Lambda > \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a saddle point.
- $\Lambda < \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a minimum.

Critical points of energy surface B

I-B: $\varphi = \beta = 0$ ($E_B/(j\varepsilon) = -1$). Regardeless the values of Λ and χ.

- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

II-B: $\beta = 0, \cos\varphi = \frac{1}{\chi}$ ($E_B/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$). Valid for $\chi > 1$

- $\Lambda > \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a saddle point.
- $\Lambda < \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a minimum.

III-B: $|\varphi| = |\beta| = \frac{\pi}{2}$ ($E_B = -\Lambda$)

- $\Lambda > \frac{1}{4}(\chi + \sqrt{4+\chi^2})$: it's a minimum.
- $\frac{1}{4}(\chi + \sqrt{4+\chi^2}) > \Lambda > \frac{1}{4}(\chi - \sqrt{4+\chi^2})$: it's a saddle point.
- $\Lambda < \frac{1}{4}(\chi - \sqrt{4+\chi^2}) < 0$: it's a Maximum.

The phase diagram (III)

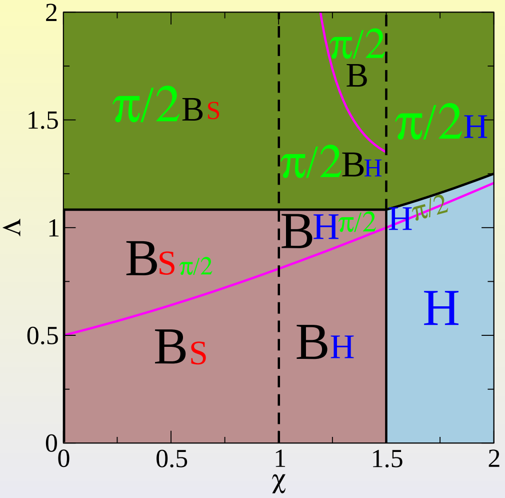


Figure: Phase diagram for the plane $\Sigma = 1.5$

The phase diagram (IV)

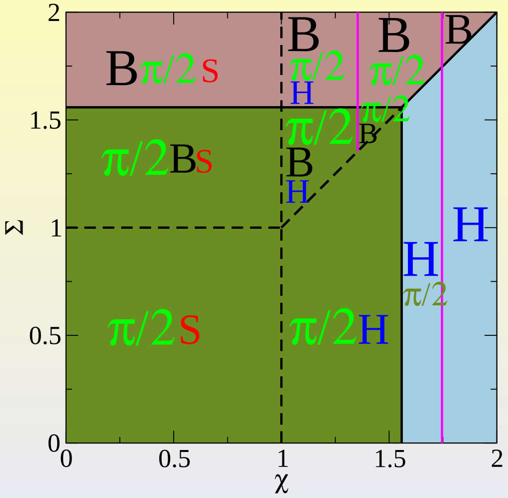
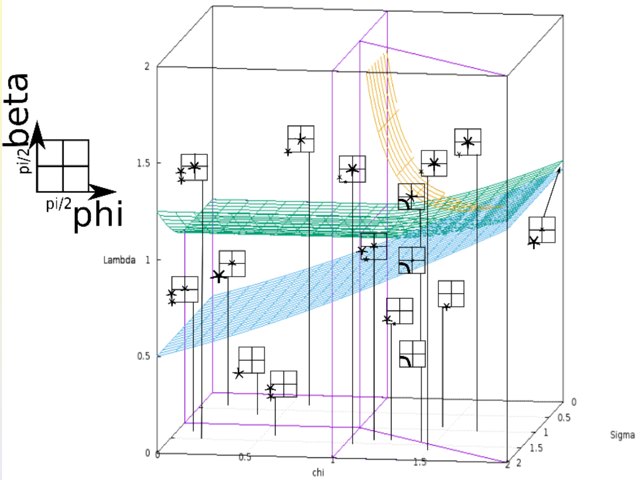


Figure: Phase diagram for the plane $\Lambda = 1.1$

The phase diagram (V)



Numerical calculations

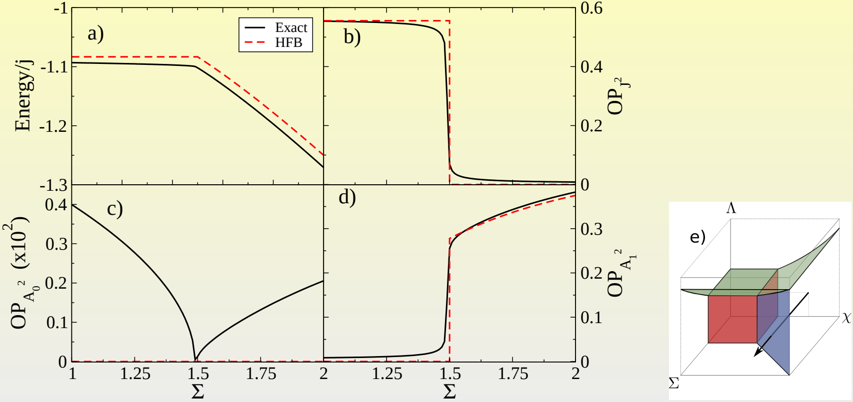


Figure: Comparison of HFB and exact results. $j = 100$ and Hamiltonian parameters $\chi = 1.5$, $\Lambda = 0.5$, as a function of Σ.