EXCEPTIONAL POINTS
FOR RANDOMLY PERTURBED CRITICAL HAMILTONIAN

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https://bit.ly/2kmewTs

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Outline

- Exceptional points (EP)
- Lipkin model: A simple critical system with both 1st and 2nd order QPTs
- EP distribution of randomly perturbed Hamiltonians

Exceptional points

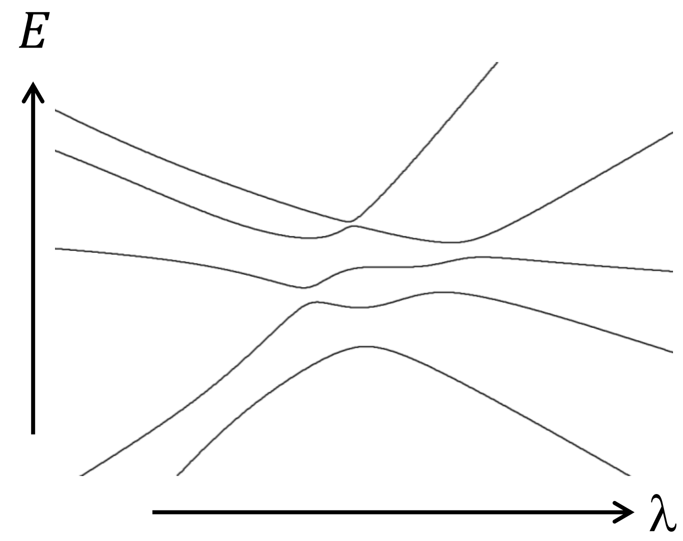
$$H(\lambda) = H_0 + \lambda V$$

$$[H_0, V] \neq 0$$

In a generic situation: no real crossings

$$E_i(\lambda) \neq E_{i+1}(\lambda)$$

(energies from the same symmetry subspace)



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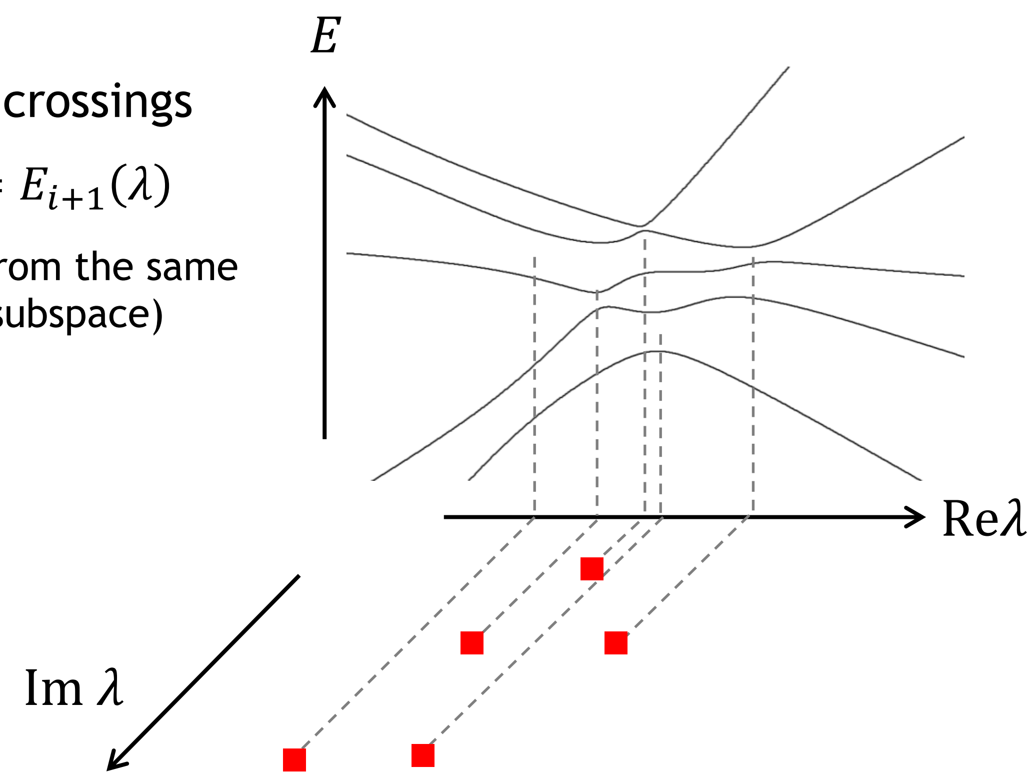
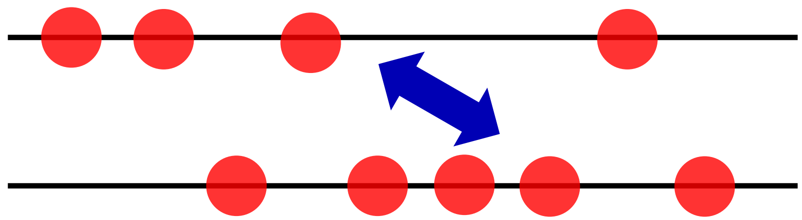


Nonhermitian extension of the Hamiltonian

Energy levels **cross** at $\frac{1}{2}n(n-1)$ complex conjugate pairs of exceptional points $\lambda^{EP}, \lambda^{EP*} \in \mathbb{C}$

Lipkin(-Meshkov-Glick) model

H.J. Lipkin, N. Meshkov, A.J. Glick, *Nucl. Phys.* **62**, 188 (1965)
N. Meshkov, A.J. Glick, H.J. Lipkin, *Nucl. Phys.* **62**, 199 (1965)
A.J. Glick, H.J. Lipkin, N. Meshkov, *Nucl. Phys.* **62**, 211 (1965)



2 levels with capacity N

N interacting fermions

$$J_z = \frac{1}{2}\sum_{i=1}^{N} a_{i+}^{+} a_{i+} - a_{i-}^{+} a_{i-}$$

$$J_+ = \sum_{i=1}^{N} a_{i+}^{+} a_{i-} \qquad J_- = (J_+)^+$$

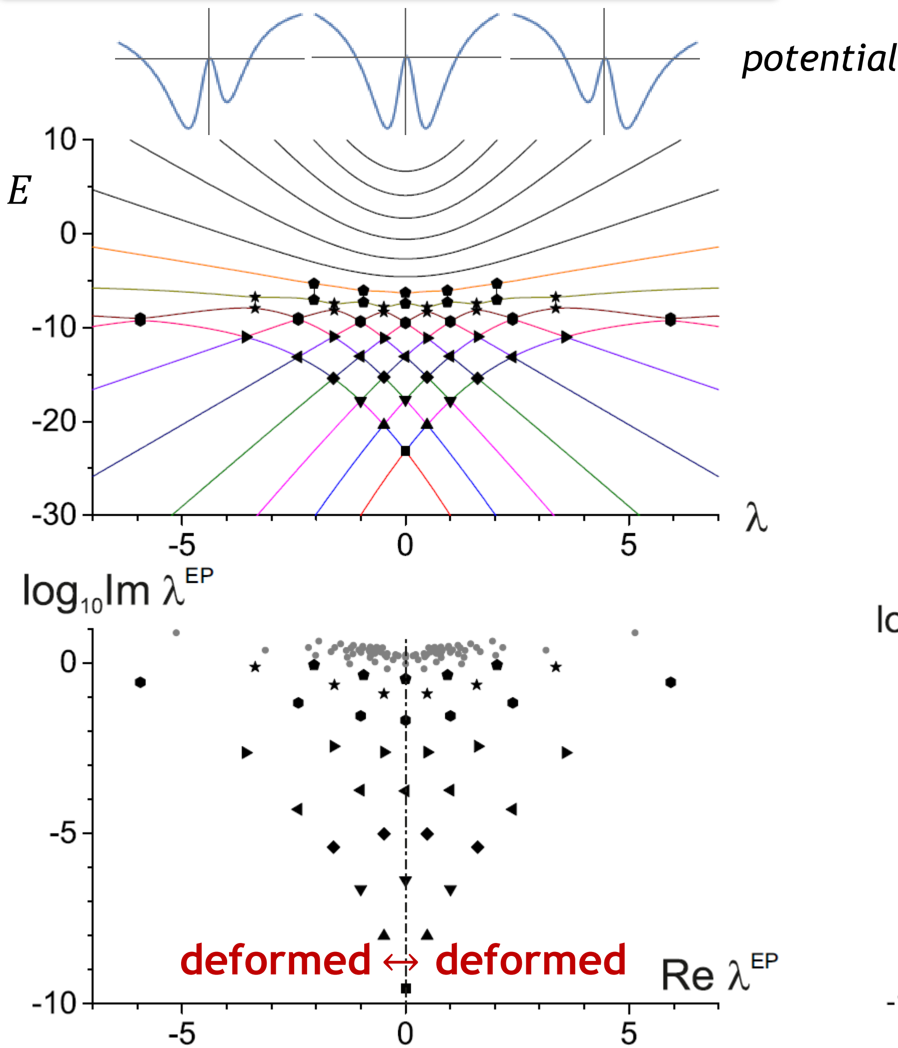
SU(2) algebra

Quasispin j is conserved

Only $j = \frac{N}{2}$ subspace considered – **1 degree of freedom**

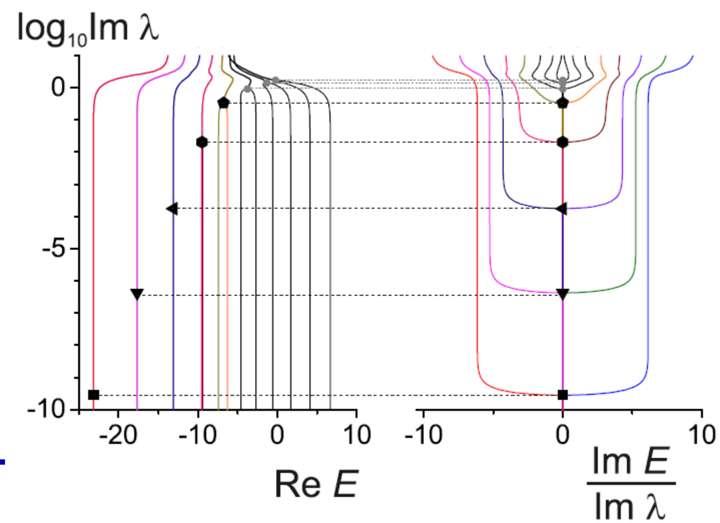
1^{st} order QPT

$$H_0 = J_3 - \frac{6}{N}J_1^2 \qquad V = -J_1 - \frac{1}{N}(J_1J_3 + J_3J_1)$$



Re E merges Im E diverges

at an individual EP
when Im λ increased

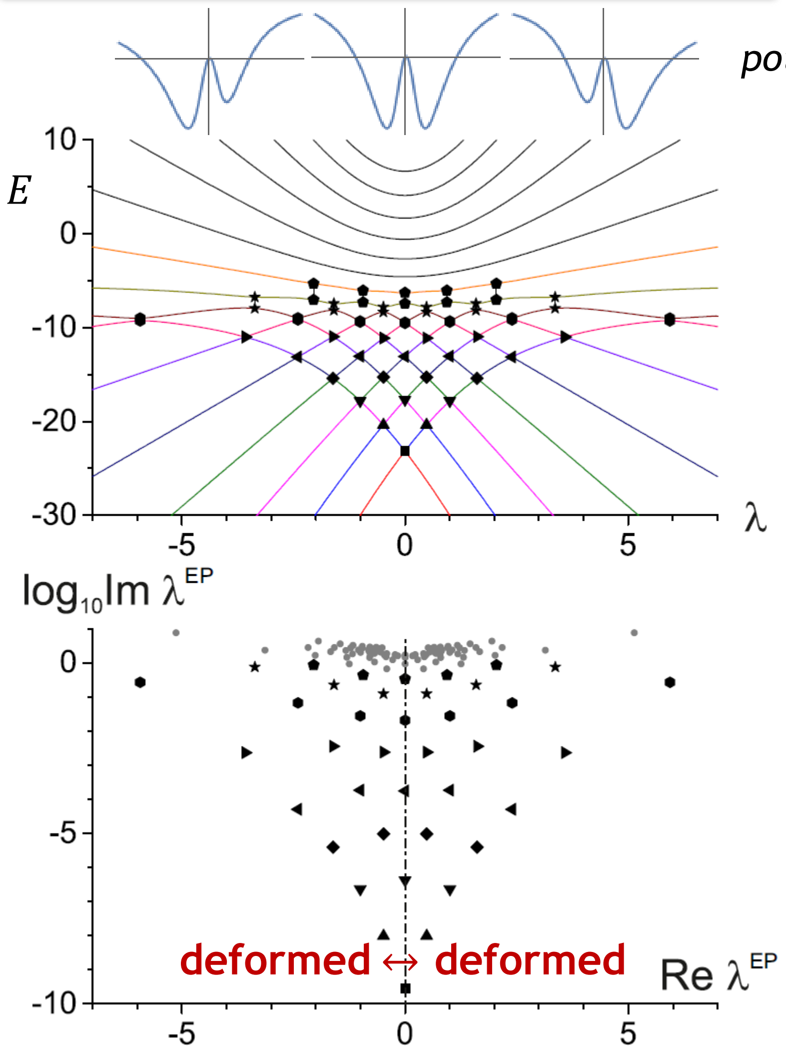


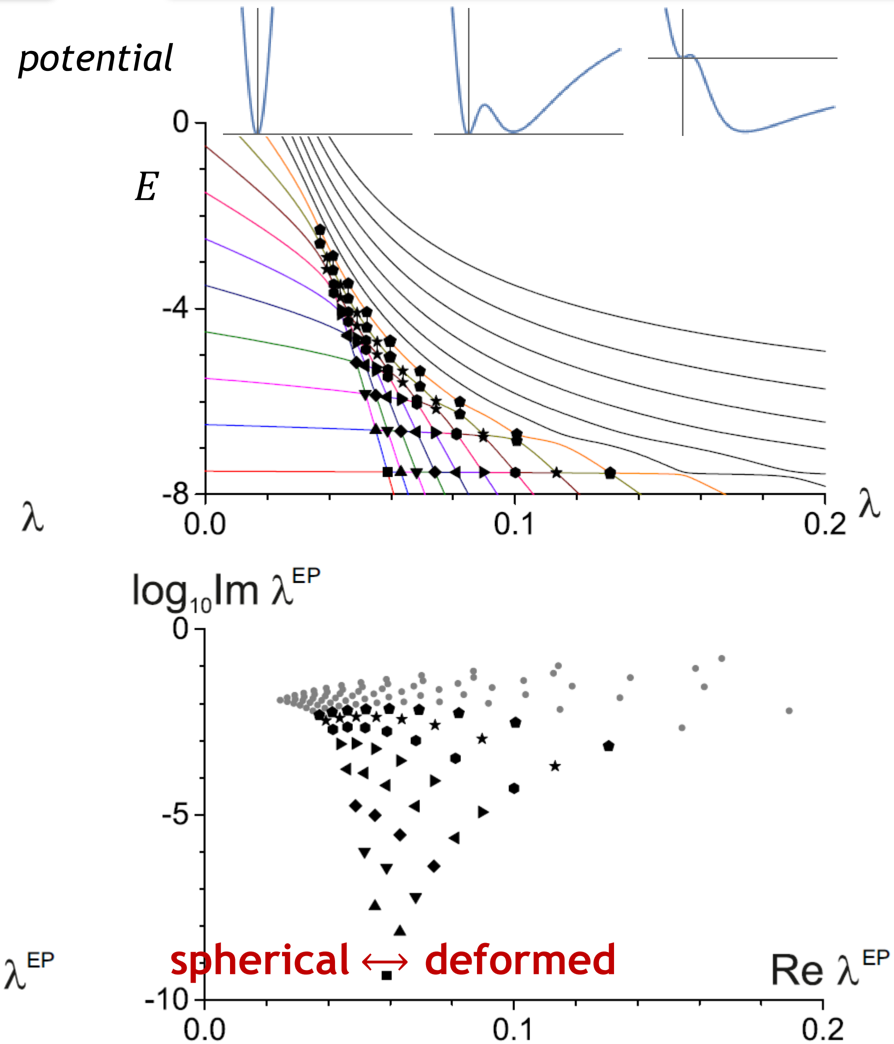
$N = 15$

1st order QPT

$$H_0 = J_3 - \frac{6}{N}J_1^2 \qquad V = -J_1 - \frac{1}{N}(J_1J_3 + J_3J_1)$$

$$H_0 = J_3 \qquad V = -\frac{1}{N}\left[J_1 + 4\left(J_3 + \frac{N}{2}\right)\right]^2$$





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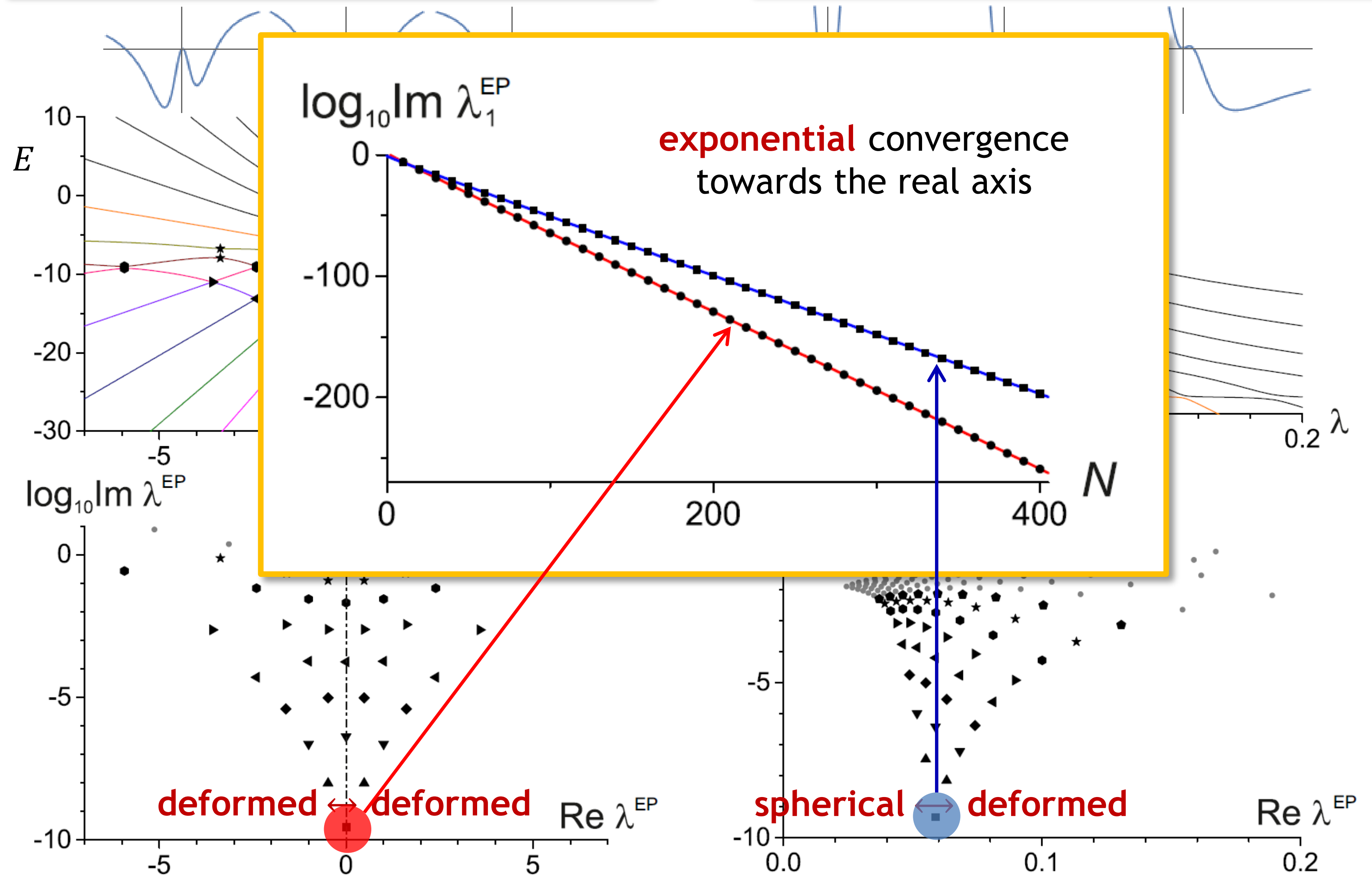
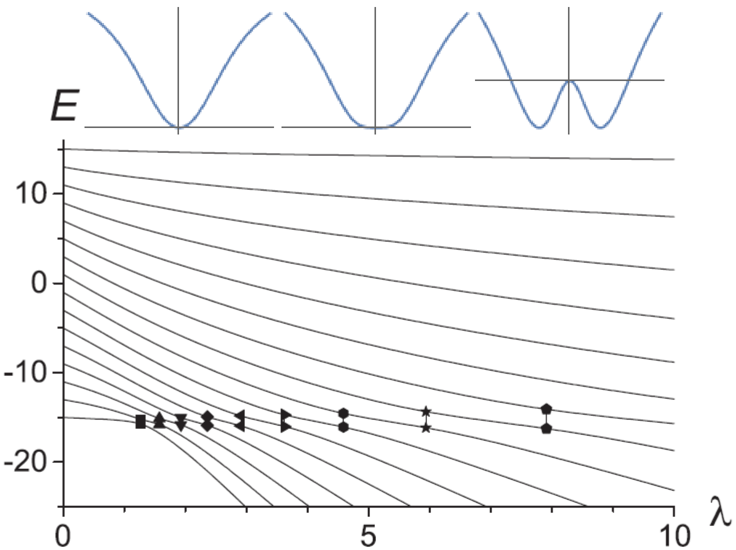
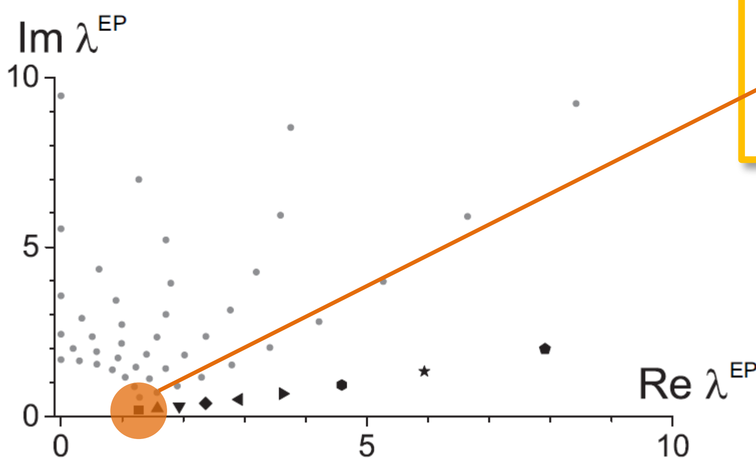


$N = 15$

2nd order QPT

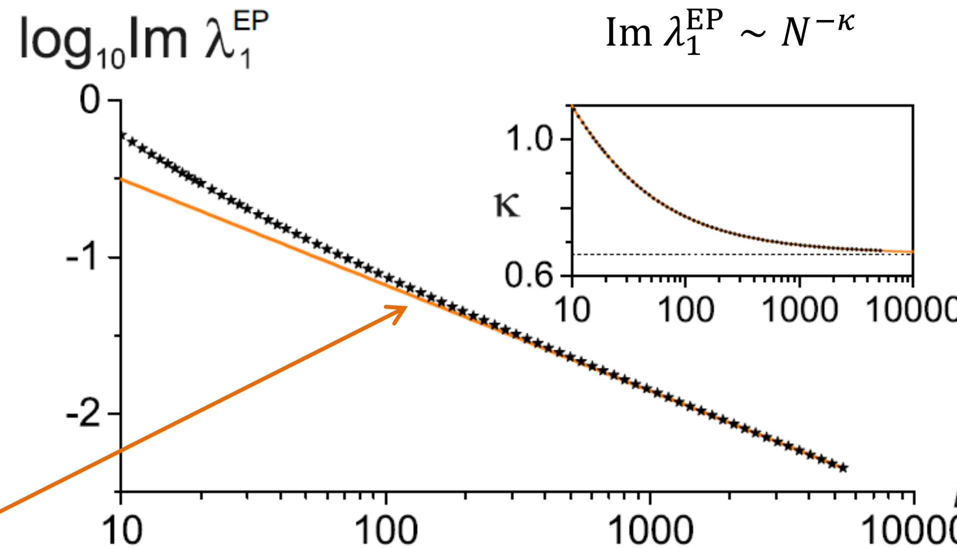
$$H_0 = J_3 \qquad V = -\frac{1}{N} J_1^2$$





algebraic convergence towards the real axis

$$\mathrm{Im}\,\lambda_1^{\mathrm{EP}} \sim N^{-\kappa}$$



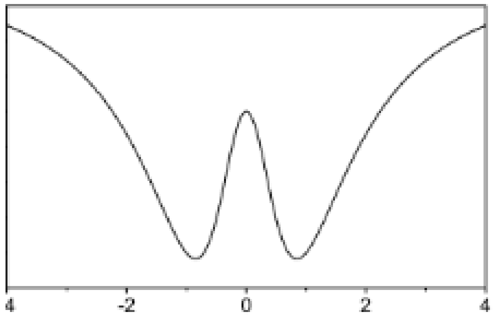
$N = 15$ even parity

W.D. Heiss, F.G. Scholz, H.B. Geyer, J. Phys. A: Math. Gen. **38**, 1843 (2005)
M. Šindelka, L.F. Santos, N. Moiseyev, Phys. Rev. A **95**, 010103 (2017)

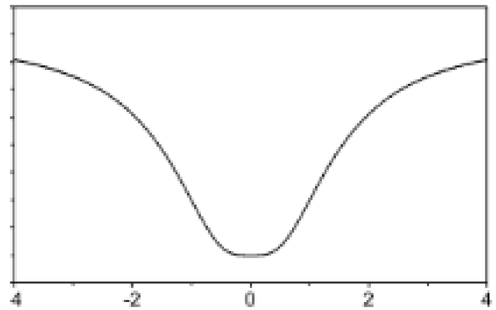
Critical Hamiltonians H_0

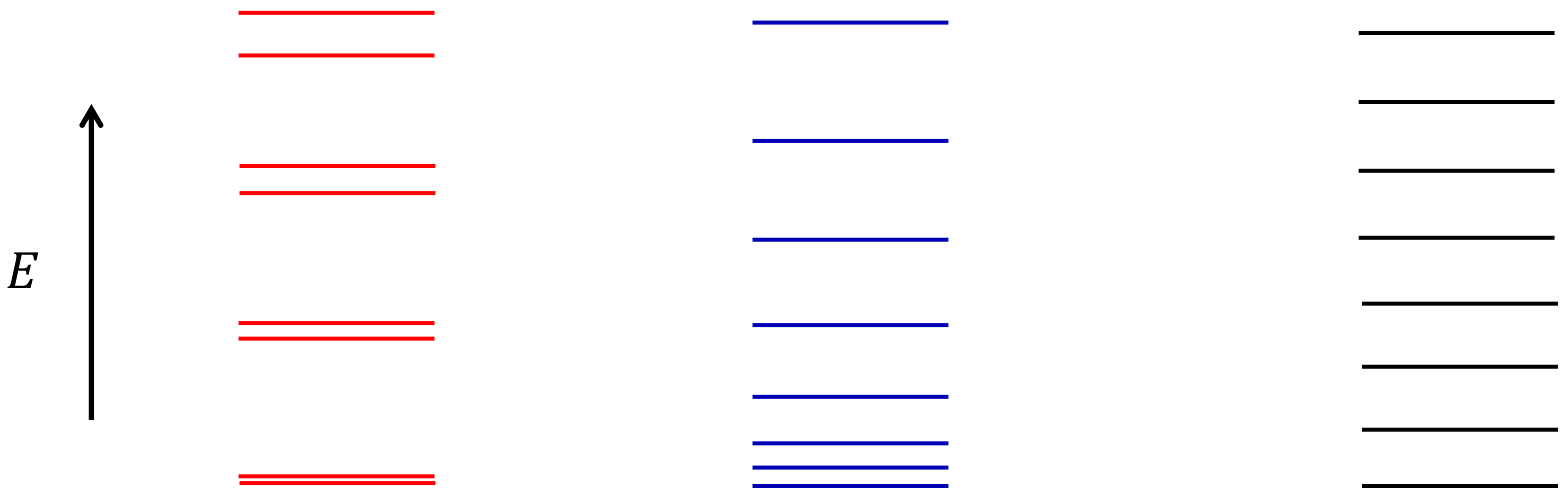
... to be compared with the Harmonic Oscillator

C1



C2



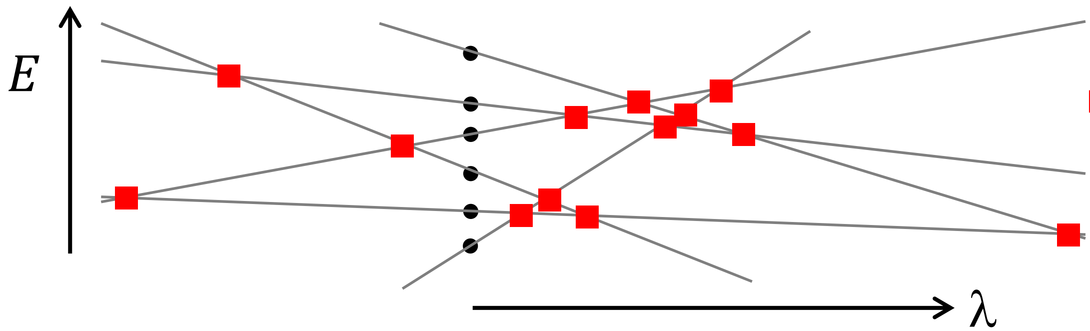


+ random perturbation V

(averaged over the whole ensemble of interactions V)

Diagonal perturbation

- corresponds to perturbations preserving all the symmetries of the original Hamiltonian

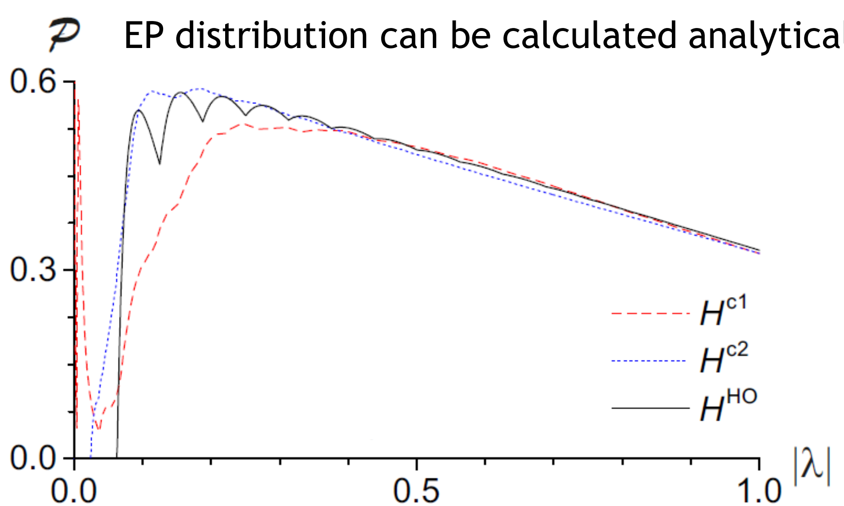


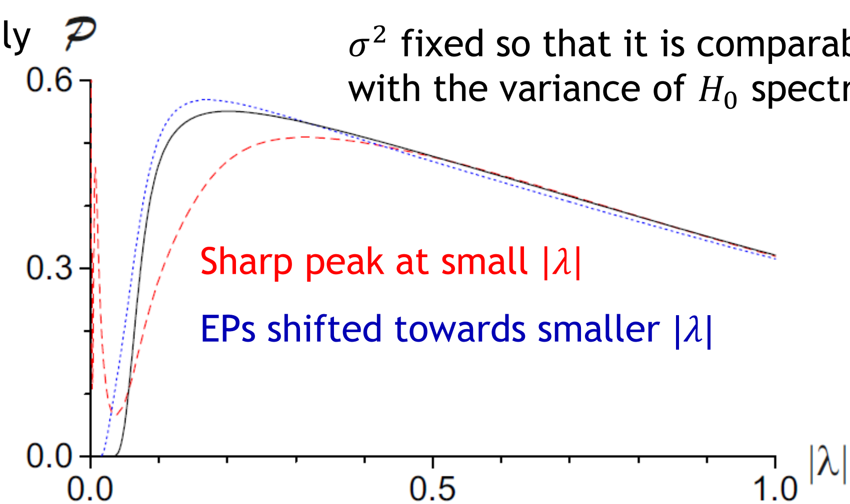
$$V = \begin{pmatrix} R(0,\sigma^2) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R(0,\sigma^2) \end{pmatrix}$$

$$V = \begin{pmatrix} N(0,\sigma^2) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N(0,\sigma^2) \end{pmatrix}$$

EP distribution can be calculated analytically

σ^2 fixed so that it is comparable with the variance of H_0 spectrum



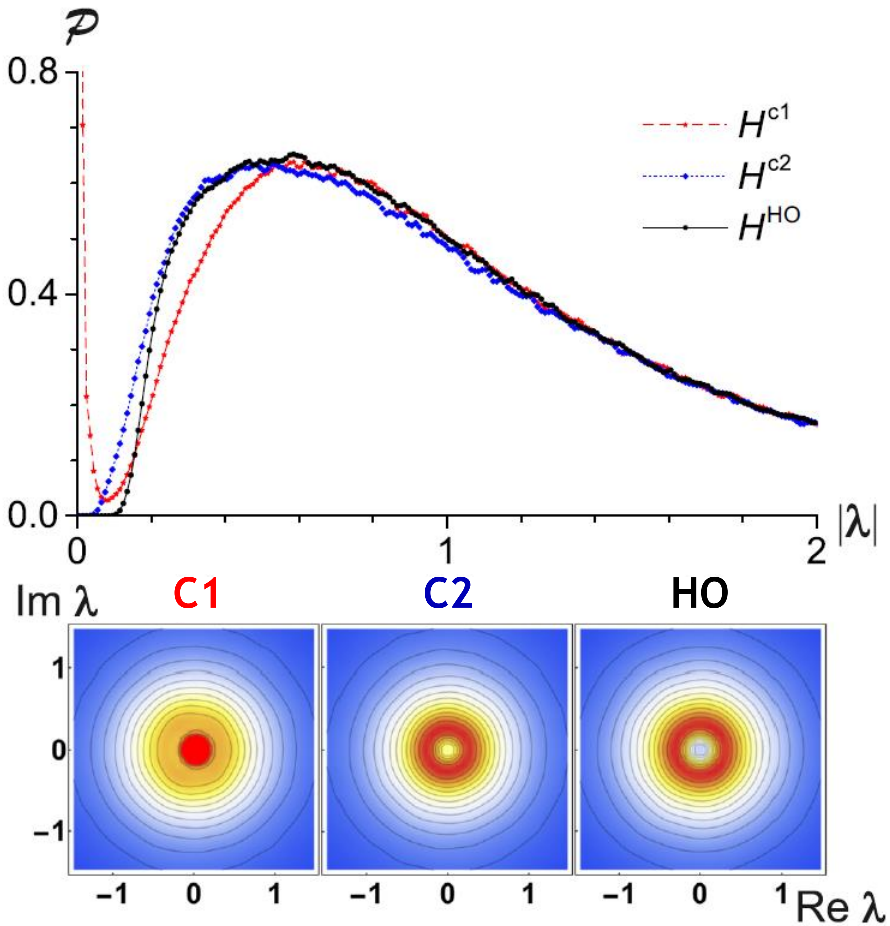


$d = 16$

GOE perturbation

$$V = \begin{pmatrix} N(0,2\sigma^2) & \cdots & N(0,\sigma^2) \\ \vdots & \ddots & \vdots \\ N(0,\sigma^2) & \cdots & N(0,2\sigma^2) \end{pmatrix} \in \mathrm{GOE}$$

eigenbasis = random rotation of the unperturbed basis



EP distribution in the complex λ plane is rotationally symmetric around the origin $\lambda = 0$

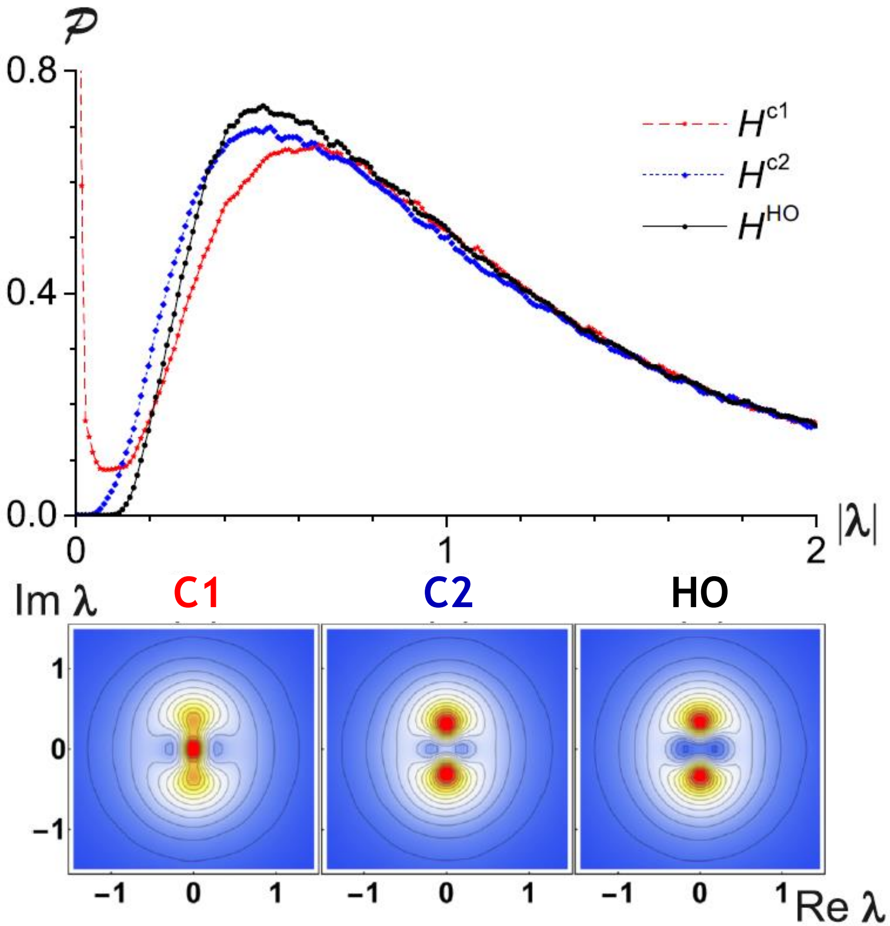
B. Shapiro, K. Zarembo,
J. Phys. A: Math. Theor. **50**, 045201 (2017)

$d = 16$, $\sim 10^6$ EPs

Off-diagonal perturbation

$$V = \begin{pmatrix} 0 & \cdots & N(0,\sigma^2) \\ \vdots & \ddots & \vdots \\ N(0,\sigma^2) & \cdots & 0 \end{pmatrix}$$

- iInitial symmetries are violated in a maximal way
- Expected results partly similar to the matrices from GUE



Small $|\lambda|$ behaviour similar with the diagonal and GOE perturbation

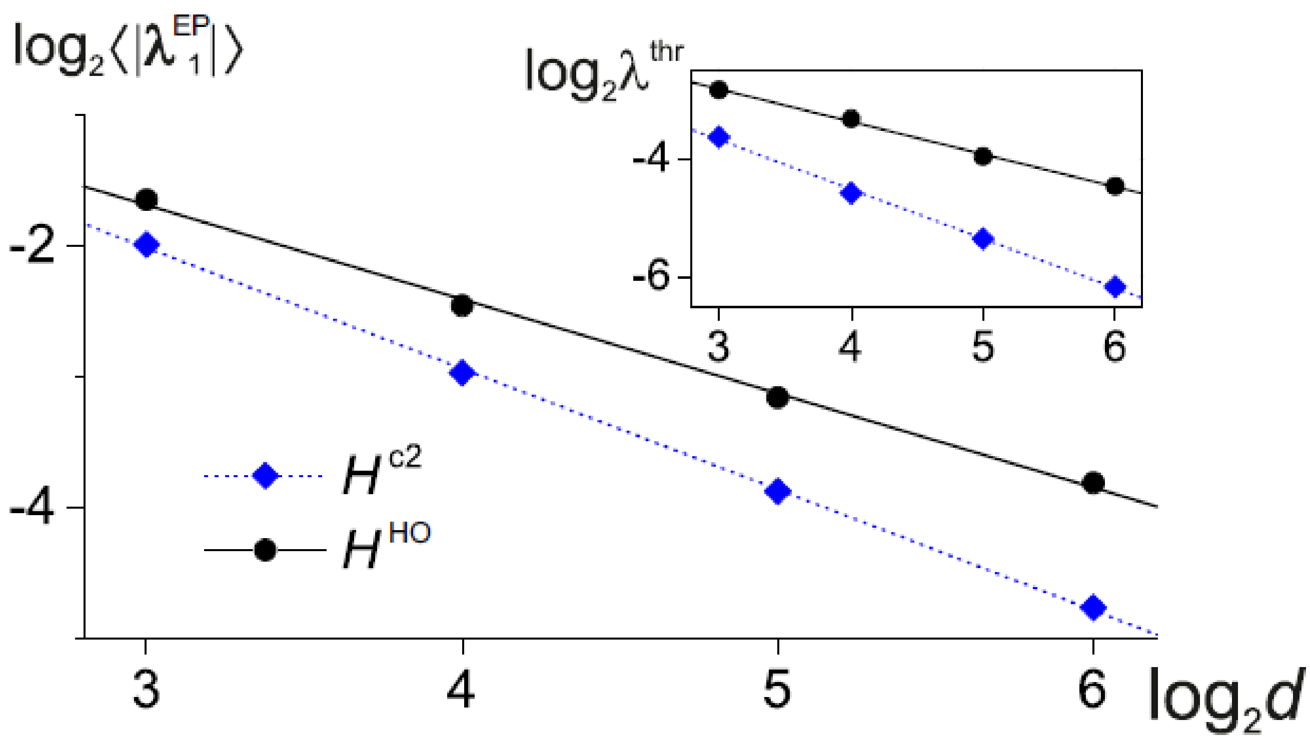
EPs the furthest from the real axis

$d = 16$, $\sim 10^6$ EPs

Scaling with matrix dimension d

Ensemble average of the distance of the closest EP from the origin

Distance of the closest EP from the origin in an ensemble of 10^6 generated EPs



Power-law behaviour d^{-c}, where c is higher for the critical Hamiltonian

Conclusions

1st order QPT

Connected with a single pair of EPs.

Exponential

2nd order QPT

Ground-state affected by several EPs located at comparable distances from the real axis.

Algebraic

- convergence of the EPs to the real axis when the system's size grows
- accumulation of the EPs near $\lambda = 0$ when the system is arbitrarily perturbed

- EP distribution represents a strong signature of quantum criticality allowing us to discriminate between the first- and higher-order critical Hamiltonians
- EP distribution may have consequences for the superradiance phenomenon in open quantum systems (*work in progress*)

P. Stránský, M. Dvořák, P. Cejnar, Physical Review E **97**, 012112 (2018)

THANKS FOR YOUR ATTENTION

Resultant $R(\lambda)$

$$P(E, \lambda) = \det[E - H(\lambda)] = 0$$

$$Q(E, \lambda) = \frac{\partial P(E, \lambda)}{\partial E} = 0$$

- polynomial of degree $N(N-1)$
- pairs of complex conjugate roots
- no roots on the real axis

σ^2 adjustment

$$D_H = \frac{1}{d-1}\sum_{n=1}^{d}|E_n - M_E|^2 = \frac{\mathrm{Tr}\, HH^+}{d-1} - \frac{\mathrm{Tr}\, H\, \mathrm{Tr}\, H^+}{d(d-1)}$$

- quadratic spread of the spectrum E_n
- characterizes an average diameter of the cloud of complex eigenvalues

$$M_H = \frac{1}{d}\sum_{n=1}^{d} E_n = \frac{\mathrm{Tr}\, H}{d}$$

- center of mass of the spectrum

HO: $\sqrt{D_E} \approx \frac{\omega d}{\sqrt{12}},\ E_n = \omega n$

C2: $\sqrt{D_E} \approx \frac{\omega d}{\sqrt{11.23}},\ E_n = \omega n^{\frac{4}{3}} d^{-\frac{1}{3}}$

$$\langle D_V \rangle = D_{H(0)}$$

- $\sigma^2 = D_{H(0)}$ for the diagonal perturbation
- $\sigma^2 = \frac{D_{H(0)}}{d+2}$ for the GOE perturbation
- $\sigma^2 = \frac{D_{H(0)}}{d}$ for the offdiagonal perturbation

Global properties of the spectrum

$M_{H(\lambda)} = M_{H(0)} + \lambda M_V$ - spectral mean value

$$D_{H(\lambda)} = D_{H(0)} + \mathrm{Re}\lambda \underbrace{\frac{2d}{d-1}\left[M_{H(0)V} - M_{H(0)}M_V\right]}_{K} + |\lambda|^2 D_V$$

- quadratic spread (quadratic dependence on λ)

$\mathrm{Im}\lambda = 0$

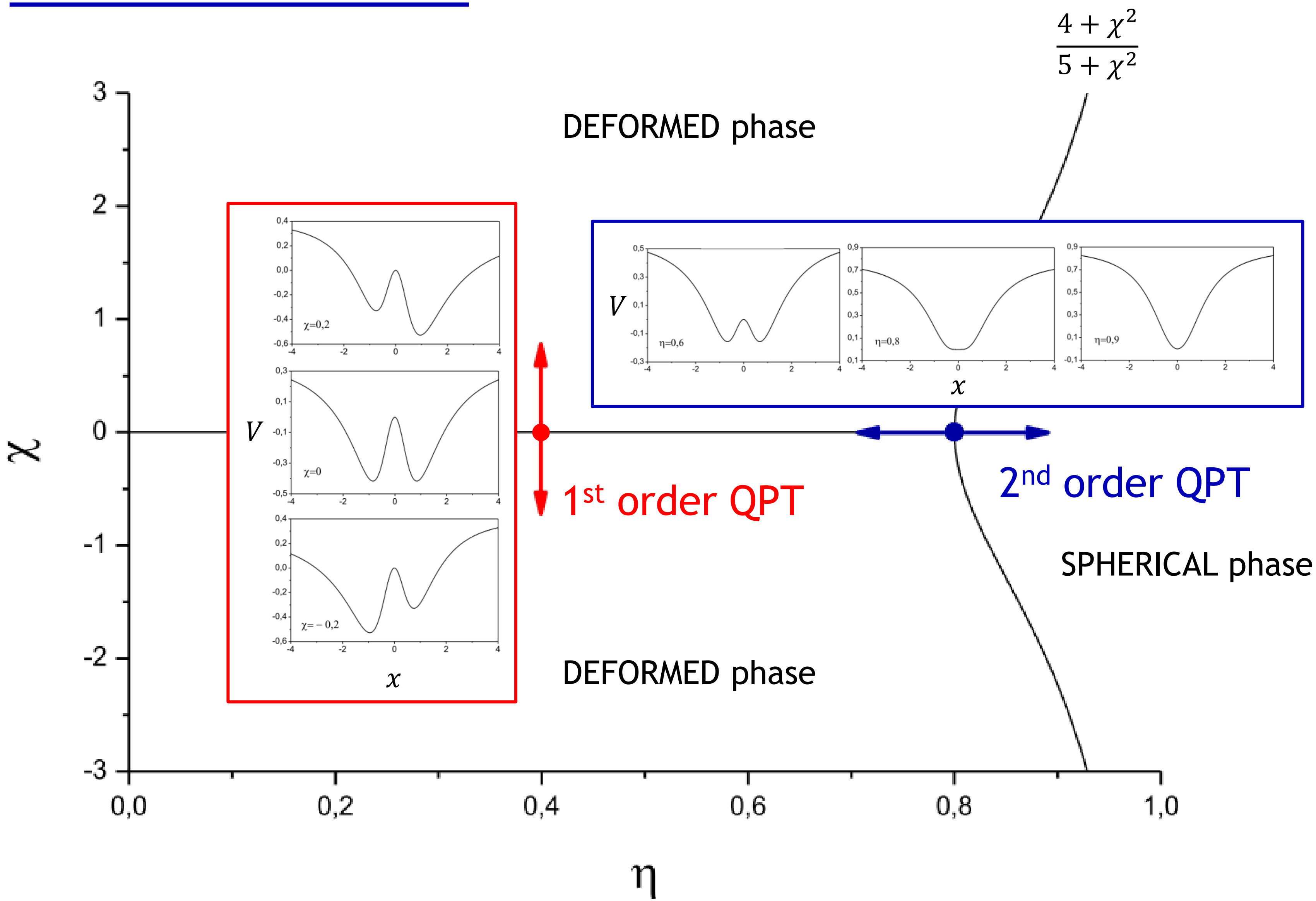
parabola with a minimum at $\lambda_0 = -\frac{K}{2D_V}$
(at this point the spectrum is maximally compressed)

Main structural changes expected for $\Delta\lambda \approx \sqrt{\frac{D_{H(\lambda_0)}}{D_V}}$ vicinity of λ_0

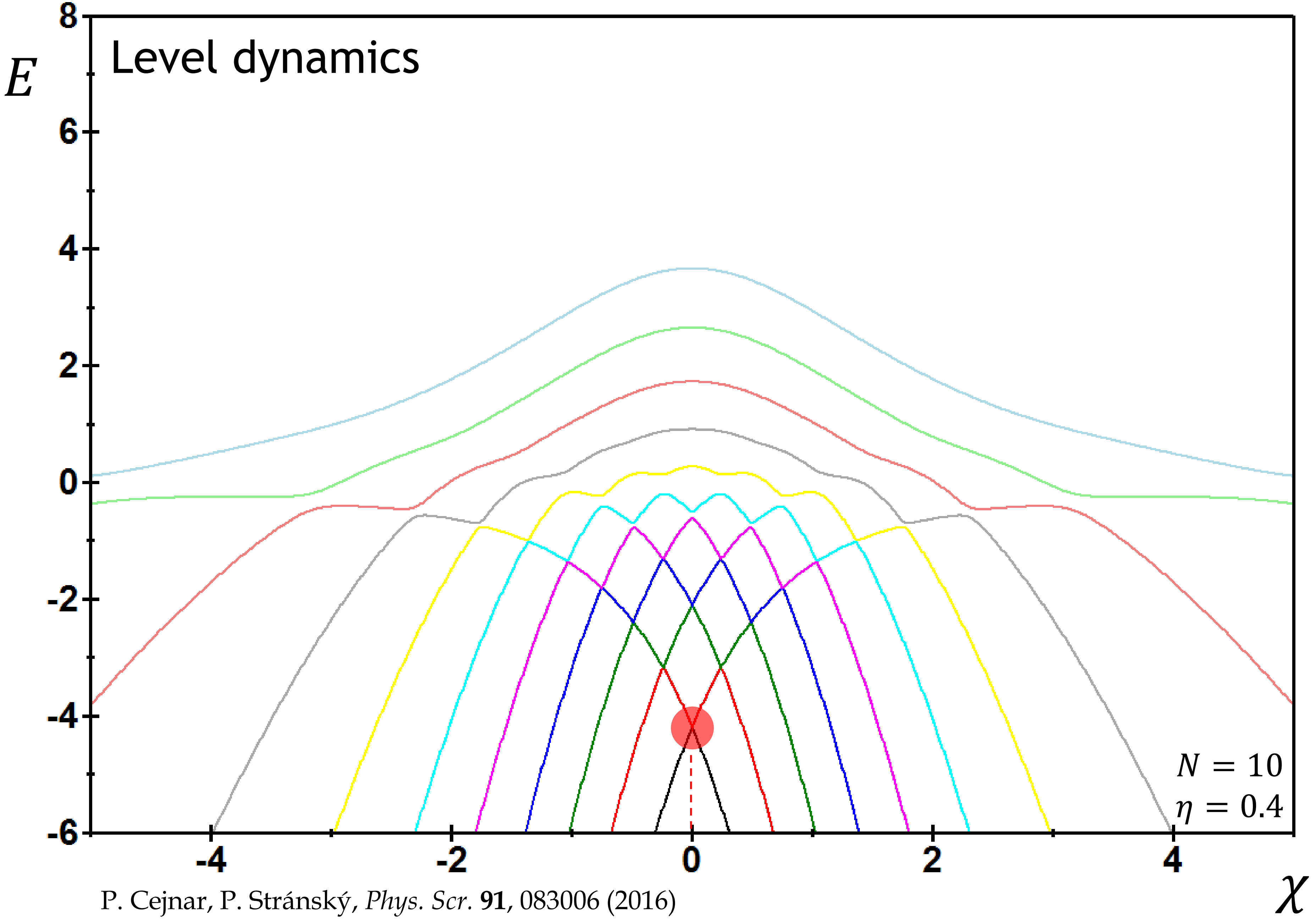
Expectation values

$\langle M_V \rangle = \langle K \rangle = 0$ $\langle D_V \rangle = D_{H(0)}$ $\langle K \rangle = 0$

Phase structure

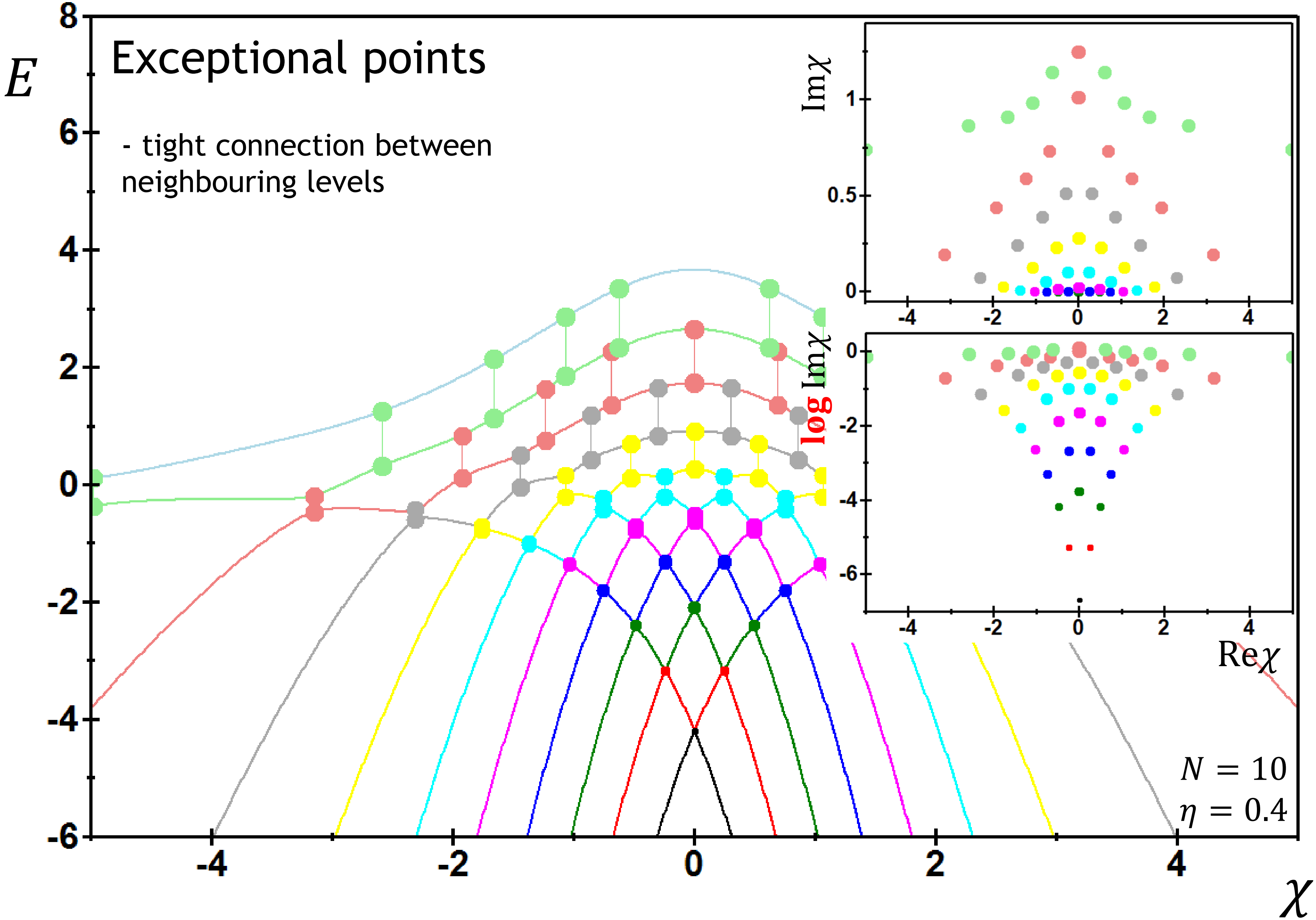


1st order QPT

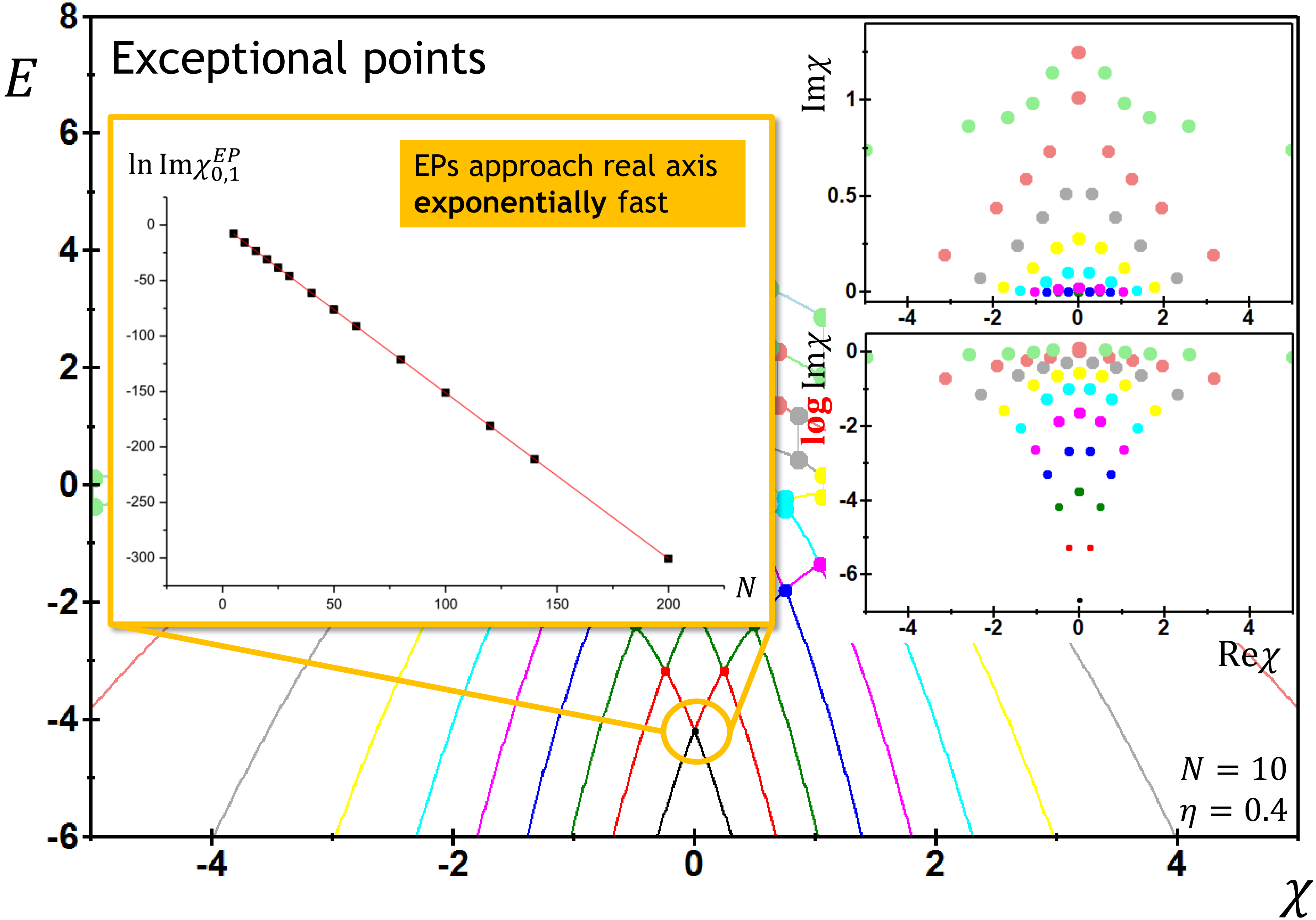


P. Cejnar, P. Stránský, *Phys. Scr.* **91**, 083006 (2016)

1st order QPT



1st order QPT



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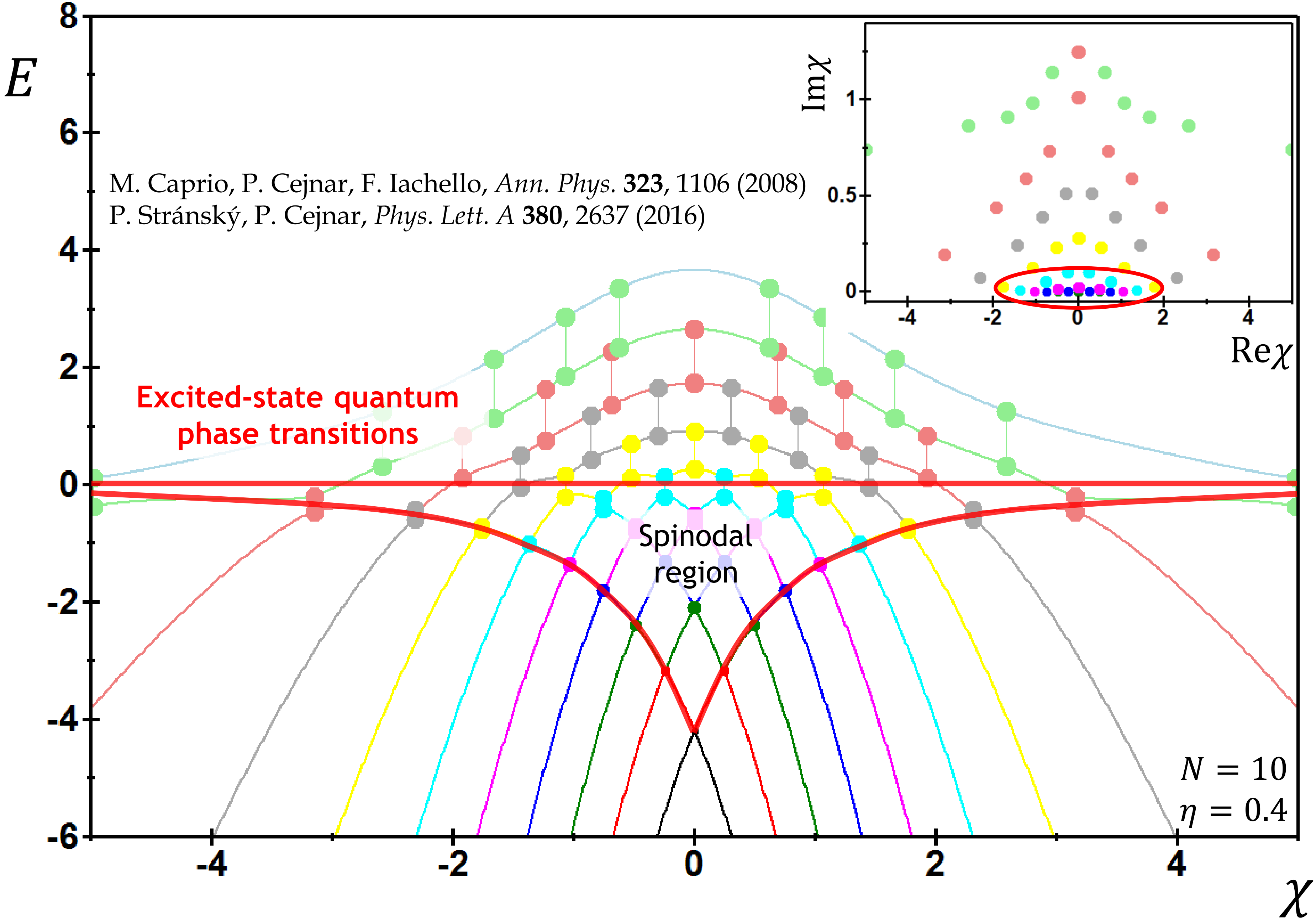
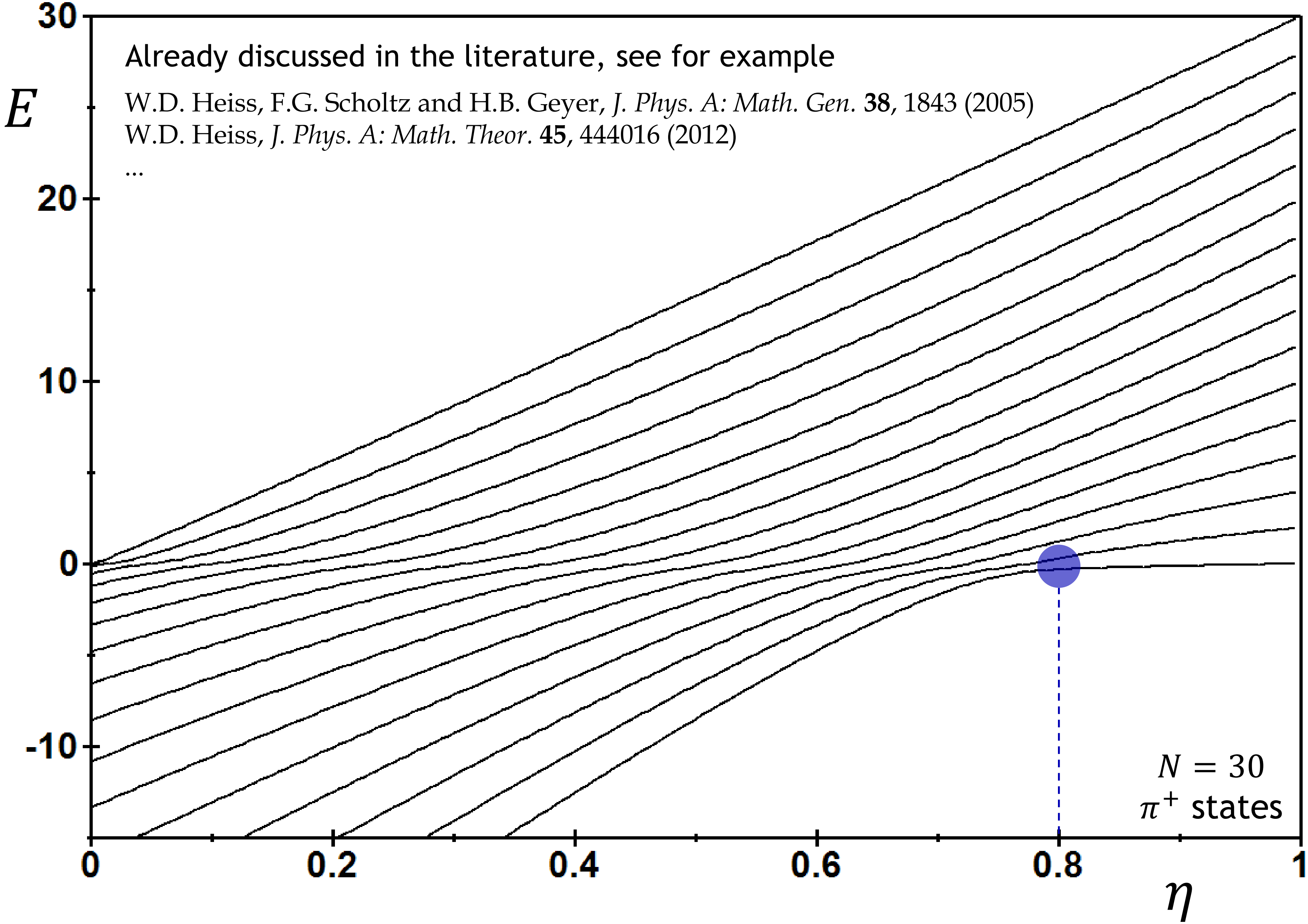
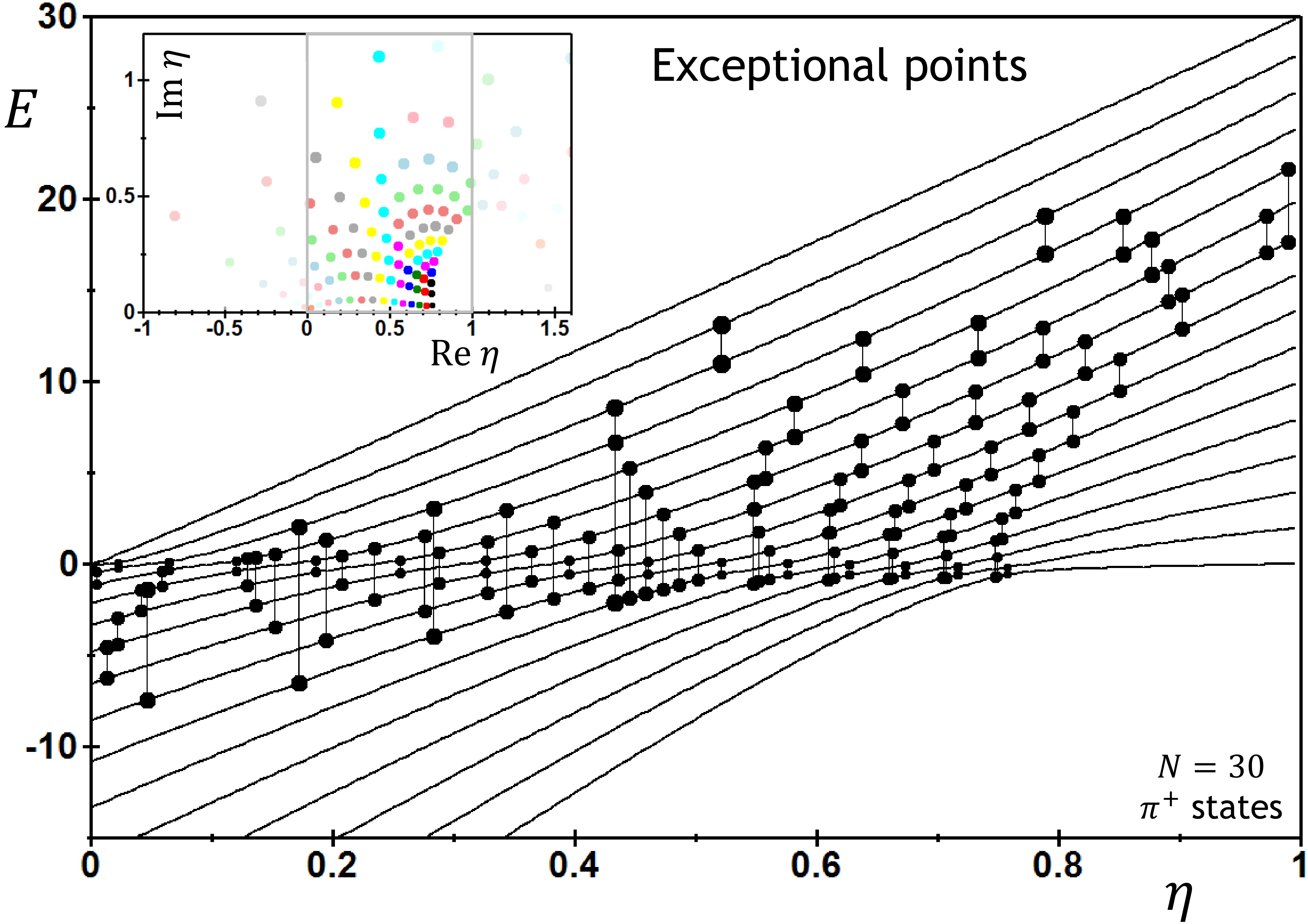


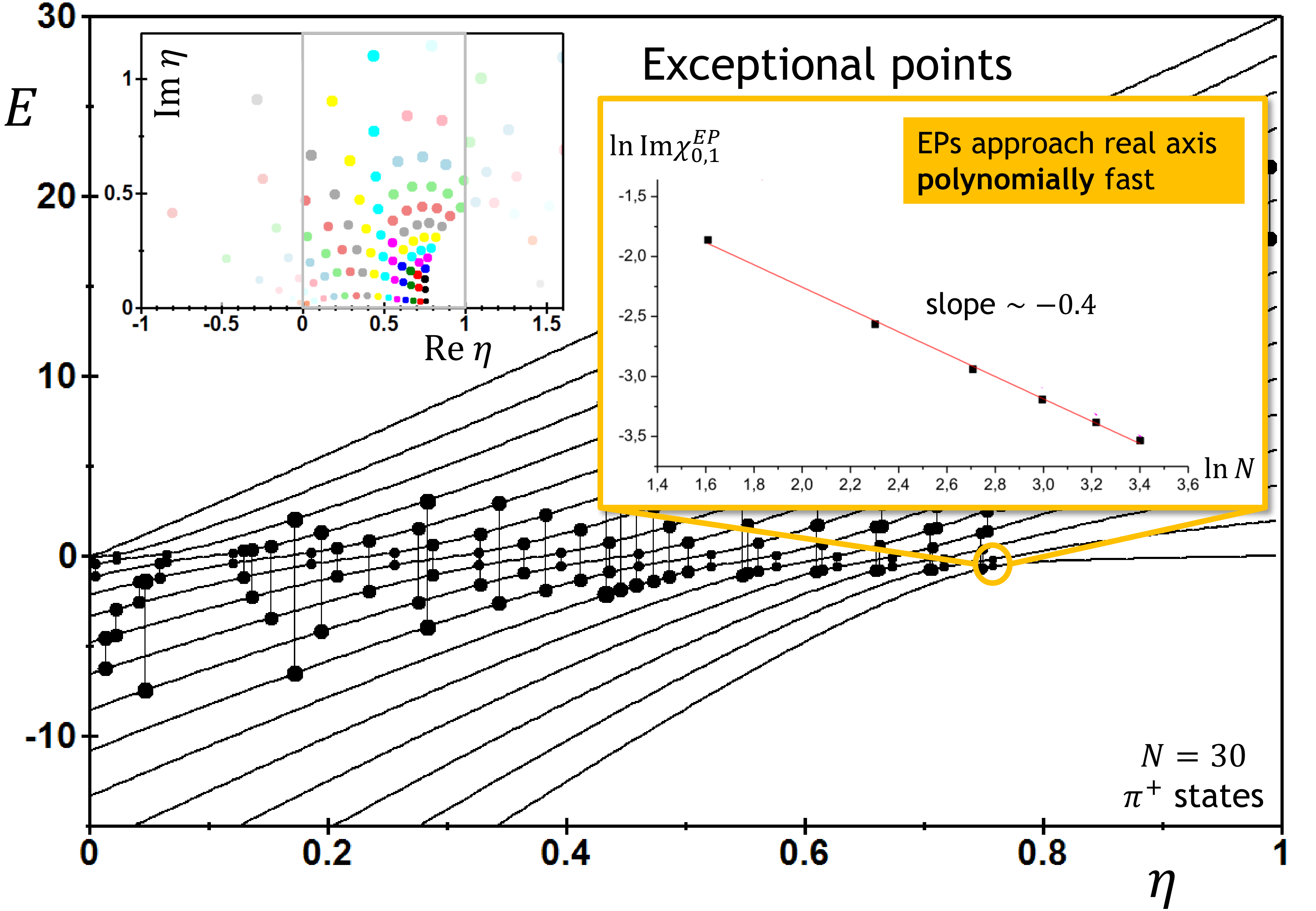
2nd order QPT



2nd order QPT



2nd order QPT



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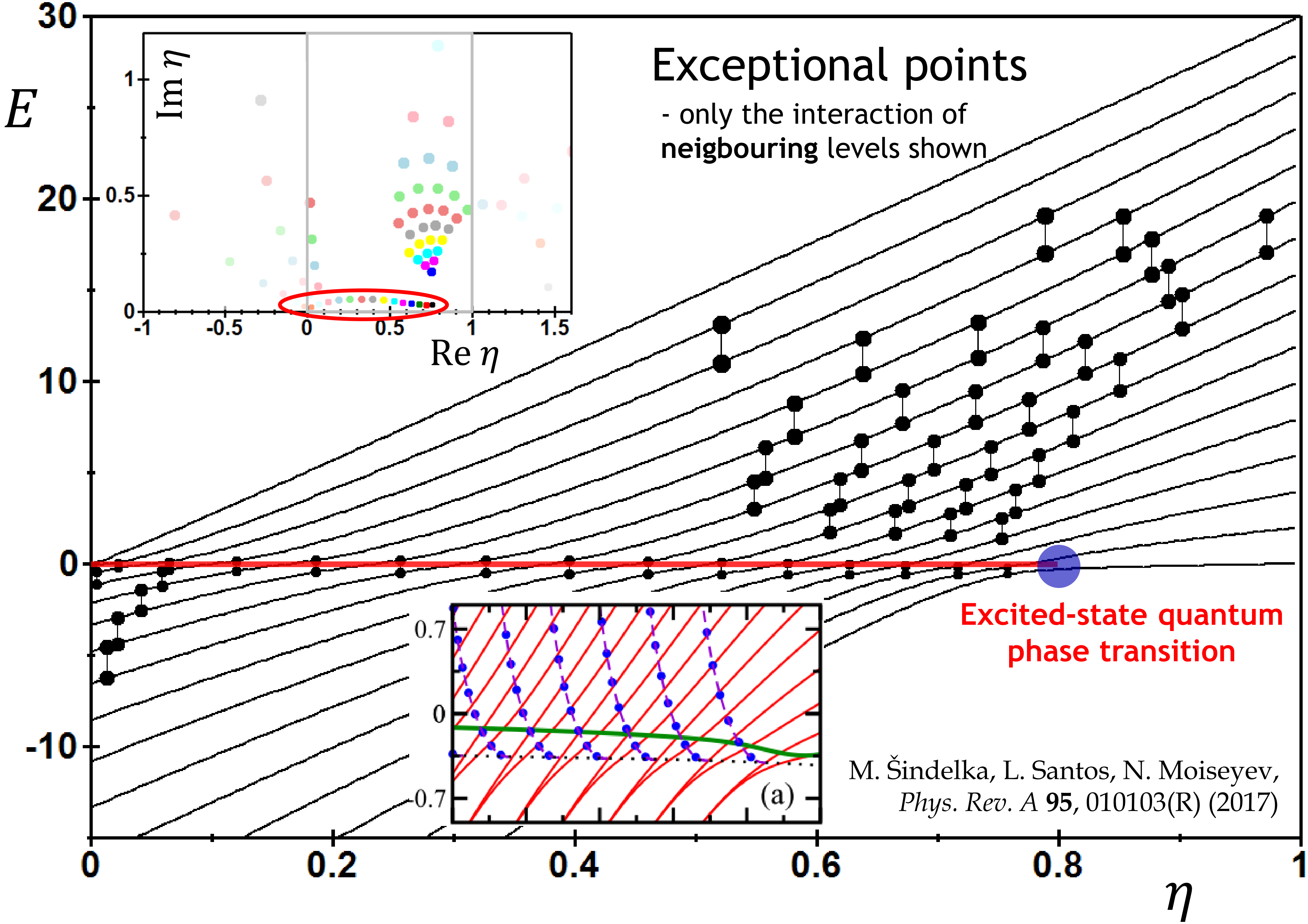
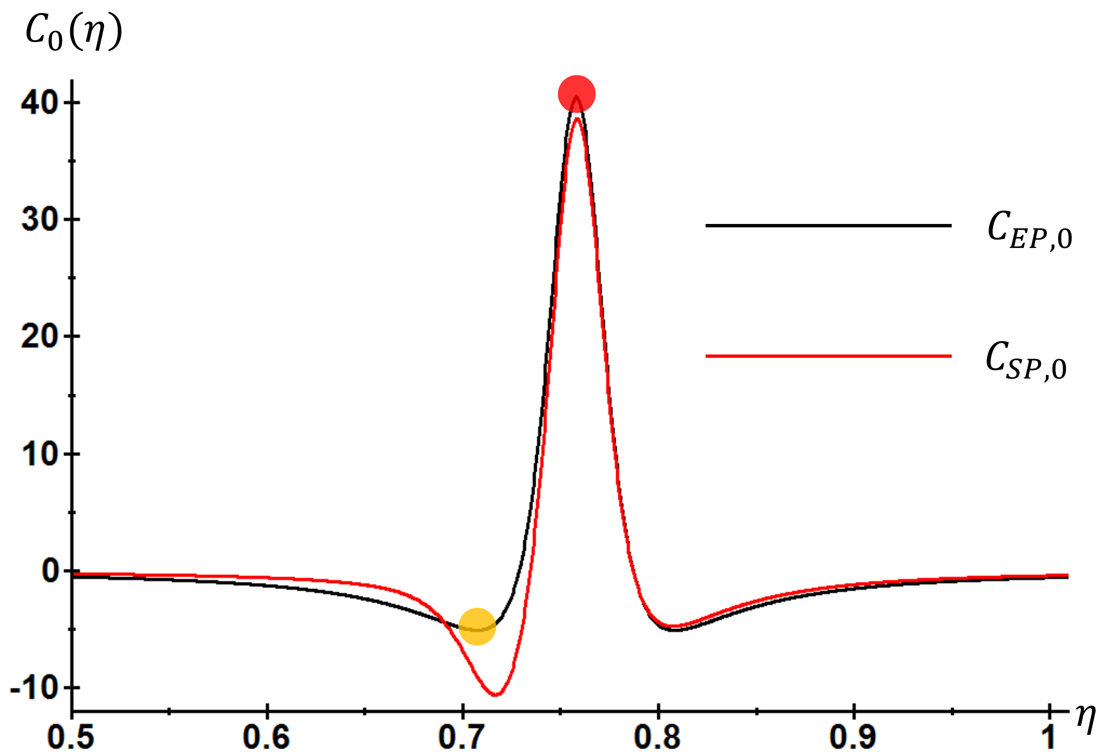
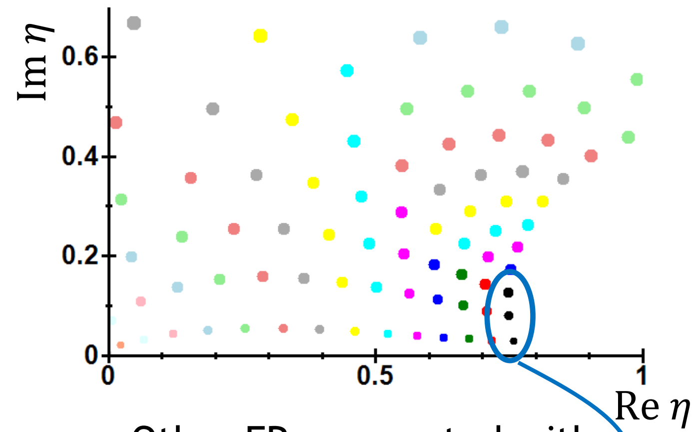


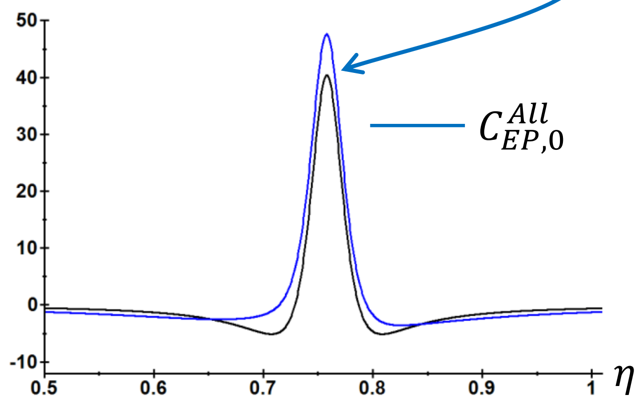
M. Šindelka, L. Santos, N. Moiseyev, *Phys. Rev. A* **95**, 010103(R) (2017)

Specific heat analogy





Other EPs connected with the ground state included

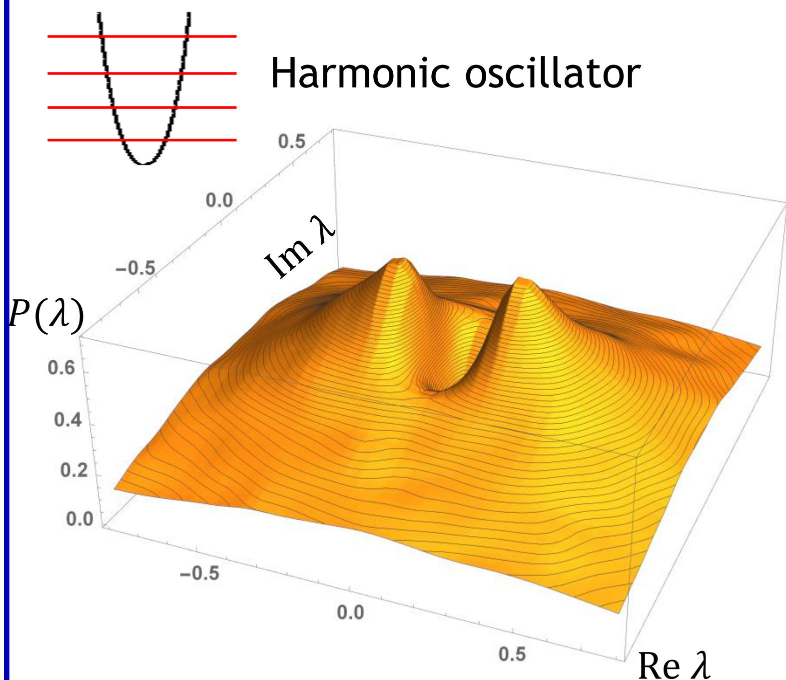


Latent heat analogy

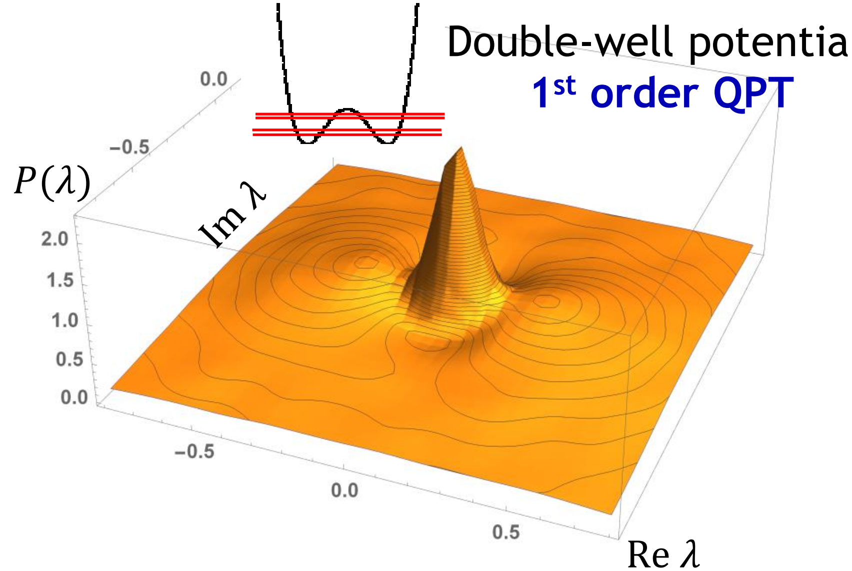
- approximated by $Q \approx \chi_{min} * C_{max} = \frac{\sqrt{3}}{2(N-1)} \frac{1}{\boldsymbol{Im}\ \boldsymbol{\chi_{0,1}^{EP}}} \sim \frac{\boldsymbol{N^{0.4}}}{N} \to 0$

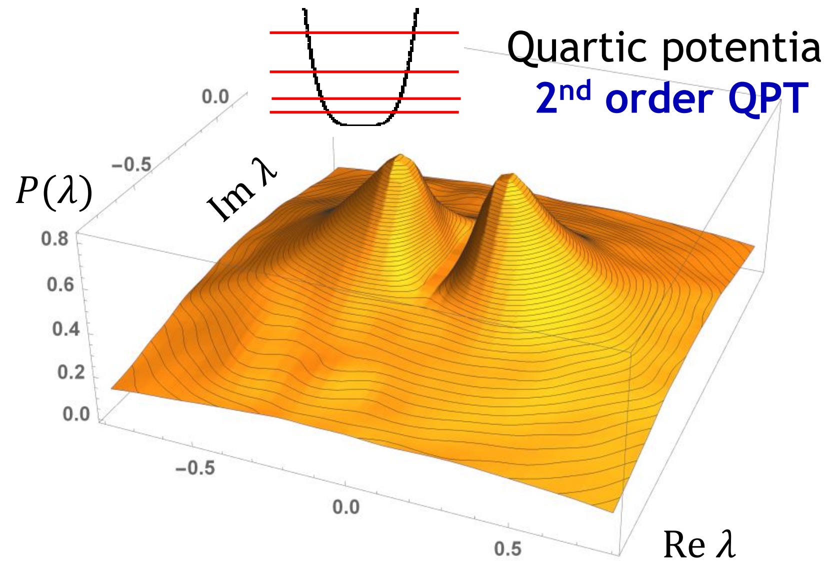
Random perturbation at a critical point

$$H = H_c + \lambda H_{\mathrm{GOE}}$$



M.R. Zirnbauer, J.J.M. Verbaarschot, H.A. Weidenmüller, *Nucl. Phys.* **A411**, 161 (1983)

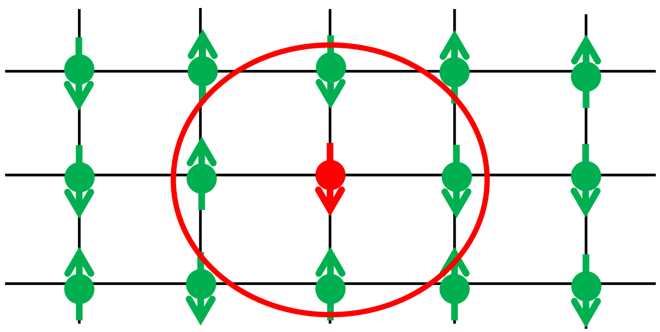


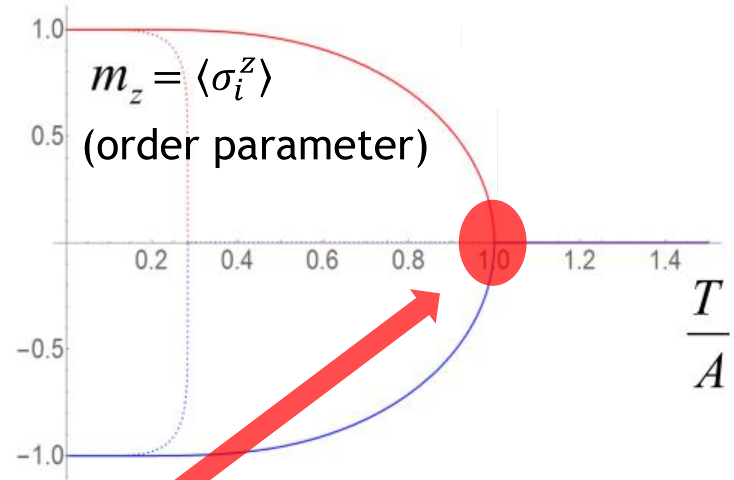


Preliminary results

Thermal phase transitions

Example: 2D Ising model $H = -\frac{A}{2}\sum_{\{ij\}} \sigma_i^z \sigma_j^z$

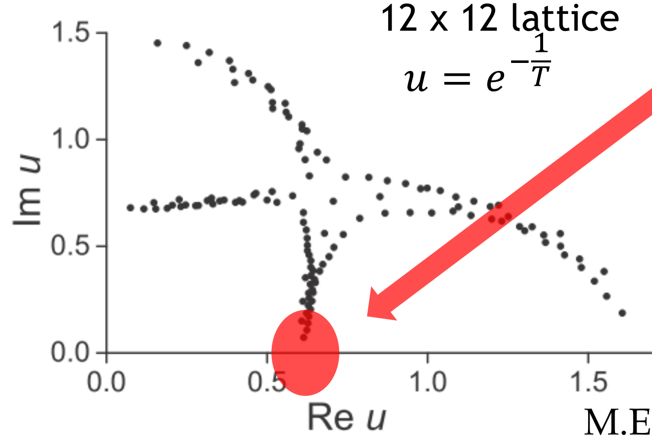




Extended into the complex plane $Z(T)$ can vanish – **Yang-Lee zeros**

12 x 12 lattice

$u = e^{-\frac{1}{T}}$



Their approaching the real axis in the TD limit indicates a TD phase transition.

T.D. Lee, C.N. Yang, *Phys. Rev.* **87**, 410 (1952)
M.E. Fischer, *Lecture Notes in Theoretical Physics* **7C**, 1 (1965)

2D electrostatics of EPs

Resultant

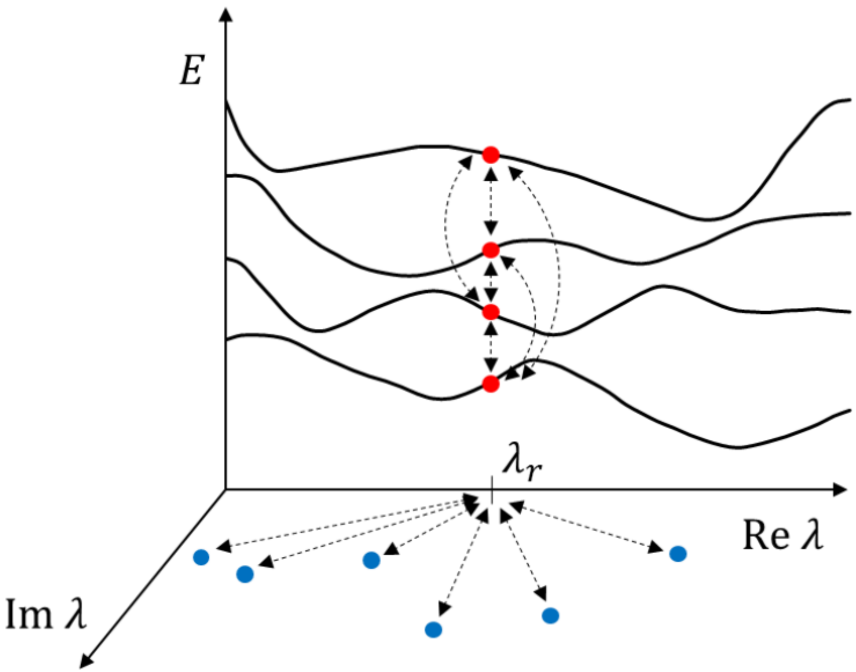
$$R(\lambda) = \prod_{i<j} \left[E_i(\lambda) - E_j(\lambda)\right]^2 = a \prod_{k=1}^{\frac{N(N-1)}{2}} \left(\lambda - \lambda_k^{EP}\right)\left(\lambda - \bar{\lambda}_k^{EP}\right)$$

$$U_{SP}(\lambda_R) = -\frac{1}{N-1} \sum_{i<j} \ln\left|E_i(\lambda_R) - E_j(\lambda_R)\right|$$

Coulomb energy of charges placed on energy levels at a given λ_R on the real axis

$$U_{EP}(\lambda_R) = -\frac{\ln a}{2(N-1)} - \frac{1}{\mathrm{N}-1} \sum_{k=1}^{\frac{N(N-1)}{2}} \ln R_k(\lambda_R)$$

(Shifted) Coulomb potential at the point λ_R from charges placed in the EPs



Partial resultant

$$R_j^a(\lambda) = \prod_{i \neq j} \left[E_i(\lambda) - E_j(\lambda)\right]$$

$$R_j^b(\lambda) = a \prod_{k \neq j} \left(\lambda - \lambda_{k,j}^{EP}\right)\left(\lambda - \bar{\lambda}_{k,j}^{EP}\right)$$

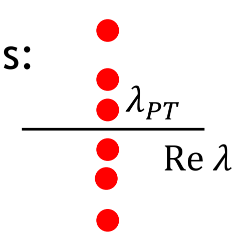
(Product over EPs on k-th Riemann sheet)

Open question: Relation between $R_j^a(\lambda)$ and $R_j^b(\lambda)$?

Thermal PT ⟷ Quantum PT

Thermal PT		Quantum PT	
Yang-Lee zeros of the partition function		Zeros of the resultant (non-Hermitian degeneracies)	
Partition function	$Z(T) = \sum_i e^{-\frac{E_i}{T}}$	Partial resultant	$R_j(\lambda) = \prod_{i \neq j} \left[E_i(\lambda) - E_j(\lambda)\right]^2$
Free energy	$F(T) = -T \ln Z(T)$	Coulomb energy	$U_j(\lambda) = -\frac{1}{\Omega} \ln R_j(\lambda)$
Specific heat	$C(T) = -T \frac{\partial^2 F(T)}{\partial T^2}$		$C_j(\lambda) = -\frac{\partial^2 U_j(\lambda)}{\partial \lambda^2}$
Latent heat	$Q(T) = \lim_{\epsilon \to 0+} \int_{T-\epsilon}^{T+\epsilon} C(T') dT'$		$Q(\lambda) = \lim_{\epsilon \to 0+} \int_{\lambda-\epsilon}^{\lambda+\epsilon} C(\lambda') d\lambda'$

Order of the PT is given by the **density of zeros** in the vicinity of the real axis:



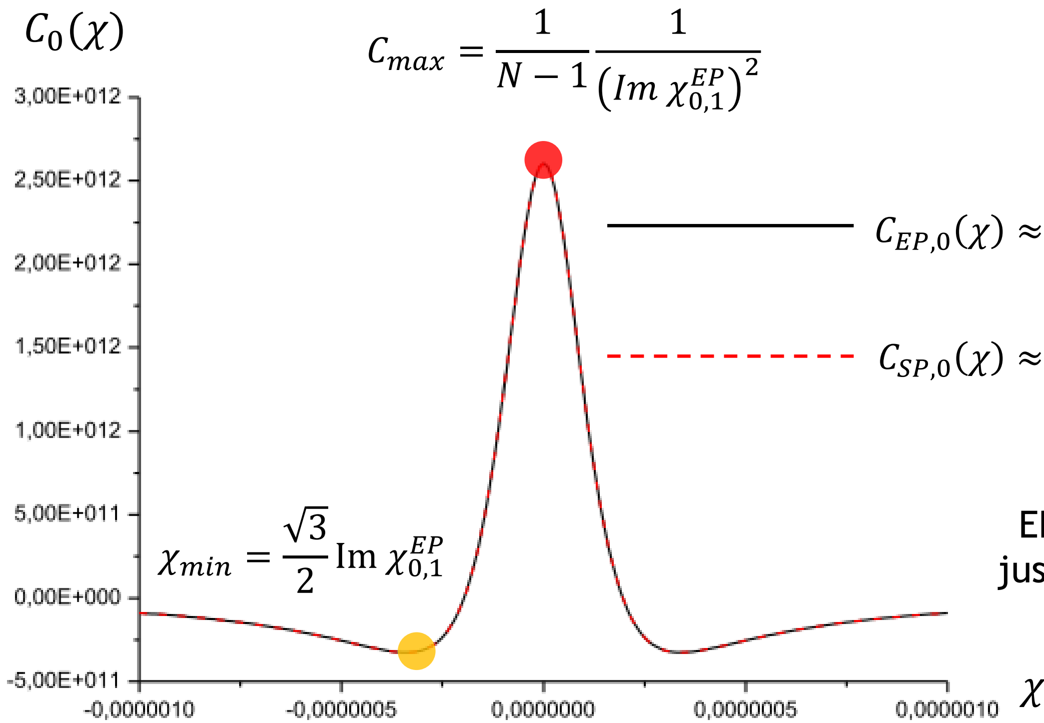
$$\rho^{EP} \propto (\mathrm{Im}\,\lambda)^{\alpha}$$

$\alpha = 0$ - 1st order
$0 < \alpha < 1$ - 2nd order

P. Cejnar, S. Heinze, J. Dobeš, *Phys. Rev. C* **71**, 011304(R) (2005)
P. Cejnar, S. Heinze, M. Macek, *Phys. Rev. Lett.* **99**, 100601 (2007)

Specific heat analogy

$$C_{max} = \frac{1}{N-1}\frac{1}{\left(Im\ \chi_{0,1}^{EP}\right)^2}$$



$$\chi_{min} = \frac{\sqrt{3}}{2}\text{Im}\ \chi_{0,1}^{EP}$$

$$C_{EP,0}(\chi) \approx \frac{1}{N-1}\frac{d^2}{d\chi^2}\ln\sqrt{\left(\chi - \text{Re}\ \chi_{0,1}^{EP}\right)^2 + \left(\text{Im}\ \chi_{0,1}^{EP}\right)^2}$$

$$C_{SP,0}(\chi) \approx \frac{1}{N-1}\frac{d^2}{d\chi^2}\ln|E_1(\chi) - E_0(\chi)|$$

EPs are well separated, therefore there is just one EP connected with the ground state

⬇

We can neglect the influence of other EPs

Latent heat analogy

- approximated by $Q \approx \chi_{min} * C_{max} = \frac{\sqrt{3}}{2(N-1)}\frac{1}{\mathbf{Im}\ \boldsymbol{\chi}_{\mathbf{0,1}}^{\boldsymbol{EP}}} \sim \frac{\boldsymbol{e}^{\boldsymbol{N}}}{N} \rightarrow \infty$