Relationship between the ESPQT and the thermal phase transition in the Dicke model

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Introduction

- **Quantum phase transitions** (QPT's) describe the change in the ground state wave function of a many particle system due to quantum fluctuations.
- An **excited state quantum phase transition** (ESQPT) is similar to a QPT but affecting to excited states. (M. A. Caprio, P. Cejnar and F. Iachello, Ann. Phys. (NY) **323**, 1106 (2008))
- It has been suggested that precursors of QPT's can be observed in finite-size mesoscopic systems, although the concepts of phase transition and critical points have been defined, strictly speaking, for macroscopic systems.

Motivation

The Dicke model displays several phase transitions:

- **A Thermal Phase Transition**. (G. Comer Duncan, Phys. Rev. A **9**, 418 (1974).)
- **A Quantum Phase Transition**. (C. Emery and T. Brandes, Phys. Rev. E **67** 066203 (2003); C. Emery and T. Brandes, Phys. Rev. Lett. **90** 044101 (2003).)
- **An Excited State Quantum Phase Transition**. (PPF, A. Relaño, J. M. Arias, P. Cejnar, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. E **83**, 046208 (2011); PPF, A. Relaño, J. M. Arias, P. Cejnar, J. Dukelsky, and J. E. García-Ramos, Phys. Rev. E **83**, 046208 (2011).)

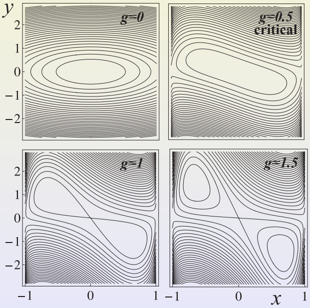
The Dicke model

The Hamiltonian

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{2\lambda}{\sqrt{N}} J_x \left(a^\dagger + a \right)$$

There are two conserved quantities, apart from H:

- The parity, $\Pi = \exp\left(i\pi\left[j + J_z + a^\dagger a\right]\right)$. So $\Pi \left| E_i, \pm \right\rangle = \pm \left| E_i, \pm \right\rangle$.
- The total angular momentum J^2 of the N $1/2$-spin particles.



Highly-symmetric Dicke states, $|j = N/2, M\rangle$

Microcanonical ensemble

The density of states is proportional to the size of the phase space available to the system,

$$\rho(E, j) = C \int dq_1 dq_2 dp_1 dp_2 \, \delta\left[E - H(q_1, q_2; p_1, p_2)\right]$$

$$\langle J_z(E, j)\rangle = \frac{1}{\rho(E, j)} \int dq_1 dq_2 dp_1 dp_2 \, \delta\left[E - H(q_1, q_2; p_1, p_2)\right] J_z$$

$$\langle J_x(E, j)\rangle = \frac{1}{\rho(E, j)} \int dq_1 dq_2 dp_1 dp_2 \, \delta\left[E - H(q_1, q_2; p_1, p_2)\right] J_x$$

Canonical ensemble

Considering that the system weakly interacts with a thermal bath which commutes with J^2, the partition function is given by:

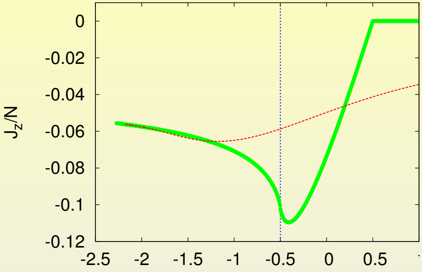
$$Z(N,\beta) = \sqrt{\frac{1}{\pi\beta\omega}}\int_{-\infty}^{\infty} dx\, \exp\left[-\beta\left(\omega x^2 + \frac{N}{2}\sqrt{\omega_0^2 + \frac{4\lambda^2 x^2}{N}}\right)\right] \times \frac{\exp\left(\beta(N+1)\sqrt{\omega_0^2 + \frac{4\lambda^2 x^2}{N}}\right) - 1}{\exp\left(\beta\sqrt{\omega_0^2 + \frac{4\lambda^2 x^2}{N}}\right) - 1}$$

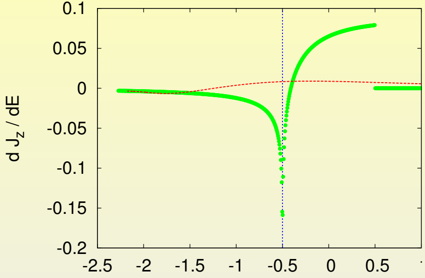
It has been shown that there is no thermal phase transition for this case. (M. Aparicio Alcalde *et. al.*, Phys. Rev. E **86**, 012101 (2012))

Other thermodynamic results can be obtained from the partition function:

$$\langle E\rangle = -\frac{\partial \log Z}{\partial\beta} \qquad \langle J_z\rangle = -\frac{1}{\beta}\frac{\partial \log Z}{\partial\omega_0} \qquad \langle J_x\rangle = 0$$

Results for $\omega = \omega_0 = 1$, $\lambda = 3\lambda_c = 1.5$, and $N = 10^5$





- The different ensembles are not equivalent in the thermodynamic limit. This is due to the finite number of (semiclassical) degrees of freedom that the system has in the thermodynamic limit, $N \to \infty$. (P. Stránský and P. Cejnar, Phys. Lett. A **381**, 984 (2017))

Dicke model including all the j-sectors

The Hamiltonian

$$H_j = \omega a^\dagger a + \omega_0 J_z + \frac{2\lambda_j^{\text{eff}}}{\sqrt{2j}} J_x \left(a + a^\dagger\right)$$

$$\lambda_j^{\text{eff}} = \lambda \sqrt{2j/N}$$

- H_j, is formally identical than the one of the highly-symmetric sector, $j = N/2$, but with a different effective coupling λ_j^{eff}.
- Critical λ for the ESQPT:

$$\lambda_c^{(j)} = \sqrt{\frac{N\omega\omega_0}{8j}}$$

- The $j = 0$ sector does not exhibit an ESQPT in any case ($\lambda_c^{(j)} \to \infty$).

Microcanonical ensemble

$$\rho(E) = \sum_{j=0}^{N/2} g(N,j)\rho(E,j)$$

$$g(N,j) = \frac{1+2j}{1+j+N/2}\binom{N}{N/2-j}$$

- We define $x = j/N$ that can be taken as a continuous variable $x \in [0, 1/2]$ in the thermodynamical limit, $N \to \infty$.

$$\rho(E,N) = \int_0^{1/2} dx\, g(N,x)\rho(E,Nx)$$

$$\langle J_z(E,N)\rangle = \frac{1}{\rho(E,N)}\int_0^{1/2} dx\, g(N,x)\rho(E,Nx)J_z(E,Nx)$$

$$\langle J_x(E,N)\rangle = \frac{1}{\rho(E,N)}\int_0^{1/2} dx\, g(N,x)\rho(E,Nx)J_x(E,Nx)$$

Canonical ensemble

$$Z(N,\beta) = \frac{2^N}{\sqrt{\pi\beta\omega}} \int_{-\infty}^{\infty} dx \exp\left(-\beta\omega x^2\right) \left[\cosh\left(\frac{\beta\sqrt{N\omega_0^2 + 16\lambda^2 x^2}}{2\sqrt{N}}\right)\right]^N$$

(G. Comer Duncan, Phys. Rev. A **9**, 418 (1974).)

$$\langle J_\alpha(N,\beta)\rangle = \frac{1}{Z(N,\beta)} \mathrm{Tr}\left[J_\alpha \exp\left(-\beta H\right)\right]$$

In the thermodynamical limit:

$$\lim_{N\to\infty} Z(N,\beta) = \sqrt{\frac{2}{\beta\left|\Psi''(y_0)\right|}} \exp\left[N\Psi(y_0)\right]$$

$$\Psi(y) = -\beta\omega y^2 + \log\left(2\cosh\left[\frac{\beta\omega_0}{2}\sqrt{1+\frac{16\lambda^2 y^2}{\omega_0^2}}\right]\right)$$

We find that the partition function becomes nonanalytic at the critical temperature β_c

$$\beta_c = \frac{2}{\omega_0}\tanh^{-1}\left(\frac{\omega\omega_0}{4\lambda^2}\right)$$

- If $\lambda < \lambda_c$, there is no thermal phase transition.
- At $\lambda = \lambda_c$, the phase transition takes place at at $T \to 0$ (QPT).
- If $\lambda > \lambda_c$, there exists a thermal phase transition at a critical temperature $T_c = 1/\beta_c$.

Spontaneus symmetry breaking

- Usual order parameters for the superradiant phase transition: J_z or $a^\dagger a$.
- Those observables do not break the parity symmetry $\mathbb{Z}_2$. This symmetry comes from the invariance of H under $J_x \to -J_x$ and $a \to -a$.
- $\langle J_x \rangle = 0$ in any eigenstate with well-defined parity.

$$H_\epsilon = \omega a^\dagger a + \omega_0 J_z + \frac{2\lambda}{\sqrt{N}} J_x \left(a^\dagger + a \right) + \epsilon J_x$$

$$\epsilon \ll \omega, \omega_0, \lambda$$

- Motivation: to study the behavior of $\langle J_x \rangle$ when crossing the thermal phase transition.

Partition function and observables

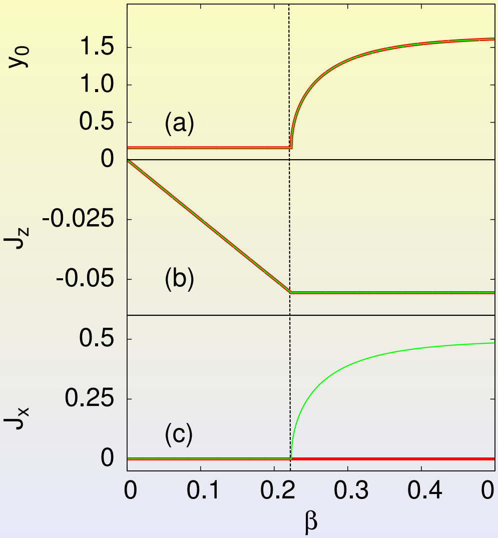
$$\lim_{N\to\infty} Z_\epsilon(N,\beta) = \sqrt{\frac{2}{\beta\,|\Psi''_\epsilon(y_0)|}}\exp\left[N\Psi_\epsilon(y_0)\right]$$

$$\Psi_\epsilon(y) = -\beta\omega y^2 + \log\left(2\cosh\left[\frac{\beta\omega_0}{2}\sqrt{1+\left(\frac{\epsilon+4\lambda y}{\omega_0}\right)^2}\right]\right)$$

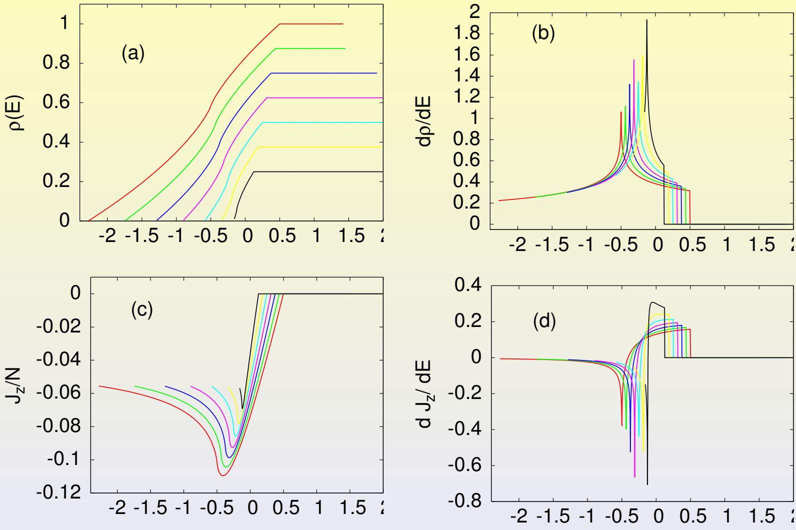
$$\begin{aligned}\langle J_z\rangle_\epsilon &= \frac{1}{\beta}\frac{\partial \log Z_\epsilon}{\partial\omega_0}\\ \langle J_x\rangle_\epsilon &= \frac{1}{\beta}\frac{\partial \log Z_\epsilon}{\partial\epsilon}\end{aligned}$$

- We study both observables in the limit $\epsilon \to 0$.

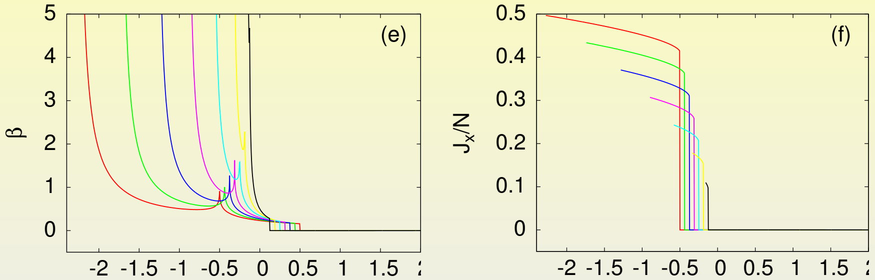
Numerical results



Numerical results for different *j*-sectors



Numerical results for different j-sectors



- Microcanonical temperature:

$$\beta = \frac{\partial \log \rho(E)}{\partial E}$$

Link between ESQPT and Thermal Phase Transition

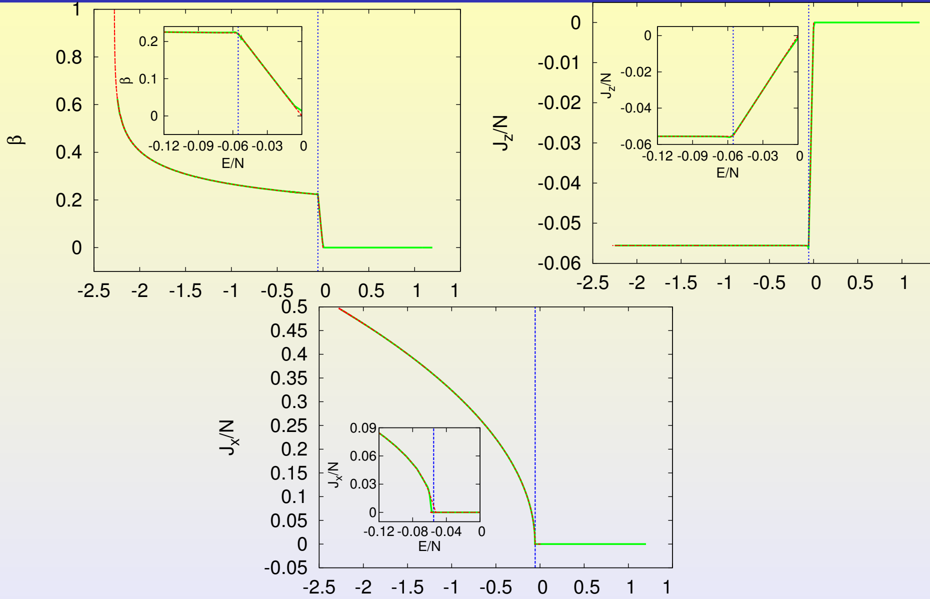
Critical Energy and critical J_z

$$\langle E_c \rangle = -\left.\frac{\partial \log Z}{\partial \beta}\right|_{\beta_c} \quad \langle J_{z,c} \rangle = \frac{1}{\beta} \left.\frac{\partial \log Z}{\partial \omega_0}\right|_{\beta_c}$$

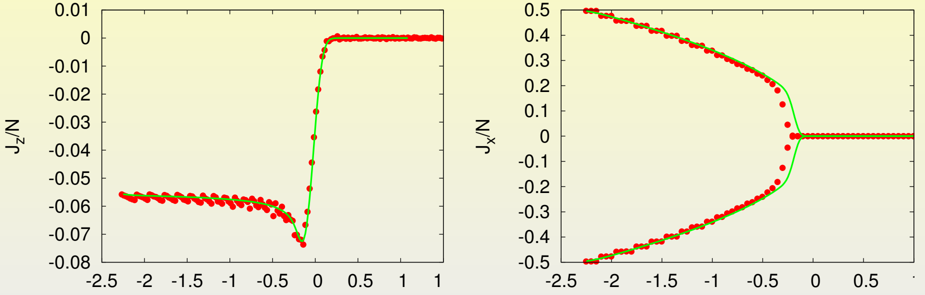
Numerical case: $\omega = \omega_0 = 1$ and $\lambda = 1.5$

$$\begin{aligned} \beta_c &= 0.223144 \\ \langle E_c \rangle / N &= -0.055 \\ \langle J_{z,c} \rangle &= -0.055 = \langle E_c \rangle / N \end{aligned}$$

Numerical results



Exact numerical results for $N = 50$



Summary and conclusions

- We have analyzed the relationship between the thermal phase transition and the ESQPT in the Dicke model when all the j-sectors are taken into account.
- Canonical and microcanonical ensembles are incompatible if we considerer just the highly-symmetric representation, i. e. $j = N/2$.
- In order to get a correct description of the thermodynamics properties it is necessary to include all the j-sectors.
- To perform the microcanonical analysis we must adopt a semiclassical approach. For $N = 50$ works pretty well.

- We have shown that each j-sector is equivalent to the one with $j = N/2$, but with a smaller effective coupling strength.
- The main signatures of the ESQPT are ruled out when we collect all the j-sectors.
- $\langle J_x \rangle$ still behaves as an order parameter.
- We have shown that the parity symmetry is spontaneously broken in the thermodynamical limit, if the system is in contact with a thermal bath.

Reference: PPF. and A. Relaño, Phys. Rev. E **96**, 012121 (2017).

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Thank you!