Wobbling phases of odd mass nuclei - a semiclassical description

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Wobbling excitations
- predicted for even-even nuclei, Bohr&Mottelson
- observed in odd A nuclei ^{161,163,165}Lu, ^{167}Ta, ^{135}Pr

Particle-rotor Hamiltonian $\boxed{H = H_R + H_{sp}}$

$$H_R = \sum_{k=1,2,3} A_k(\hat{I}_k - \hat{j}_k)^2$$ triaxial rotor Hamiltonian

$$\vec{R} = \vec{I} - \vec{j}$$ core angular momentum

A_k the inertial parameters related to the MOI by $A_k = \frac{1}{2\mathcal{J}_k}$

One fully aligned particle with $\hat{j}_1 \approx j \equiv const.$

$$H_{align} = \underbrace{A_1\hat{I}_1^2 + A_2\hat{I}_2^2 + A_3\hat{I}_3^2 - 2A_1 j\hat{I}_1}_{\textbf{to be treated}} + const.$$

- Treating the full degrees of freedom **is difficult**.

Solution ☞ **Time-dependent variational approach** selects a limited set of degrees of freedom relevant for the studied phenomenon.

- Relies on a time-dependent variational principle applied to a variational state which is constructed according to the problem.

$$\delta \int_0^t \langle \psi(z)| H_{align} - \frac{\partial}{\partial t'} |\psi(z)\rangle dt' = 0$$

- The variational principle provides the time-dependence of some restricted set of complex variables which parametrize the variational state.

$$|\psi(z)\rangle = \sum_{K=-I}^{I} \sqrt{\frac{(2I)!}{(I-K)!(I+K)!}} \frac{z^{I+K}}{(1+|z|^2)^I} |IMK\rangle = \frac{1}{(1+|z|^2)^I} e^{z\hat{I}_-} |IMI\rangle$$

Classical energy function

$$\begin{aligned} \langle H_{align} \rangle &= \frac{I}{2}(A_1 + A_2) + A_3 I^2 - \frac{2A_1 j I (z + z^*)}{1 + zz^*} \\ &+ \frac{I(2I-1)}{2(1+zz^*)^2} \left[A_1 (z+z^*)^2 - A_2 (z - z^*)^2 - 4 A_3 z z^* \right] \end{aligned}$$

Equations of motion

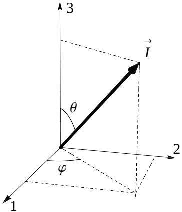
$$\frac{\partial \mathcal{H}}{\partial z} = -\frac{2iI\dot{z}^*}{(1+zz^*)^2}, \quad \frac{\partial \mathcal{H}}{\partial z^*} = \frac{2iI\dot{z}}{(1+zz^*)^2}$$

Canonical variables

Stereographic representation:

$$z = \tan\frac{\theta}{2} e^{i\varphi}, \quad 0 \le \theta < \pi, \quad 0 \le \varphi < 2\pi$$

Change of variable $r = 2I\cos^2\frac{\theta}{2}, \quad 0 < r \le 2I$



The full structure of the classical system is reproduced if the variables are canonical:

$$\frac{\partial \mathcal{H}}{\partial r} = \dot{\varphi}, \quad \frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{r} \quad \text{or} \quad \{r, \mathcal{H}\} = \dot{r}, \quad \{\varphi, \mathcal{H}\} = \dot{\varphi}$$

$$\{\varphi, r\} = 1$$

φ the generalized coordinate

r the generalized momentum

Classical energy function in terms of the canonical variables:

$$\begin{aligned} \mathcal{H}(r,\varphi) &= \frac{I}{2}(A_1 + A_2) + A_3 I^2 - 2A_1 j\sqrt{r(2I - r)}\cos\varphi \\ &+ \frac{(2I-1)r(2I-r)}{2I}(A_1\cos^2\varphi + A_2\sin^2\varphi - A_3) \end{aligned}$$

- The classical trajectory of the angular momentum vector $\vec{I}$ is a curve in the space of its classical projections

$$\begin{aligned} I_1 &= \sqrt{r(2I-r)}\cos\varphi, \\ I_2 &= \sqrt{r(2I-r)}\sin\varphi, \\ I_3 &= r - I. \end{aligned}$$

- It is determined by the intersection of the constant energy surfaces provided by the constants of motions:

$$\begin{aligned} \text{Shifted ellipsoid} \quad \mathcal{H} &= A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2 - 2A_1 j I_1, \\ \text{Sphere} \quad I^2 &= I_1^2 + I_2^2 + I_3^2. \end{aligned}$$

- The classical orbits are closed curves in the phase space of the canonical coordinates which are concentrically positioned around the stationary points of the constant energy surface.

☞ Choosing different sets of complex canonical coordinates as functions of φ and r and making an homeomorphism between classical algebra with Poisson bracket and a boson algebra with a commutator one arrives at various boson expansions.

[A.A. Raduta, **RB**, C.M. Raduta, PRC **76**, 064309 (2007)]

Boson realizations of the angular momentum operators

$$\left(\mathcal{C}, \tilde{\mathcal{C}}^*, \{,\}\right) \longrightarrow \left(a, a^\dagger, -i[,]\right)$$

$$\left\{\tilde{\mathcal{C}}^*, \mathcal{C}\right\} = i, \ \{\mathcal{C}, \mathcal{H}\} = \dot{\mathcal{C}}, \ \left\{\tilde{\mathcal{C}}^*, \mathcal{H}\right\} = \dot{\tilde{\mathcal{C}}}^*$$

Holstein-Primakoff

$$\mathcal{C} = \sqrt{2I - r} \cdot e^{i\varphi}, \ \tilde{\mathcal{C}}^* = \mathcal{C}^\dagger$$

$$\begin{aligned} \hat{J}_+ &= \hat{J}_-^\dagger = \sqrt{2I}\left(1 - \frac{a^\dagger a}{2I}\right)^{1/2} \\ \hat{J}_3 &= I - a^\dagger a \end{aligned}$$

Dyson

$$\mathcal{C} = \sqrt{\frac{r(2I - r)}{2I}} e^{i\varphi}, \ \tilde{\mathcal{C}}^* = \sqrt{\frac{2I - r}{2Ir}} e^{-i\varphi}$$

$$\begin{aligned} \hat{J}_+ &= \sqrt{2I} a \\ \hat{J}_- &= \sqrt{2I}\left[a^\dagger - \frac{\left(a^\dagger\right)^2 a}{2I}\right] \\ \hat{J}_3 &= I - a^\dagger a \end{aligned}$$

Exponential

$$\mathcal{C} = \frac{1}{\sqrt{2}}(r - i\varphi), \ \tilde{\mathcal{C}}^* = \mathcal{C}^\dagger$$

$$\begin{aligned} \hat{J}_+ &= \hat{J}_-^\dagger = \sqrt{\frac{1}{\sqrt{2}}(a^\dagger + a)} e^{\frac{1}{\sqrt{2}}(a^\dagger - a)} \\ &\times \sqrt{2I - \frac{1}{\sqrt{2}}(a^\dagger + a)} \\ \hat{J}_3 &= \frac{1}{\sqrt{2}}(a^\dagger + a) - I \end{aligned}$$

New

$$\mathcal{C} = \frac{r}{\sqrt{2I}}, \ \tilde{\mathcal{C}}^* = i\sqrt{2I}\varphi$$

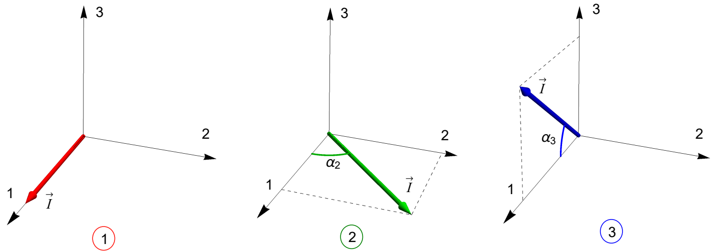
$$\begin{aligned} \hat{J}_+ &= 2I\sqrt{\frac{a}{\sqrt{2I}}} e^{\frac{a^\dagger}{\sqrt{2I}}} \sqrt{1 - \frac{a}{\sqrt{2I}}} \\ \hat{J}_- &= 2I\sqrt{1 - \frac{a}{\sqrt{2I}}} e^{-\frac{a^\dagger}{\sqrt{2I}}} \sqrt{\frac{a}{\sqrt{2I}}} \\ \hat{J}_3 &= \sqrt{2I} a - I \end{aligned}$$

Stable geometries and wobbling phases

The stationary points where $\dot{\varphi} = \dot{r} = 0$, which are stable against fluctuations are those which minimize the classical energy.

i	(r_i, φ_i)	Conditions	I_1^{cl}	I_2^{cl}	I_3^{cl}
1	$(I, 0)$	$S_{Ij} A_1 < A_2 < A_3$ $S_{Ij} A_1 < A_3 < A_2$	I	0	0
2	$(\sqrt{r_2(2I - r_2)} = I, \alpha_2)$	$A_2 < A_3 < A_1 S_{Ij}$ $A_2 < S_{Ij} A_1 < A_3$	$I \cos\alpha_2$	$I \sin\alpha_2$	0
3	$(\sqrt{r_3(2I - r_3)} = I\cos\alpha_3, 0)$	$A_3 < A_2 < A_1 S_{Ij}$ $A_3 < A_1 S_{Ij} < A_2$	$I \cos\alpha_3$	0	$I \sin\alpha_3$

Tilting angle $\quad \cos\alpha_{2,3} = \dfrac{2A_1 j}{(2I-1)(A_1 - A_{2,3})}$ Weighting factor $\quad S_{Ij} = \dfrac{2I - 1 - 2j}{2I - 1}$



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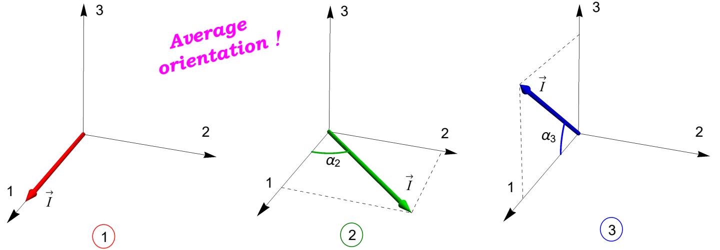


Quantized energy and wobbling frequency

Harmonic approximation on classical energy function

$$\mathcal{H}_i(r,\varphi) = \mathcal{H}(r_i,\varphi_i) + \frac{1}{2}\left(\frac{\partial^2\mathcal{H}}{\partial r^2}\right)_{r_i,\varphi_i}\tilde{r}_i^2 + \frac{1}{2}\left(\frac{\partial^2\mathcal{H}}{\partial \varphi^2}\right)_{r_i,\varphi_i}\tilde{\varphi}_i^2$$

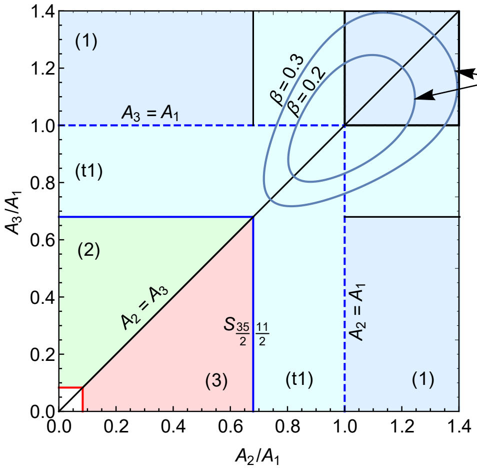
$\tilde{r}_i = r - r_i$ and $\tilde{\varphi}_i = \varphi - \varphi_i$ replaced with their operator counterparts.

Discrete energy spectra

$$\begin{aligned}
E_1(I,n) &= A_1 I^2 + \frac{I}{2}(A_2 + A_3) - 2A_1 jI + \omega_1(I)\left(n + \frac{1}{2}\right) \\
E_2(I,n) &= A_2 I^2 + \frac{I}{2}(A_1 + A_3) - A_1 jI\cos\alpha_2 + \omega_2(I)\left(n + \frac{1}{2}\right) \\
E_3(I,n) &= A_3 I^2 + \frac{I}{2}(A_1 + A_2) - A_1 jI\cos\alpha_3 + \omega_3(I)\left(n + \frac{1}{2}\right)
\end{aligned}$$

Wobbling frequencies

$$\begin{aligned}
\omega_1(I) &= \sqrt{[(2I-1)(A_3 - A_1) + 2A_1 j]\,[(2I-1)(A_2 - A_1) + 2A_1 j]} \\
&\xrightarrow[I\gg 1]{} \text{[S. Frauendorf, F. Dönau, PRC } \mathbf{89}\text{, 014322 (2014)]} \\
\omega_2(I) &= (2I-1)\sqrt{(A_3 - A_2)(A_1 - A_2)}\sin\alpha_2 \\
\omega_3(I) &= (2I-1)\sqrt{(A_2 - A_3)(A_1 - A_3)}\sin\alpha_3
\end{aligned}$$



$$\mathcal{J}_k^{rig} = \mathcal{J}_0^{rig}\left[1 - \beta\sqrt{\frac{5}{4\pi}}\cos\left(\gamma_{rig} - \frac{2}{3}k\pi\right)\right]$$

(1) and (t1) Principal axis wobbling

(2) and (3) Tilted axis wobbling

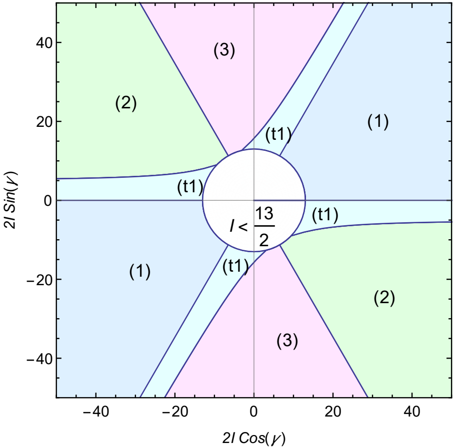
$$A_k = \frac{1}{2\mathcal{J}_k}, \quad I = 35/2, \quad j = 11/2$$

Longitudinal wobbling (1) - Quasiparticle angular momentum aligned to the axis with the largest MOI.

Transversal wobbling (t1) - Quasiparticle angular momentum aligned perpendicularly to the axis with the largest MOI.

[Y.R. Shimizu, M. Matsuzaki, K. Matsuyanagi, conf. talk (2004)]

Dynamical phase diagram



$j = 11/2$ & hydrodynamic MOI

$$\mathcal{J}_k = \frac{4}{3}\mathcal{J}_0 \sin^2\left(\gamma - \frac{2}{3}k\pi\right)$$

- (2)-(3) separatrix at $\gamma = 2\pi/3(Mod\,\pi)$
- Longitudinal wobbling phase (1) interval of existence $\gamma \in (0, \pi/3)(Mod\,\pi)$ is isolated from the tilted angle wobbling phase

☞ Transversal wobbling with alignment around an axis with the minimum MOI is restricted to a very narrow existence interval and only for $I = 13/2$ and $I = 15/2$ states.

☞ At rigid γ, as spin is increased, a transition is possible only from transversal wobbling to tilted axis wobbling.

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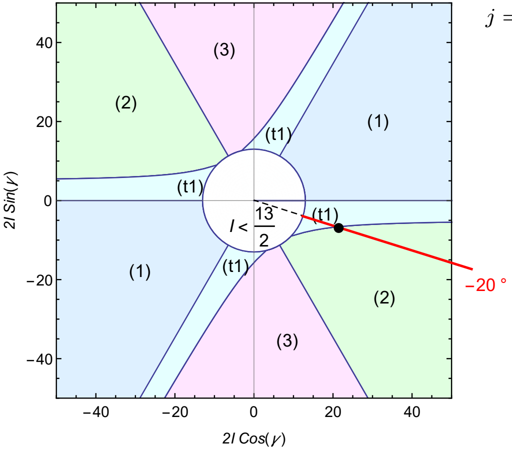

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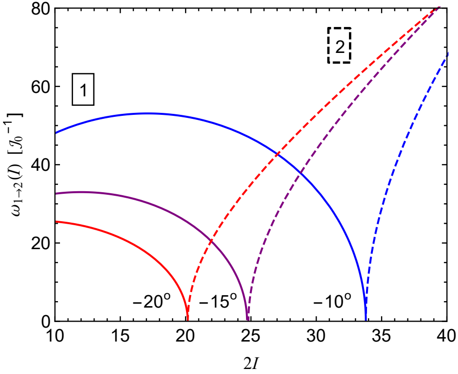
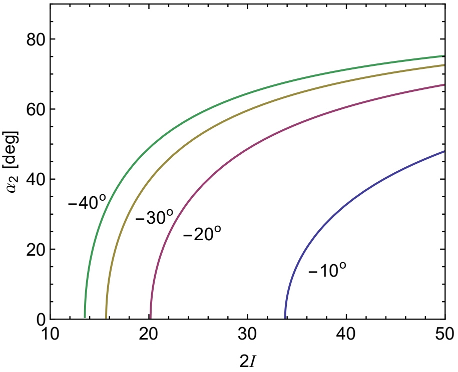
Wobbling phase transition



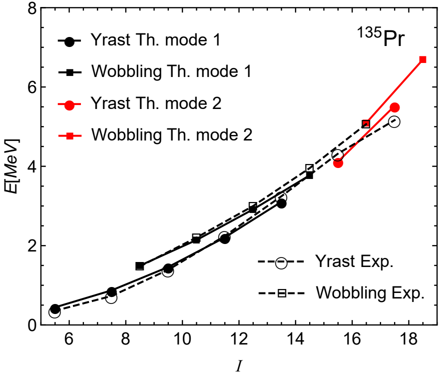
$\Leftarrow$ Evolution of wobbling frequency from the transversal regime (1) to the tilted-axis wobbling mode (2).

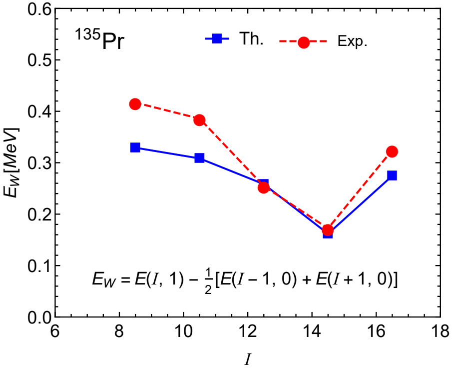
Tilting angle α_2 with hydrodynamic MOI. ☞

- Increases very rapidly for the first few spin states and then reaches a relative saturation plateau.
- γ closer to the $\frac{\pi}{3}$ separatrix $\rightarrow$ more abrupt is the increase and the plateau more level.



Wobbling excitations in ^{135}Pr [J.T. Matta *et al.*, PRL **114**, 082501 (2015)]





- $I = 31/2$ and $I = 35/2$ yrast states and the wobbling state $I = 33/2$ are considered to be part of a different rotation-wobbling regime
- The low critical spin excludes the non-independent rigid MOI description
- $\mathcal{J}_0^R = 30.96$ MeV^{-1}, $\mathcal{J}_0^W = 65.93$ MeV^{-1}, $\gamma = -11.18° \rightarrow$ rms=0.149 MeV
- Tilting $\alpha_2 = 4.22°, 20.78°, 28.36°$ for the states $I = 31/2, 33/2, 35/2$.

Summary

- In addition to transversal and longitudinal wobbling, one completed the wobbling phase space with a tilted-axis wobbling mode.
- The whole dynamics of the system is treated in a unified manner.
- Each wobbling mode follows strict conditions for MOI. This analysis put some additional constraints to the transverse wobbling regime.
- For a stable γ asymmetry, a transition from the transversal wobbling to a tilted-axis regime can occur. This transition is used to describe the wobbling excitations in ^{135}Pr nucleus.
- A similar semiclassical approach can be used to study the transition from chiral vibration to static chirality in nuclear systems with two single-particle generated spins aligned perpendicularly to the core angular momentum.

Transition probabilities

I	$\frac{B(E2,I\to I-1)_{\text{out}}}{B(E2,I\to I-2)_{\text{in}}}$		$\frac{B(M1,I\to I-1)_{\text{out}}}{B(E2,I\to I-2)_{\text{in}}}\left(\frac{\mu_N}{eb}\right)^2$	
	Expt.	Th.	Expt.	Th.
$\frac{17}{2}$		0.313		0.164
$\frac{21}{2}$	0.843(32)	0.270	0.164(14)	0.164
$\frac{25}{2}$	0.500(25)	0.258	0.035(9)	0.183
$\frac{29}{2}$	$\geqslant$0.261(14)	0.318	$\geqslant$0.016(4)	0.279

$$
\begin{aligned}
B(E2; n, I \to n, I \pm 2)_i &= \frac{5e^2}{16\pi}\left|Q_2^{(i)}\right|^2 \\
B(E2; n, I \to n-1, I-1)_i &= \frac{5e^2}{16\pi}\frac{n}{2I}\left|Q_0^{(i)}\sqrt{\frac{3}{2}}\left(\frac{1}{k_i}+k_i\right) - Q_2^{(i)}\left(\frac{1}{k_i}-k_i\right)\right|^2 \\
B(M1; n, I \to n-1, I-1)_i &= \frac{3}{4\pi}\frac{n}{4I}\left|j(g_j-g_R)\left(\frac{1}{k_i}+k_i\right)\right|^2
\end{aligned}
$$

$$
k_i = \sqrt{m_i\omega_i}, \quad m_i = \left[I\left(\frac{\partial^2\mathcal{H}}{\partial r^2}\right)_{r_i,\varphi_i}\right]^{-1}
$$

Separatrix

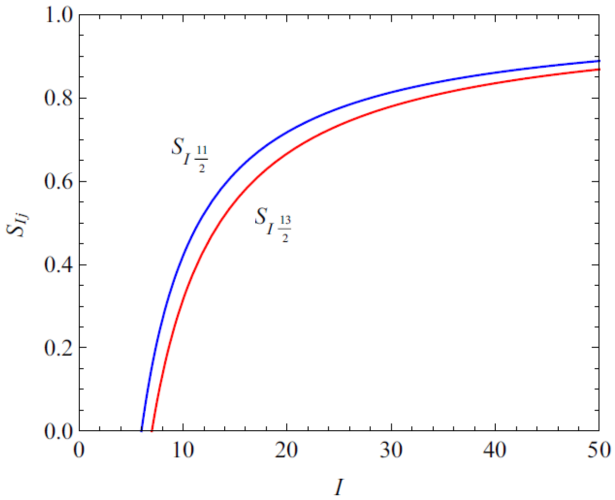


FIG. 2. Evolution of the separatrix S_{Ij} as a function of angular momentum for $j = 11/2$ and $13/2$.