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Nuclear shape phase transitions described in terms of the sextic oscillator

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I. MOTIVATION AND PRELIMINARIES

- Well-known **dynamical symmetries** correspond to unique nuclear shape phases
  - U(5): spherical vibrator
  - O(6): γ-unstable shape
  - SU(3): axially deformed rotor
- New symmetries in nuclear collective motion
- Shape phase transitions described by the infinite square well potential
  - from spherical to deformed γ-unstable shapes: E(5) *[Iachello, 2000]*
  - from spherical to axially deformed shapes: X(5) *[Iachello, 2001]*
  - from axially deformed to triaxial shapes: Y(5) *[Iachello, 2003]*
  - from prolate to oblate shapes: Z(5) *[Bonatsos, 2004]*
- Symmetries imply predictions for the spectroscopic properties.

Why not use other exactly solvable potentials to study shape phase transitions?

Phase transitions can then be analysed in a controllable way.

The Bohr Hamiltonian

$$H = -\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_k \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)}\right) + V(\beta,\gamma)$$

Assume that $V(\beta,\gamma) = U(\beta)$ and separate the γ and angular variables:

$$\Psi(\beta,\gamma,\theta_i) = \beta^{-2}\phi(\beta)\Phi(\gamma,\theta_i)$$

Then a Schrödinger-like equation is obtained

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}\beta^2} + \left(\frac{(\tau+1)(\tau+2)}{\beta^2} + u(\beta)\right)\phi = \epsilon\phi ,$$

where $\epsilon = \frac{2B}{\hbar^2}E$ and $u(\beta) = \frac{2B}{\hbar^2}U(\beta)$

Note the presence of the $(\tau+1)(\tau+2)\beta^{-2}$ term

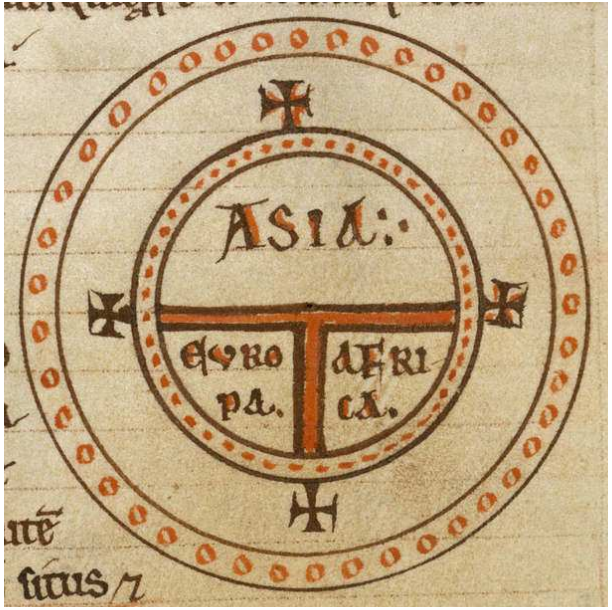
Find solvable $u(\beta)$ potentials for any τ

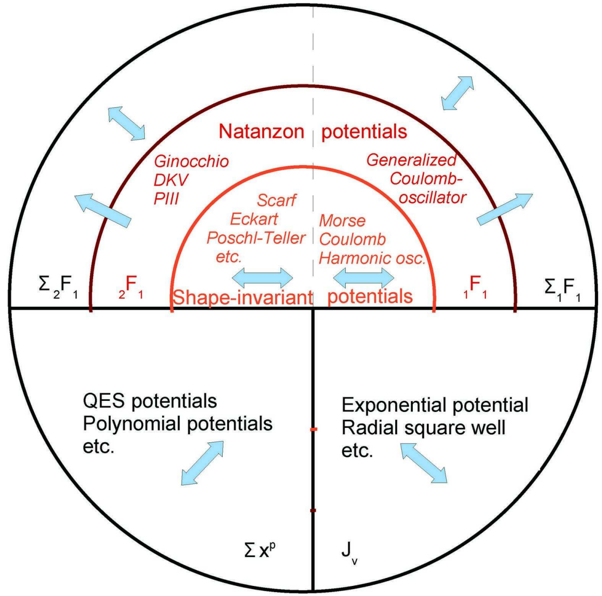
There is a limited variety of such potentials...

Typical γ-independent potentials proposed: *A perhaps incomplete list...*

Potential	Ref.	Minimum	Number of parameters	Solution
Square well	*Wilets, Jean 1956*	flat	0+scale	$J_\nu(z)$
Harmonic oscillator	*Bohr 1952*	$\beta = 0$	1+scale	$L_n^{(\alpha)}(z)$
H.O. $+\beta^{-2}$	*Elliott et al. 1986*	$\beta > 0$	2+scale	$L_n^{(\alpha)}(z)$
Coulomb	*Fortunato, Vitturi 2003*	$\beta = 0$ singular	1+scale	$L_n^{(\alpha)}(z)$
Kratzer	*Fortunato, Vitturi 2003*	$\beta > 0$	2+scale	$L_n^{(\alpha)}(z)$
β^4	*Arias et al. 2003*	$\beta = 0$	0+scale	numerical
Sextic oscillator	*Lévai, Arias 2004,2010*	$\beta = 0$ and/or $\beta > 0$	2+scale	QES (Heun)
Morse	*Bonatsos et al. 2008*	$\beta > 0$	1+scale	iterative

The world map of exactly solvable potentials





Only a few of the radial ($x \in [0, \infty)$) potentials are solvable for any l (or τ)

II. THE SEXTIC OSCILLATOR AS A γ-INDEPENDENT POTENTIAL

$$u(\beta) = (b^2 - 4ac^\pi)\beta^2 + 2ab\beta^4 + a^2\beta^6 + u_0^\pi \ ,$$

Properties: *[G. Lévai and J. M. Arias, PRC* **69***, 014304 (2004)]*

- Two free parameters, a and b, plus a scale; c is fixed.
- Flexible shape: $\beta_{\rm min} = 0$ OR $\beta_{\rm min} > 0$ OR $\beta_{\rm min} > \beta_{\rm max} > 0$
- Quasi-Exactly Solvable (QES): solvable for the lowest $M + 1$ levels ($M = 0,\ 1,\ 2,\ \ldots$).

$$\phi_n(\beta) = N_n P_n(\beta^2)(\beta^2)^{s-\frac{1}{4}} \exp\left(-\frac{a}{4}\beta^4 - \frac{b}{2}\beta^2\right) \qquad n = 0,\ 1,\ 2,\ \ldots$$

- $a \geq 0$ is necessary for normalizability. For $a = 0$ $u(\beta) \Longrightarrow b^2\beta^2$ H.O. U(5)
- N_n can be calculated exactly in terms of Kummer's functions (or ${}_1F_1(p, q; z)$)
- c^π and thus $u(\beta)$ is slightly different for even and odd values of τ:

$$\frac{1}{2}\left(\tau + 2M + \frac{7}{2}\right) \equiv c^\pi = const.$$

Set u_0^+ and u_0^- to give the same minimum

- $B(E2)$ rates can be calculated exactly from

$$T^{(\rm E2)} = t\alpha_{2\mu} = t\beta \left[D^{(2)}_{\mu,0} \cos\gamma + 2^{-1/2}(D^{(2)}_{\mu,2} + D^{(2)}_{\mu,-2}) \sin\gamma\right] \ .$$

Explicit form of the lowest few energy eigenvalues and wavefunctions

ξ	τ	M	$E^*_{\xi,\tau} = E_{\xi,\tau} - E_{1,0}$	$\phi_{\xi,\tau} / \exp(-\frac{a\beta^4}{4} - \frac{b\beta^2}{2})$
1	0	1	0	$N_{10}(\beta^2 - \lambda_-/10)$
1	1	1	$2b + \tilde{\lambda}_- - \lambda_- + u_0^-$	$N_{11}\beta(\beta^2 - \tilde{\lambda}_-/14)$
1	2	0	$4b - \lambda_-$	$N_{12}\beta^4$
1	3	0	$6b - \lambda_- + u_0^-$	$N_{13}\beta^5$
2	0	1	$\lambda_+ - \lambda_-$	$N_{20}(\beta^2 - \lambda_+/10)$
2	1	1	$2b + \tilde{\lambda}_+ - \lambda_- + u_0^-$	$N_{21}\beta(\beta^2 - \tilde{\lambda}_+/14)$

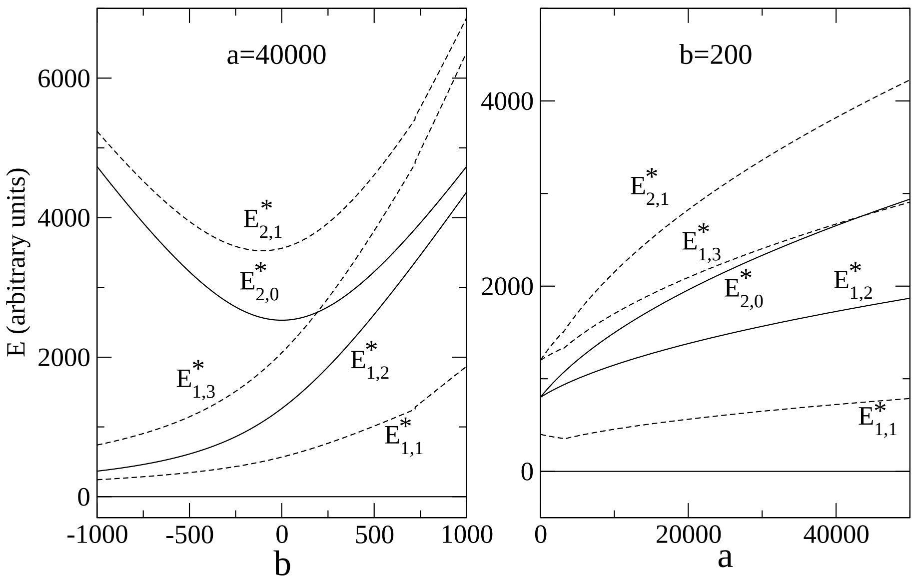
$$\lambda_\pm = 2b \pm 2(b^2 + 10a)^{1/2} \qquad \tilde{\lambda}_\pm = 2b \pm 2(b^2 + 14a)^{1/2}$$

Note: a and b can be expressed from

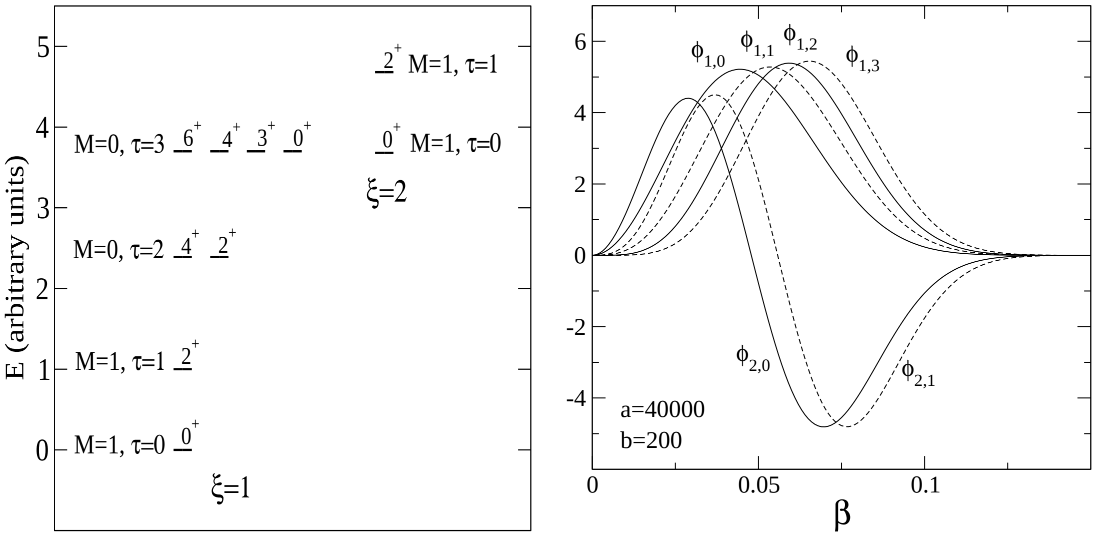
$$a = \frac{E^*_{1,2}}{40}\left(E^*_{2,0} - E^*_{1,2}\right) \qquad b = \frac{E^*_{1,2}}{2} - \frac{E^*_{2,0}}{4}$$

Note: $a > 0$ implies $E^*_{2,0} > E^*_{1,2}$

Excitation energies $E^*_{\xi,\tau} = E_{\xi,\tau} - E_{1,0}$ for fixed a and fixed b

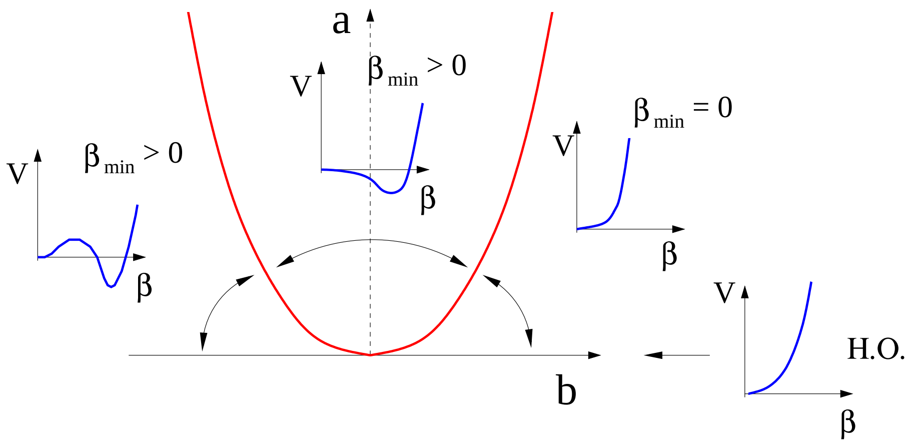


Example for the $M = 0$ and 1 case



Schematic spectrum for the sextic oscillator and wavefunctions $\phi_{\xi,\tau}$ for $a = 40000$ and $b = 200$.

Potential shapes in various regions of the (a, b) prameter space



Regions are separated by the critical parabola $a = b^2/11$

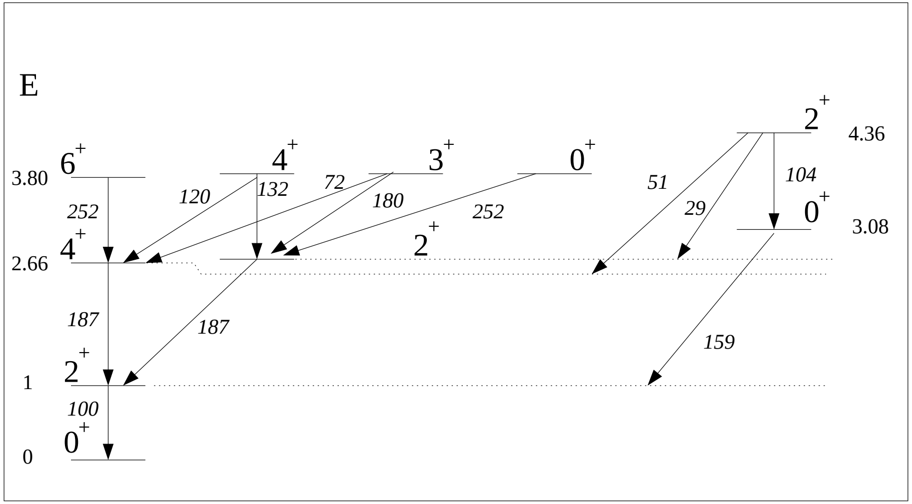
Note that for odd τ *the critical parabola is* $a = b^2/13$

Benchmark values and their comparison with those of other models

Many key values are constant along the critical parabola

	$\frac{E(4^+_{1,2})}{E(2^+_{1,1})}$	$\frac{E(0^+_{2,0})}{E(2^+_{1,1})}$	$\frac{E(6^+_{1,3})}{E(2^+_{1,1})}$	$\frac{B(\mathrm{E2};4^+_{1,2}\to 2^+_{1,1})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(\mathrm{E2};0^+_{2,0}\to 2^+_{1,1})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(\mathrm{E2};0^+_{1,3}\to 2^+_{1,2})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$
U(5)	2.00	2.00	3.00	2.00	2.00	3.00
β^4	2.09	2.39	3.27	1.83	1.42	2.56
β^6	2.14	2.62	3.39	1.77	1.19	2.40
$E(5)$	2.20	3.03	3.59	1.68	0.87	2.17
sextic osc.	2.66	3.08	3.80	1.87	1.59	2.52

Benchmark values along the critical parabola



III. APPLICATION TO Ru, Pd AND Cd NUCLEI

Even–even nuclei below the $Z = 50$ shell closure and mid-shell N values

Identify candidates for the model states

Use available $B(E2)$ values to identify band structure

0^+, 2^+ *and* 4^+ *levels with different characteristics*

Use a least-square fit of a and b to the energy spectrum

For unambigious J^π *use W=0.5*

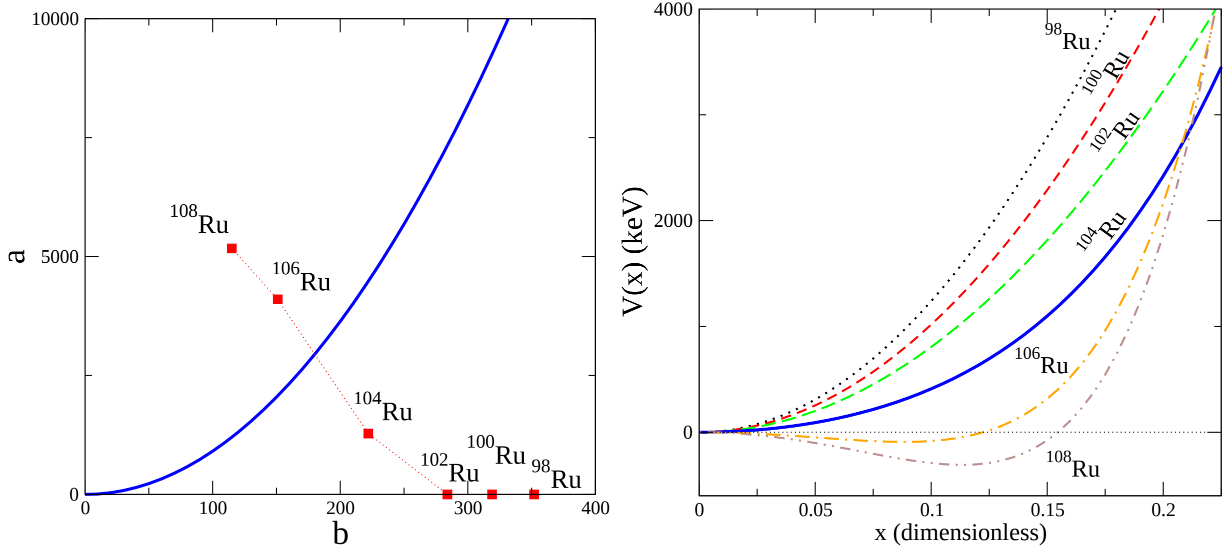
If $E^{\mathrm{Exp.}}_{2,0}(0^+) < E^{\mathrm{Exp.}}_{1,2}(2^+)$, *then set* $a = 0$ *(a cannot be negative)*

[G. Lévai and J. M. Arias, PRC **81***, 044304 (2010)]*

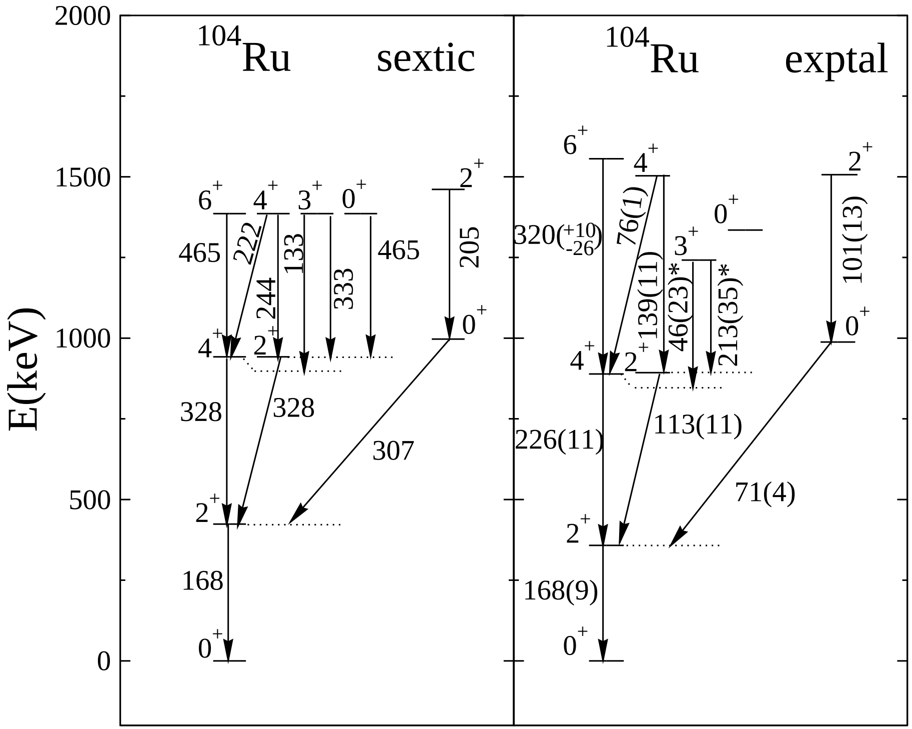
The Ru chain

A	^{98}Ru	^{100}Ru	^{102}Ru	^{104}Ru	^{106}Ru	^{108}Ru
N	54	56	58	60	62	64
$E_{1,1}^{\text{Exp.}}(2^+)$	652	540	475	358	270	242
$E_{1,2}^{\text{Exp.}}(2^+)$	1414	1362	1103	893	792	(708)
$E_{1,2}^{\text{Exp.}}(4^+)$	1398	1227	1106	889	(715)	665
$E_{1,3}^{\text{Exp.}}(0^+)$	$(2375)_3$	1741_3	1837_3	1335_3		
$E_{1,3}^{\text{Exp.}}(3^+)$	1797	1881	1522	1242	(1092)	(975)
$E_{1,3}^{\text{Exp.}}(4^+)$	2267	2063	1799	1503	(1307)	(1183)
$E_{1,3}^{\text{Exp.}}(6^+)$	2223	2076	1873	1556	(1296)	1241
$E_{2,0}^{\text{Exp.}}(0^+)$	1322_2	$\mathbf{1130}_2$	$\mathbf{944}_2$	$\mathbf{988}_2$	991_2	$(976)_2$
a_{Fit}	0	0	0	1279	4101	5170
b_{Fit}	352	319	284	222	151	115
$E_{1,1}^{\text{Fit}}$	704	638	569	424	315	298
$E_{1,2}^{\text{Fit}}$	1409	1276	1138	942	808	739
$E_{1,3}^{\text{Fit}}$	2113	1914	1707	1385	1184	1113
$E_{2,0}^{\text{Fit}}$	1409	1276	1138	996	1011	1019

The location of the fitted (a, b) and the corresponding potential shapes



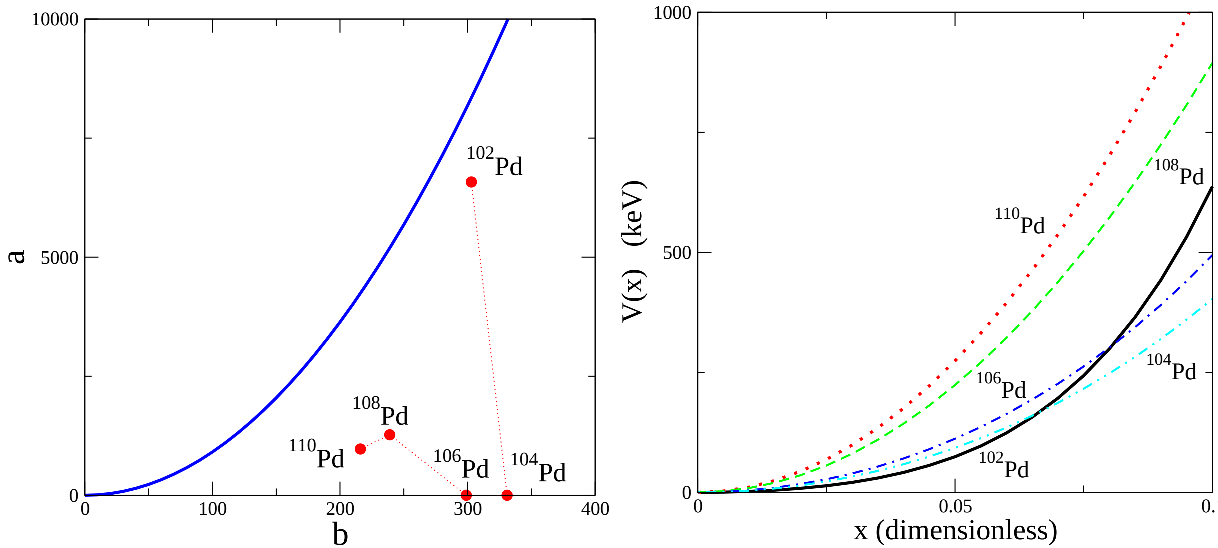
The spectrum of the ^{104}Ru nucleus and its interpretation.



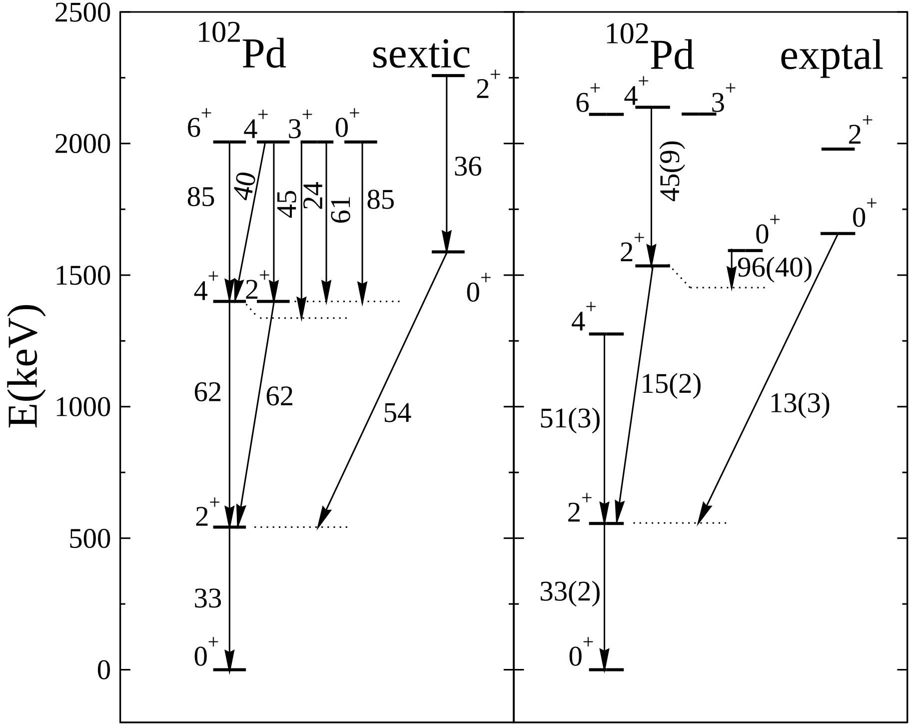
The Pd chain

A	^{102}Pd	^{104}Pd	^{106}Pd	^{108}Pd	^{110}Pd
N	56	58	60	62	64
$E_{1,1}^{\mathrm{Exp.}}(2^+)$	556	556	512	434	374
$E_{1,2}^{\mathrm{Exp.}}(2^+)$	1535	1342	1128	931	814
$E_{1,2}^{\mathrm{Exp.}}(4^+)$	1276	1324	1229	1048	921
$E_{1,3}^{\mathrm{Exp.}}(0^+)$	1658_3	1793_3	1706_3	1314_3	1171_3
$E_{1,3}^{\mathrm{Exp.}}(3^+)$	2112	1821	1558	1335	(1212)
$E_{1,3}^{\mathrm{Exp.}}(4^+)$	2138	2082	1932	(1624)	1398
$E_{1,3}^{\mathrm{Exp.}}(6^+)$	2111	2250	2077	1771	1574
$E_{2,0}^{\mathrm{Exp.}}(0^+)$	1593_2	$\mathbf{1334}_2$	$\mathbf{1134}_2$	$\mathbf{1053}_2$	$\mathbf{947}_2$
a_{Fit}	6579	0	0	1270	971
b_{Fit}	303	331	299	239	216
$E_{1,1}^{\mathrm{Fit}}$	543	661	598	459	416
$E_{1,2}^{\mathrm{Fit}}$	1401	1322	1196	1009	906
$E_{1,3}^{\mathrm{Fit}}$	2008	1983	1795	1485	1338
$E_{2,0}^{\mathrm{Fit}}$	1589	1322	1196	1057	949

The location of the fitted (a, b) and the corresponding potential shapes



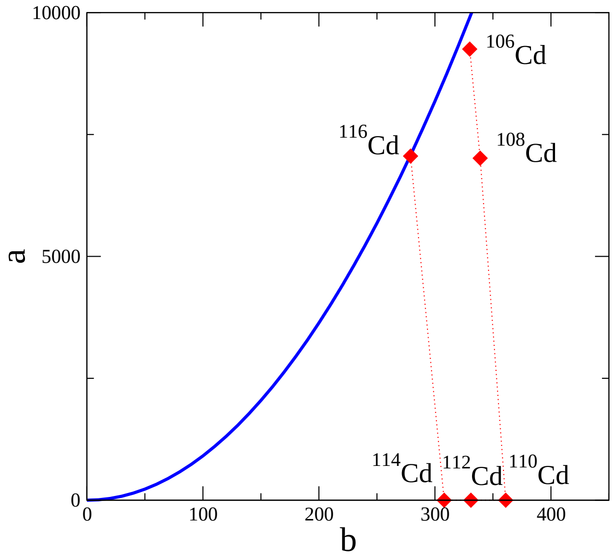
The spectrum of the ^{102}Pd nucleus and its interpretation.

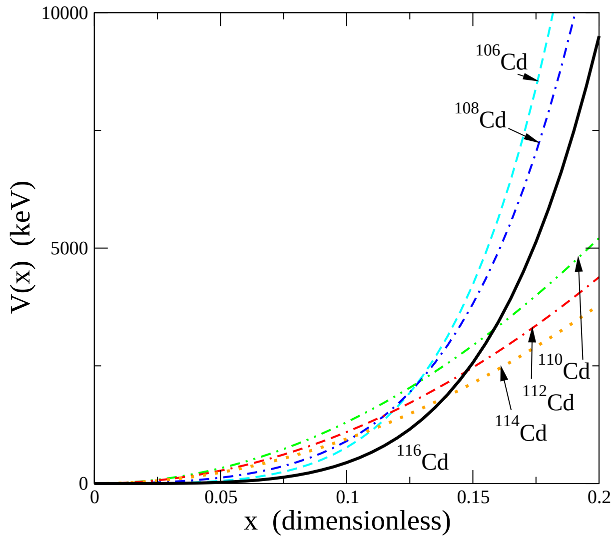


The Cd chain

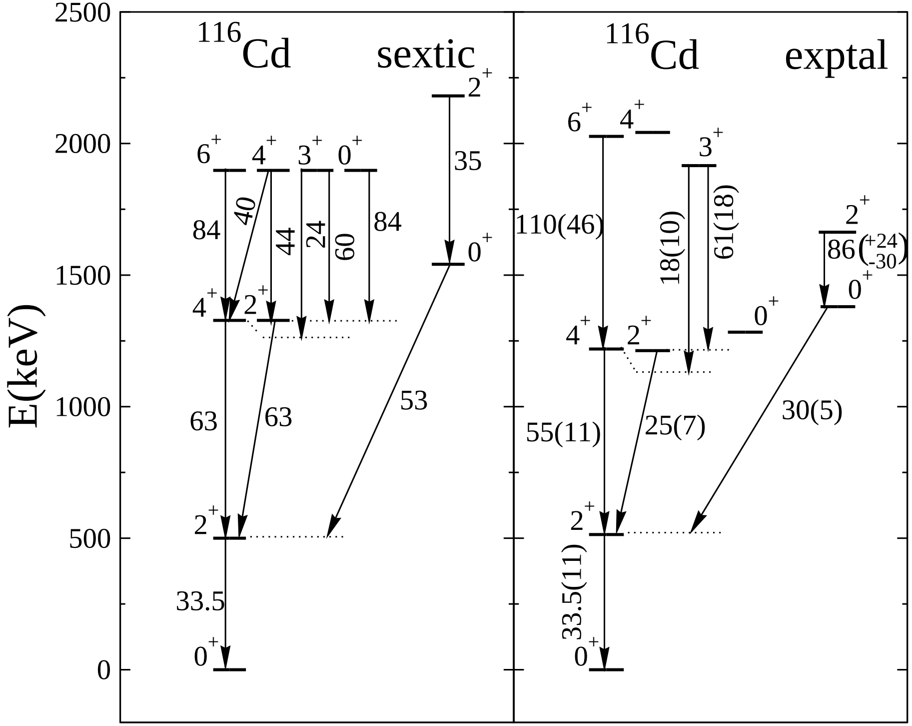
A	^{106}Cd	^{108}Cd	^{110}Cd	^{112}Cd	^{114}Cd	^{116}Cd
N	58	60	62	64	66	68
$E^{\mathrm{Exp.}}_{1,1}(2^+)$	633	633	658	618	559	514
$E^{\mathrm{Exp.}}_{1,2}(2^+)$	1717	1602	1476	1312	1210	1213
$E^{\mathrm{Exp.}}_{1,2}(4^+)$	1494	1509	1543	1416	1284	1219
$E^{\mathrm{Exp.}}_{1,3}(0^+)$	2144_3	1913_3	1731_3	1433_3	1306_3	1283_2
$E^{\mathrm{Exp.}}_{1,3}(3^+)$	(2254)	2146	2163	2065	1864	(1916)
$E^{\mathrm{Exp.}}_{1,3}(4^+)$	2105	2239	2220	1871	1732	2042
$E^{\mathrm{Exp.}}_{1,3}(6^+)$	2503	2541	2480	2168	1990	2027
$E^{\mathrm{Exp.}}_{2,0}(0^+)$	1795_2	1721_2	1473_2	$\mathbf{1225}_2$	$\mathbf{1135}_2$	$\mathbf{1380}_3$
a_{Fit}	9252	7014	0	0	0	7057
b_{Fit}	330	339	361	331	308	279
$E^{\mathrm{Fit}}_{1,1}$	581	615	723	662	616	499
$E^{\mathrm{Fit}}_{1,2}$	1557	1539	1446	1324	1232	1327
$E^{\mathrm{Fit}}_{1,3}$	2217	2217	2169	1987	1848	1897
$E^{\mathrm{Fit}}_{2,0}$	1795	1721	1446	1324	1232	1540

The location of the fitted (a, b) and the corresponding potential shapes





The spectrum of the ^{116}Cd nucleus and its interpretation.



IV. DISCUSSION

- The sextic oscillator is able to reproduce γ-independent potentials.
- The lowest states ($\xi,\ \tau =$ 1,0; 1,1; 1,2; 1,3; 2,0; 2,1) are described **exactly**.
- B(E2) values can be determined **exactly**.
- Realistic nuclear spectra can be reproduced among the Ru, Pd and Cd nclei
- Including higher states, with $M > 1$ is possible

  Perhaps this can solve the $E_{2,0} < E_{1,2}$ *problem*
- The sextis oscillator has been used in other models too:

  $X(5)$, $Z(5)$, $Z(4)$, $X(3)$ *see Budaca et al. Ann. Phys. 375 (2016) 65*
- Other QES potentials might also be used in the Bohr Hamiltonian.