Excited-state quantum phase transitions studied from a non-Hermitian perspective

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Lipkin-Meshkov-Glick (LMG) model

N sites with spin $1/2$

$$\hat{H}(\alpha) \;=\; \alpha\left(\frac{N}{2}+\hat{S}_z\right) \;-\; \frac{4(1-\alpha)}{N}\,\hat{S}_x^2 \quad ; \qquad (1)$$

here

$$\alpha \in [0,1] \;\;,\;\; \hat{S} = \sum_{j=1}^{N} \hat{s}_j$$

limit of $N \to +\infty$ ... quantum phase transitions (QPT)

ground state QPT at $\alpha = 0.8$

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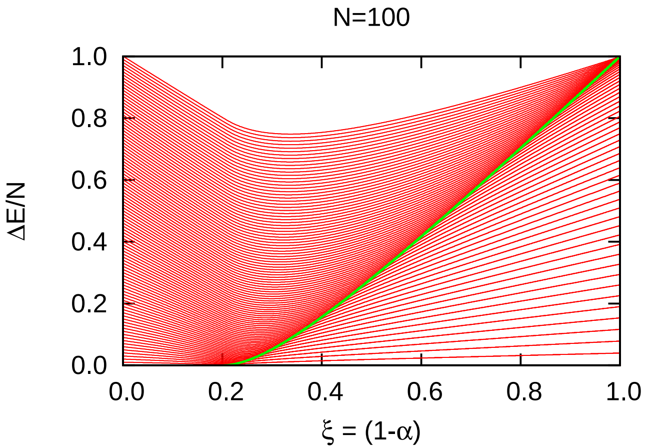
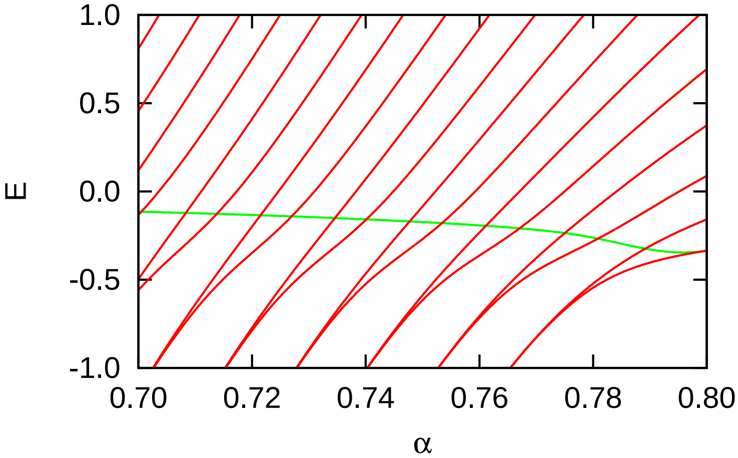
excited state quantum phase transitions (ESQPT)



Santos, Távora, Pérez-Bernal, Phys. Rev. A, **94**, *012113 (2016)* excited state quantum phase transitions (ESQPT)

N=100



Looking at the problem from a nonhermitian perspective:

$$\hat{H}(\alpha) \;=\; \alpha\left(\frac{N}{2}+\hat{S}_z\right) \;-\; \frac{4(1-\alpha)}{N}\,\hat{S}_x^2 \qquad ; \qquad\qquad (2)$$

$\alpha \in \mathbb{C}$

the Hamiltonian becomes non-hermitian

$$\left(\hat{H}(\alpha)\right)^{\dagger} \;=\; \hat{H}(\alpha^*) \neq \hat{H}(\alpha)$$

its eigenvalues, $E_n(\alpha)$, become complex valued

How does the dependence of $E_n(\alpha)$ look for $\alpha \in \mathbb{C}$?
Any surprises or insights related to QPT and ESQPT ?

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Exceptional points (EPs) of nonhermitian Hamiltonians:

there might be some special values of $\alpha_{\mathrm{EP}}^{(\nu)} \in \mathbb{C}$
such that $\hat{H}(\alpha_{\mathrm{EP}}^{(\nu)})$ is non-diagonalizable

Recall

$$\hat{H}(\alpha) \;=\; \alpha\left(\frac{N}{2}+\hat{S}_z\right) \;-\; \frac{4(1-\alpha)}{N}\,\hat{S}_x^2 \quad ; \qquad (3)$$

this can be viewed as an $(N+1)$-by-$(N+1)$ matrix.

generic choice of $\alpha \in \mathbb{C}$ ...
... $(N+1)$ linearly independent eigenvectors of $\hat{H}(\alpha)$

specific choice of $\alpha_{\mathrm{EP}}^{(\nu)} \in \mathbb{C}$ ...
... $\hat{H}(\alpha_{\mathrm{EP}}^{(\nu)})$ has only N linearly independent eigenvectors!

N. Moiseyev, Non-Hermitian Quantum Mechanics, Cambridge (2011), Chapter 9

Exceptional points (EPs) of nonhermitian Hamiltonians:

generic choice of $\alpha \in \mathbb{C}$ ...

... $(N+1)$ linearly independent eigenvectors of $\hat{H}(\alpha)$

$$\hat{H}(\alpha)\,\vec{c}_n \;=\; E_n\,\vec{c}_n \quad , \quad n \in \{1, 2, \cdots, N+1\} \tag{4}$$

$$\vec{c}_n \cdot \vec{c}_{n'} \;=\; \delta_{nn'} \quad , \quad \sum_{n=1}^{N+1} \vec{c}_n\,\vec{c}_n^{\,T} \;=\; \hat{1} \tag{5}$$

Exceptional points (EPs) of nonhermitian Hamiltonians:

specific choice of $\alpha_{\mathrm{EP}}^{(\nu)} \in \mathbb{C} \ldots$

$\ldots \hat{H}(\alpha_{\mathrm{EP}}^{(\nu)})$ has only N linearly independent eigenvectors!

$$\hat{H}(\alpha_{\mathrm{EP}}^{(\nu)})\,\vec{c}_n \;=\; E_n\,\vec{c}_n \quad , \quad n \in \{1, 2, \cdots, (N-1), EP\} \tag{6}$$

$$\vec{c}_n \cdot \vec{c}_{n'} \;=\; \delta_{nn'} \quad , \quad [\,1 \leq n, n' \leq (N-1)\,] \tag{7}$$

$$\vec{c}_n \cdot \vec{c}_{EP} \;=\; 0 \quad , \quad [\,1 \leq n \leq (N-1)\,] \tag{8}$$

$$\vec{c}_{EP} \cdot \vec{c}_{EP} \;=\; 0 \tag{9}$$

self-orthogonality!

N. Moiseyev, Non-Hermitian Quantum Mechanics, Cambridge (2011), Chapter 9

Exceptional points (EPs) of nonhermitian Hamiltonians:

consider $E_n, \vec{c}_n$ as functions of $\alpha \in \mathbb{C}$

an EP is formed when $E_n(\alpha)$ coalesces with $E_{n'}(\alpha)$
and simultaneously $\vec{c}_n(\alpha)$ coalesces with $\vec{c}_{n'}(\alpha)$

this may happen only at some specific discrete values of $\alpha = \alpha_{\rm EP}^{(\nu)}$
$[\,\nu \equiv (nn')\,]$

An EP $\alpha = \alpha_{\rm EP}^{(\nu)}$ represents a branch point of $E_n(\alpha)$.
Thus the complex surface $E_n(\alpha)$ is topologically nontrivial.

N. Moiseyev, Non-Hermitian Quantum Mechanics, Cambridge (2011), Chapter 9

How about the LMG model ?

How does the dependence of $E_n(\alpha)$ look for $\alpha \in \mathbb{C}$?

Are there any EPs for the LMG model ?

Any surprises or insights related to QPT and ESQPT ?

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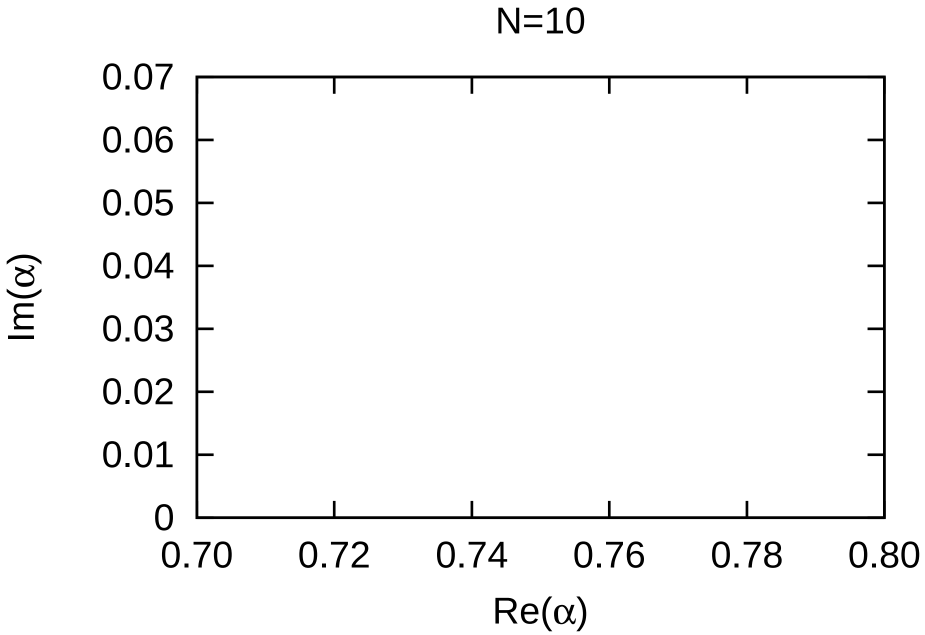
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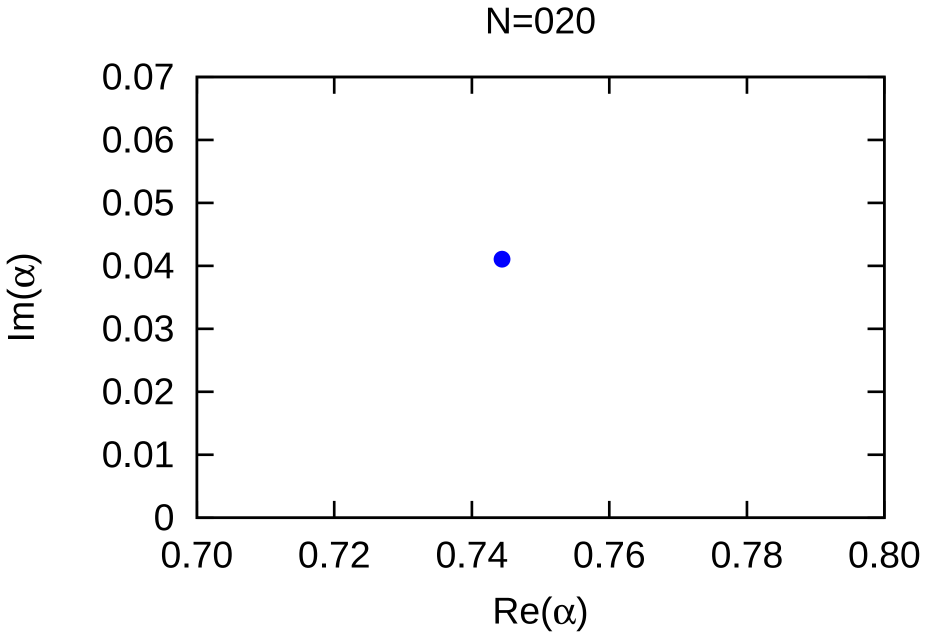
Im(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=020
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0.06
0.05
0.04
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0.02
0.01
0
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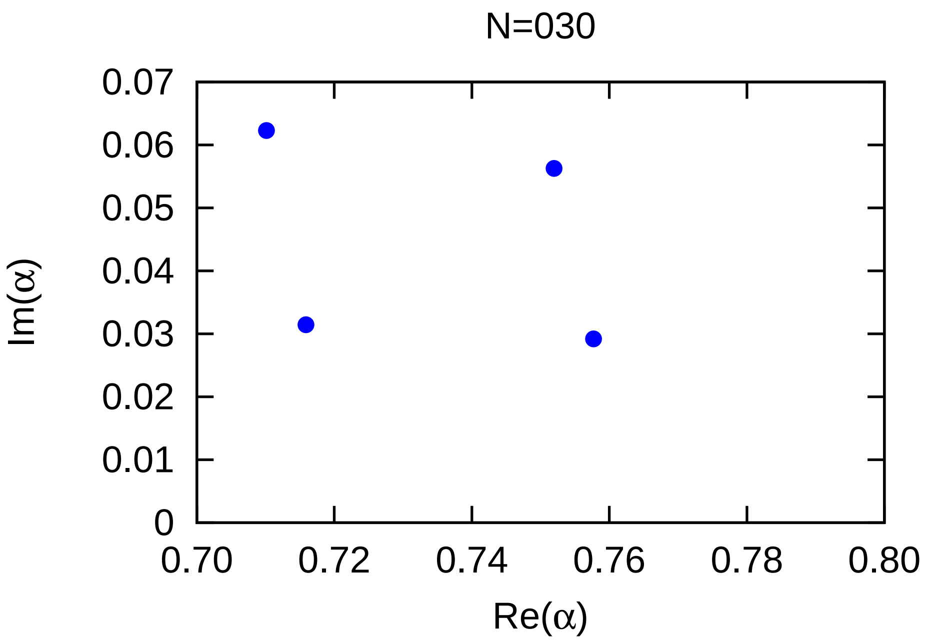
0.72
0.74
0.76
0.78
0.80
Re(α)
Im(α)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0.06
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0.04
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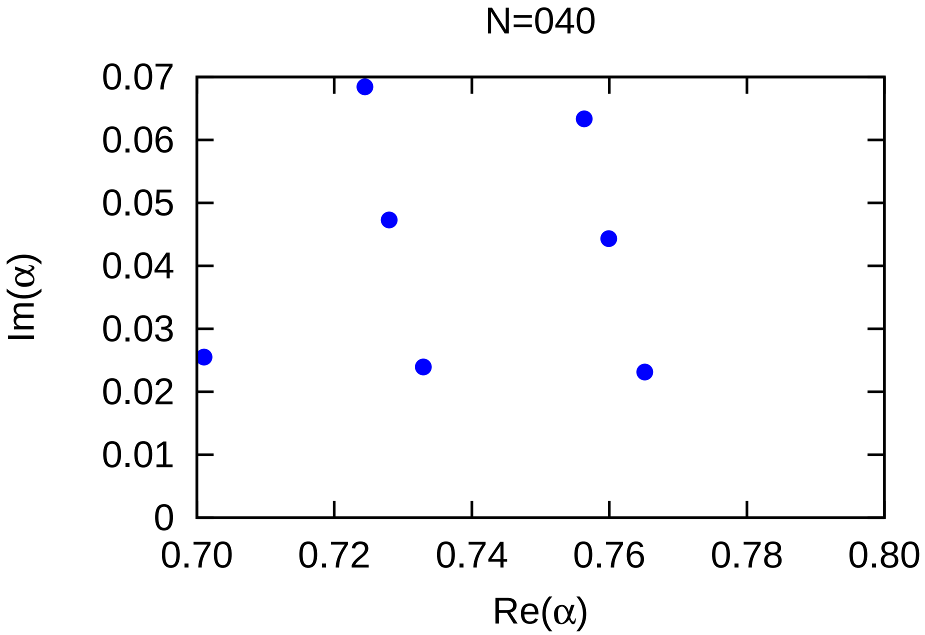
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Im(α)
Re(α)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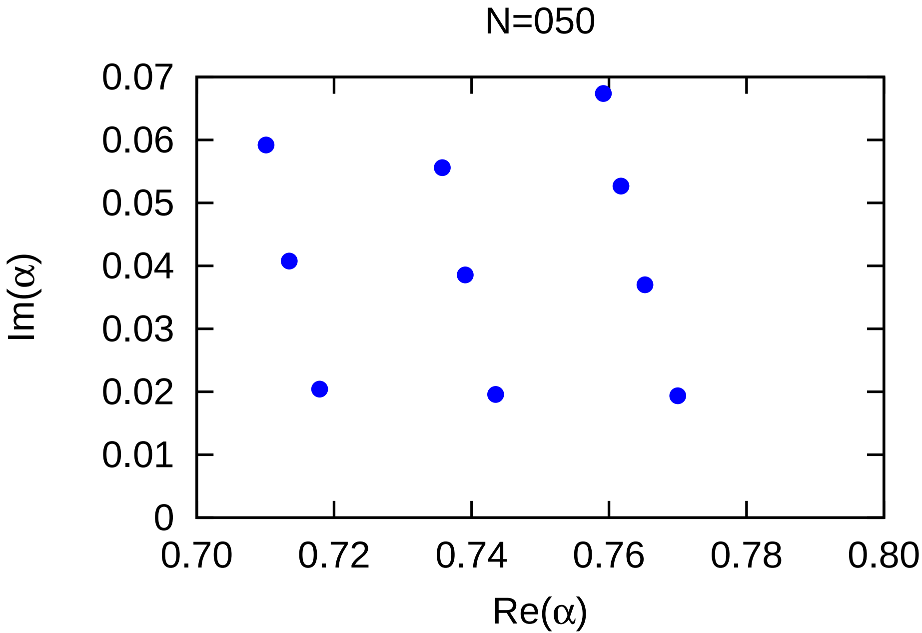
Im(α)
Re(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80


N=050
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0.06
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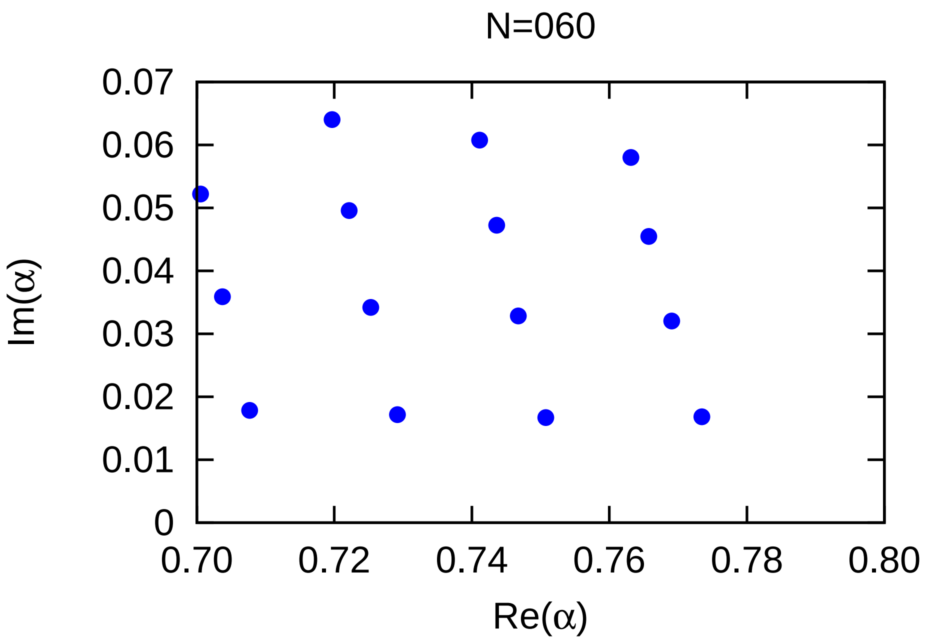
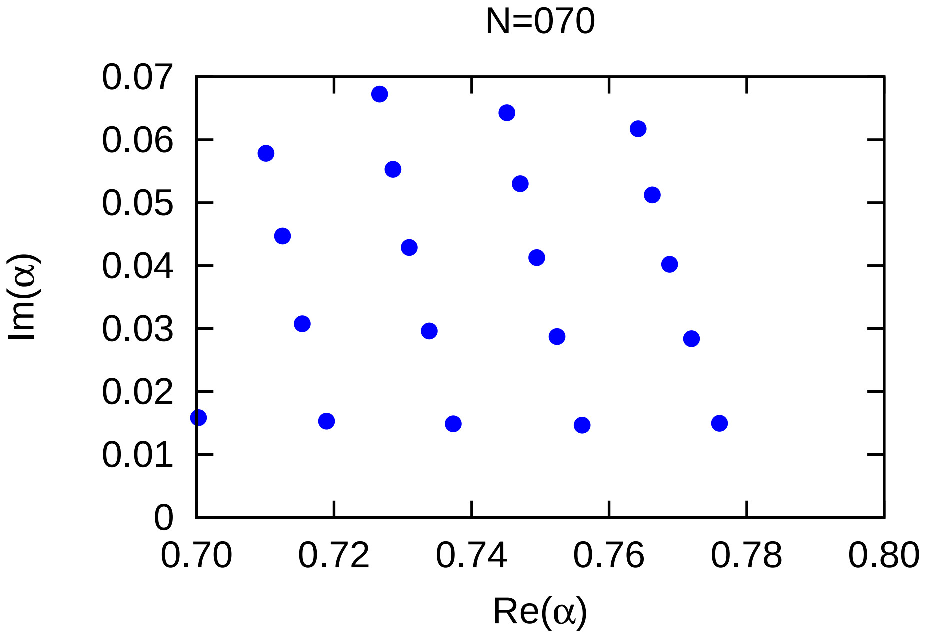
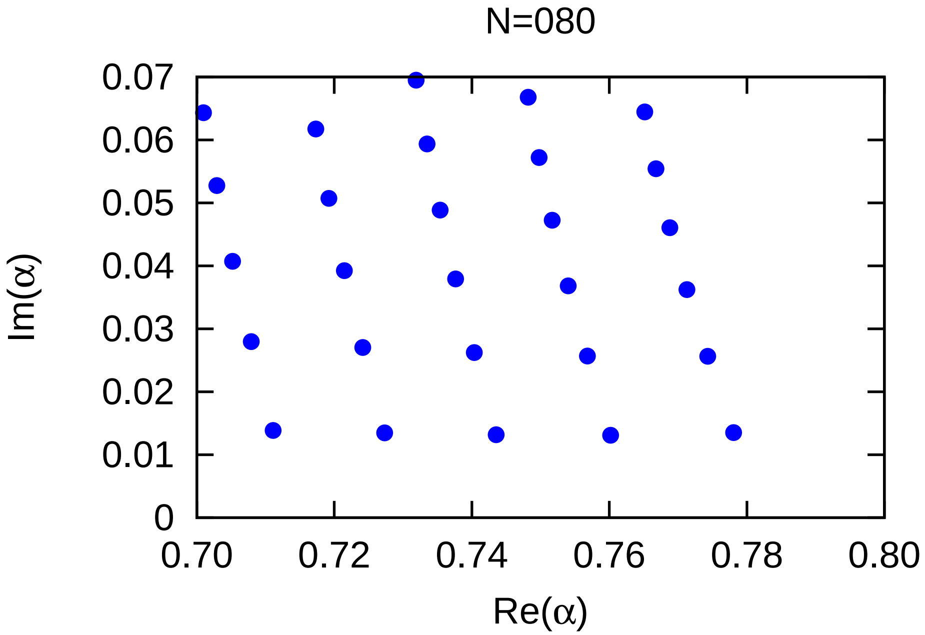
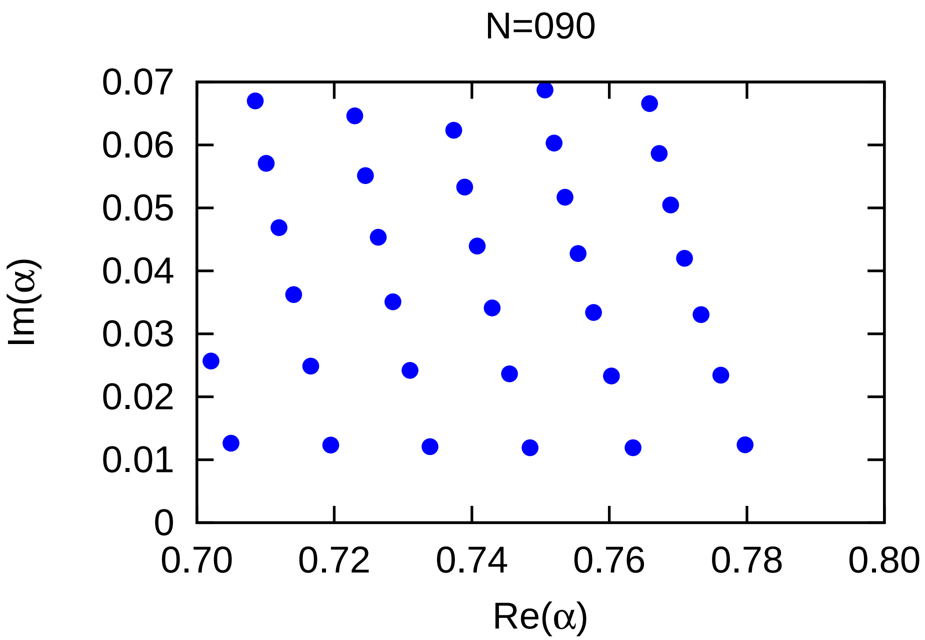
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Im(α)
Re(α)

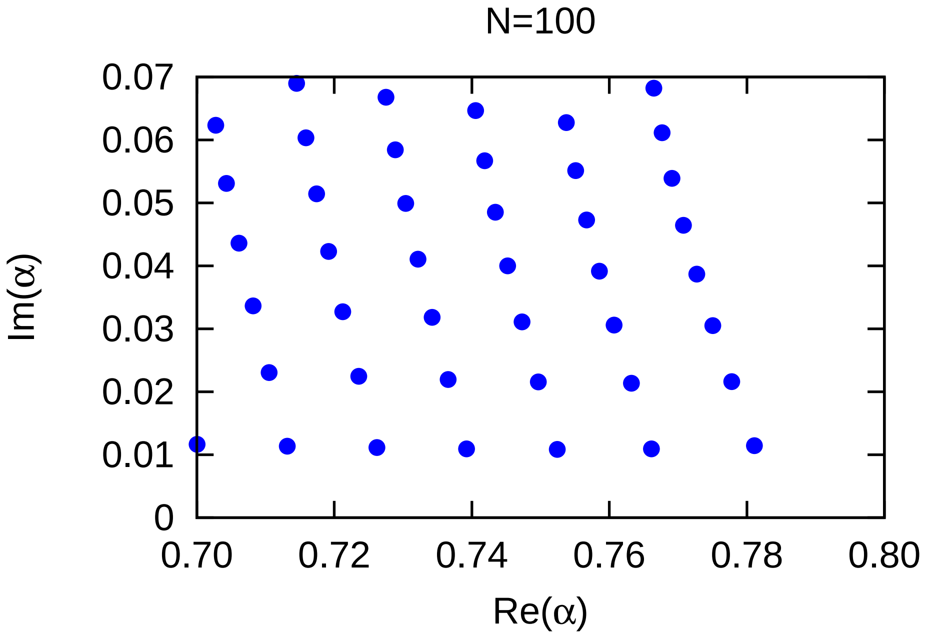


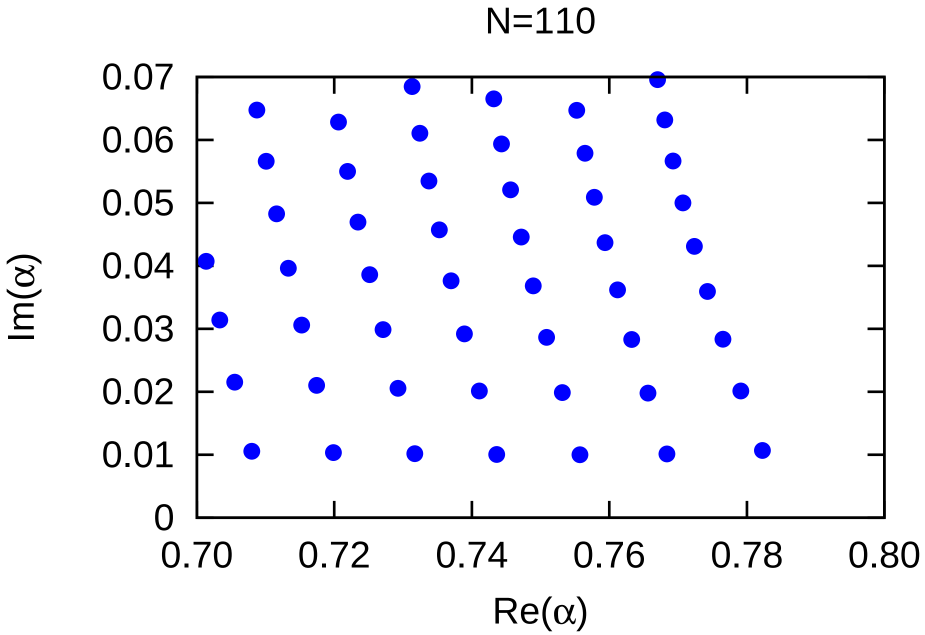
N=060
Im(α)
Re(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80

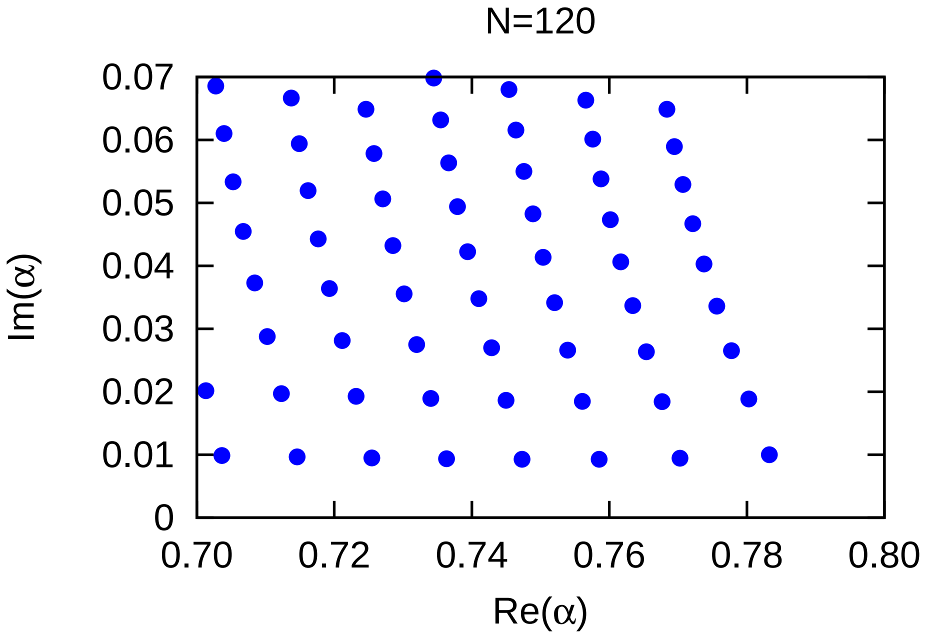

N=070
Im(α)
Re(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80


N=080
Im(α)
Re(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80


N=090
Im(α)
Re(α)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80


N=100
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Im(α)
Re(α)


N=110
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Im(α)
Re(α)


N=120
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.70
0.72
0.74
0.76
0.78
0.80
Im(α)
Re(α)

Are there any EPs for the LMG model ? . . . yes!

But only for $\mathrm{Re}\,\alpha < \alpha_{\mathrm{QPT}} = 0.8$.

As N increases,
the density of EPs in the complex α-plane increases.
The EPs form a regular pattern in the α-plane.

With an increasing N,
all the EPs approach the physical axis of real α,
and move towards $\alpha_{\mathrm{QPT}} = 0.8$.
Confirmed numerically using the Padé extrapolation technique:

$$\lim_{(1/N)\to 0} \alpha_{\mathrm{EP}}^{(\nu)}(1/N) \;=\; \alpha_{\mathrm{QPT}} \;=\; 0.8 \quad . \tag{10}$$

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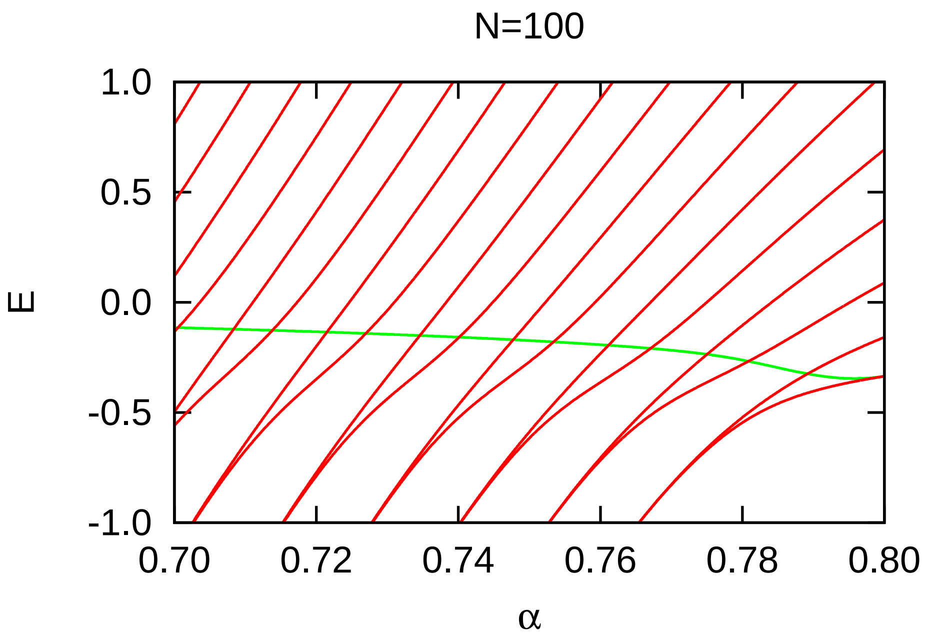
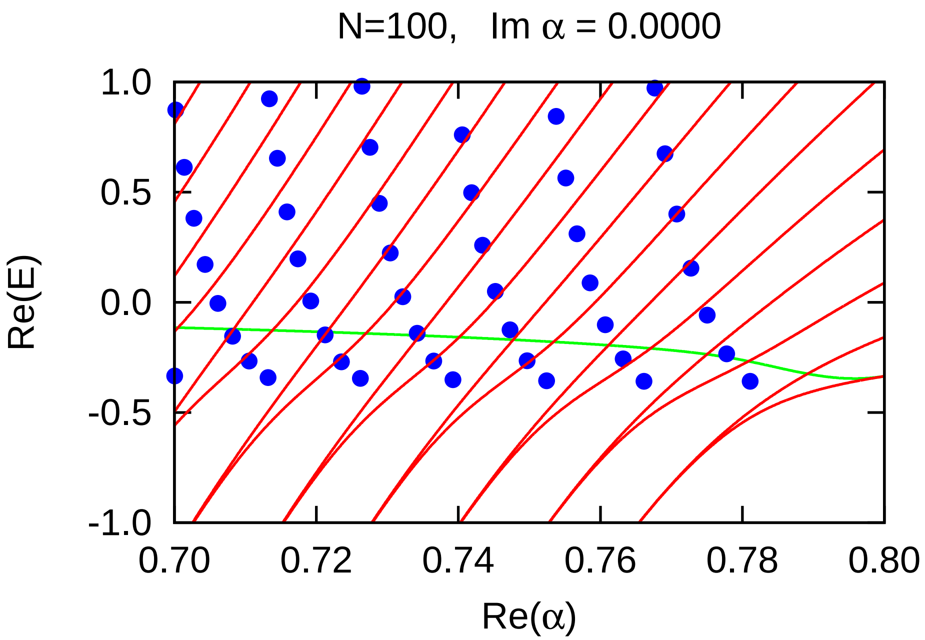
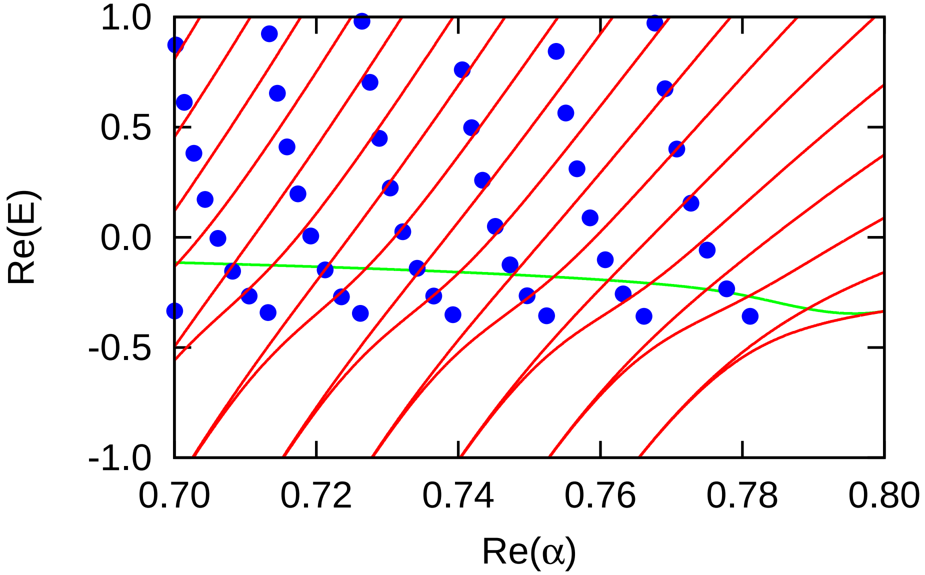
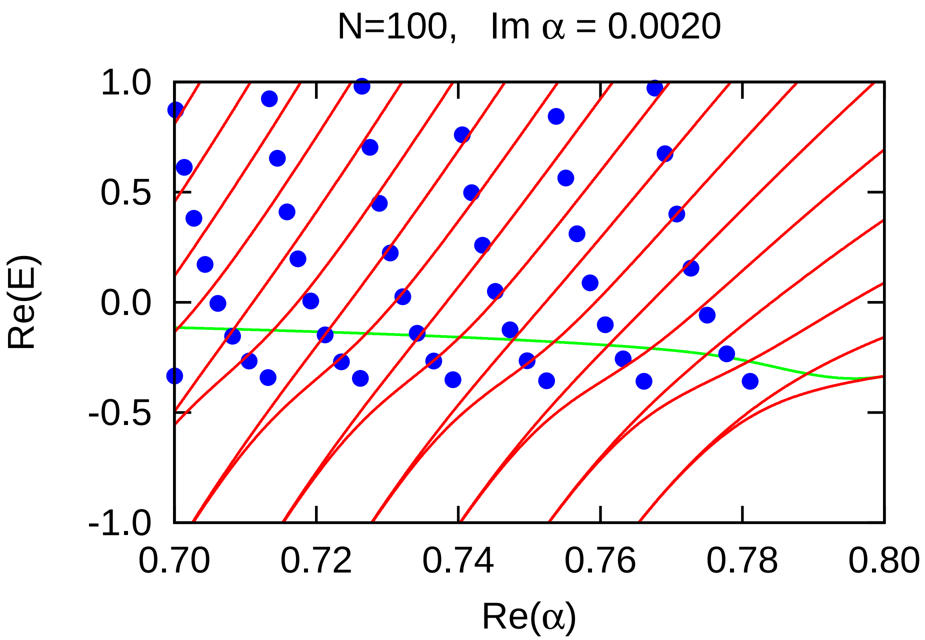
How about the ESQPT ?

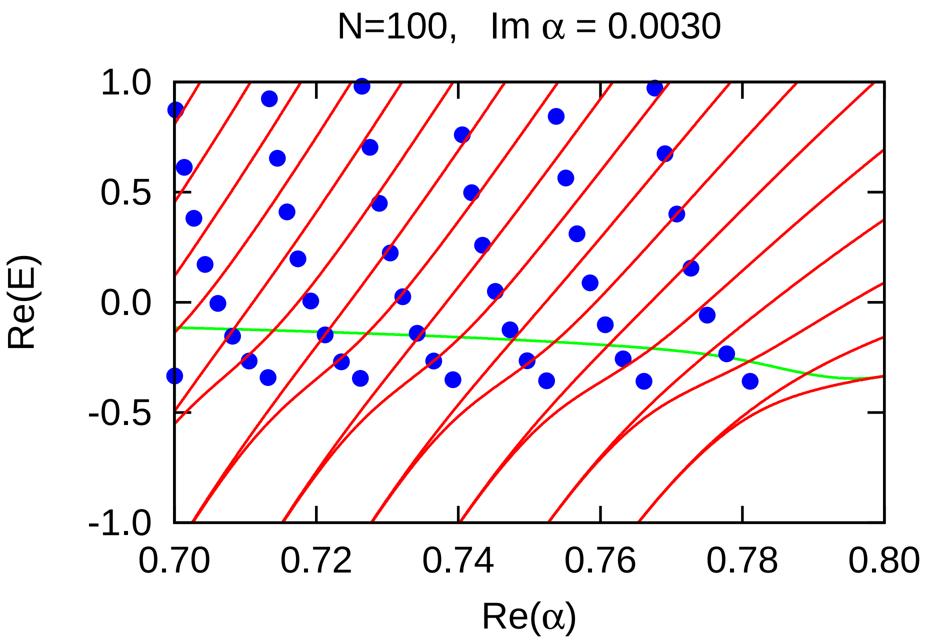

N=100
E
α
1.0
0.5
0.0
-0.5
-1.0
0.70
0.72
0.74
0.76
0.78
0.80


N=100, Im α = 0.0000
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

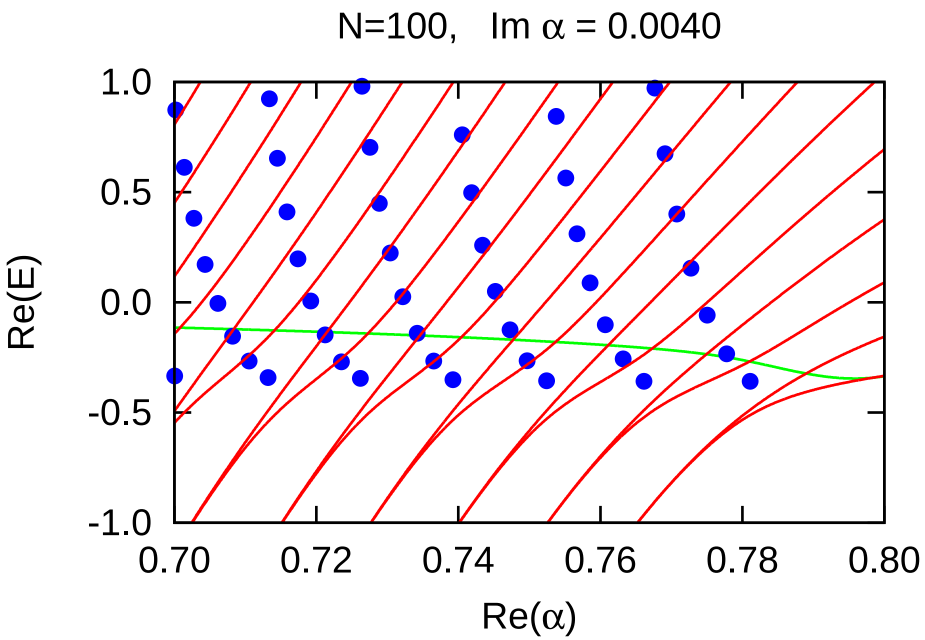
N=100,   Im α = 0.0010




N=100,   Im α = 0.0020
Re(E)
1.0
0.5
0.0
-0.5
-1.0
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)



N=100,   Im α = 0.0030
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)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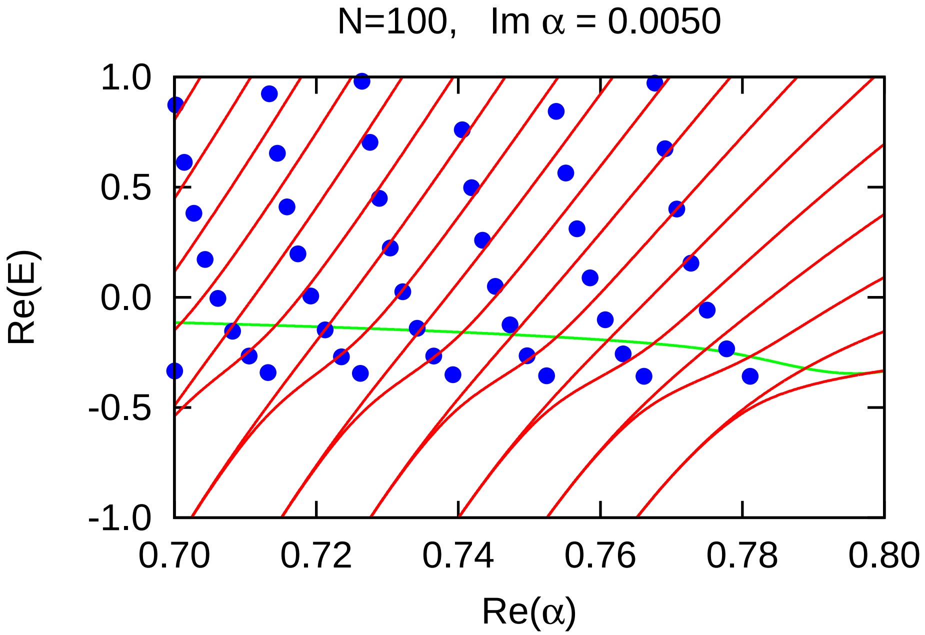
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)



N=100,   Im α = 0.0050
Re(E)
Re(α)
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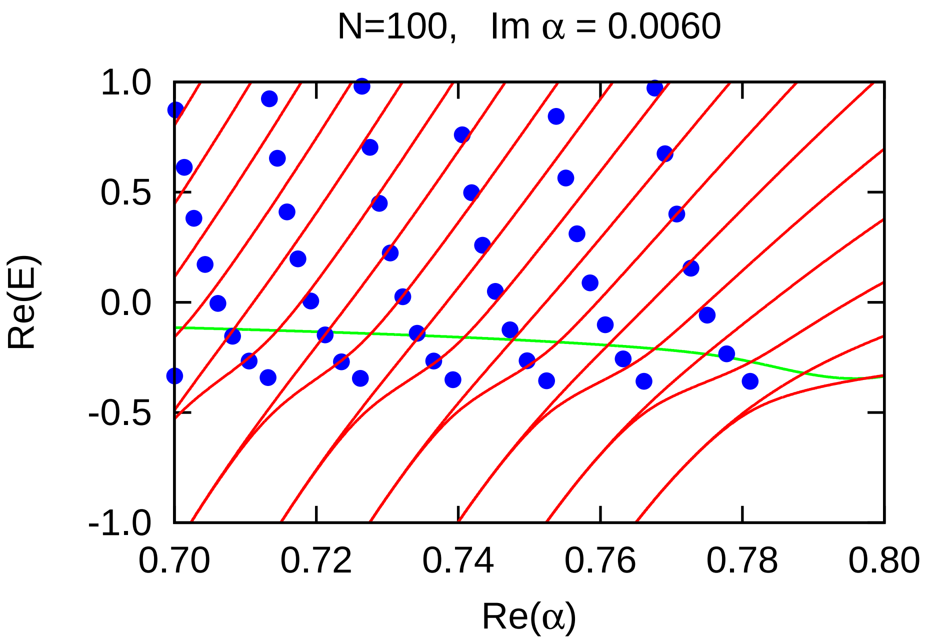
0.5
0.0
-0.5
-1.0
0.70
0.72
0.74
0.76
0.78
0.80



N=100, Im α = 0.0060
Re(E)
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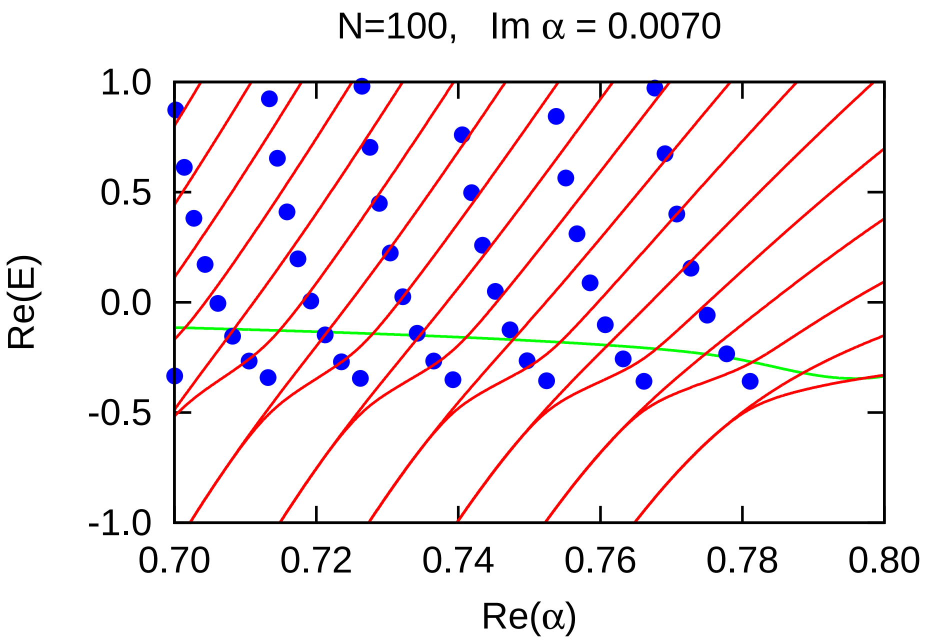
0.5
0.0
-0.5
-1.0
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)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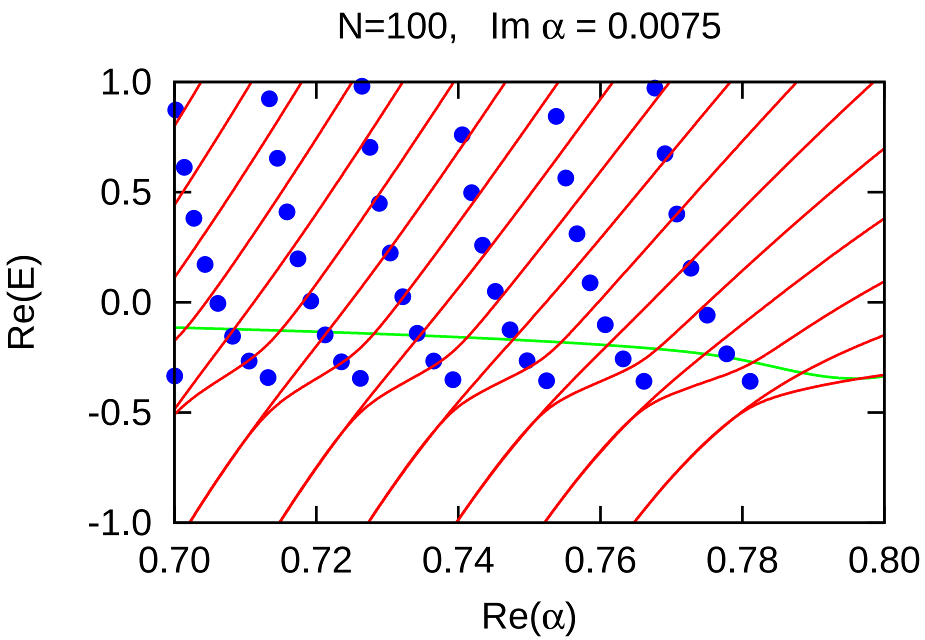
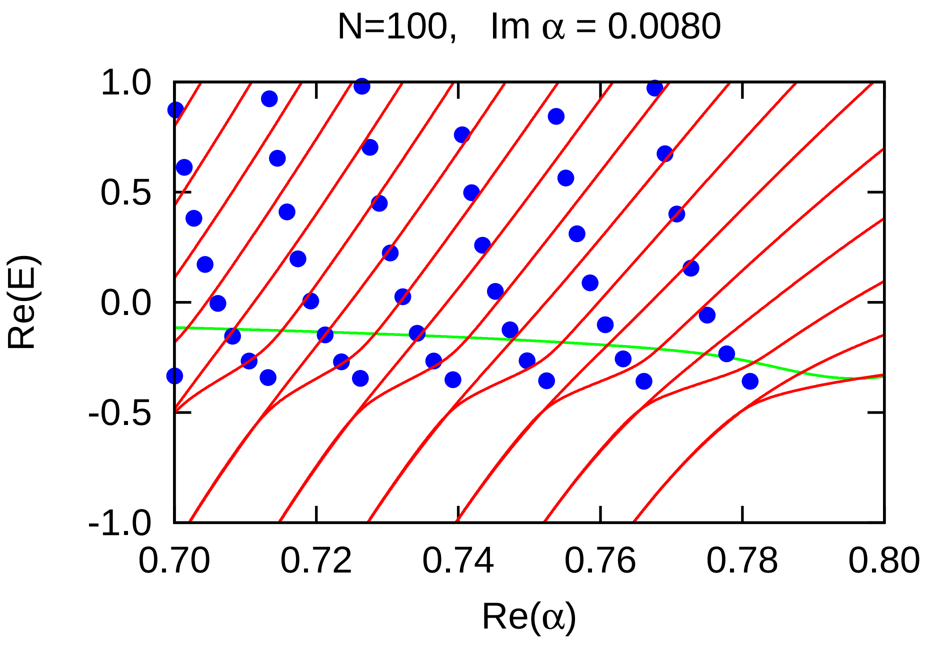
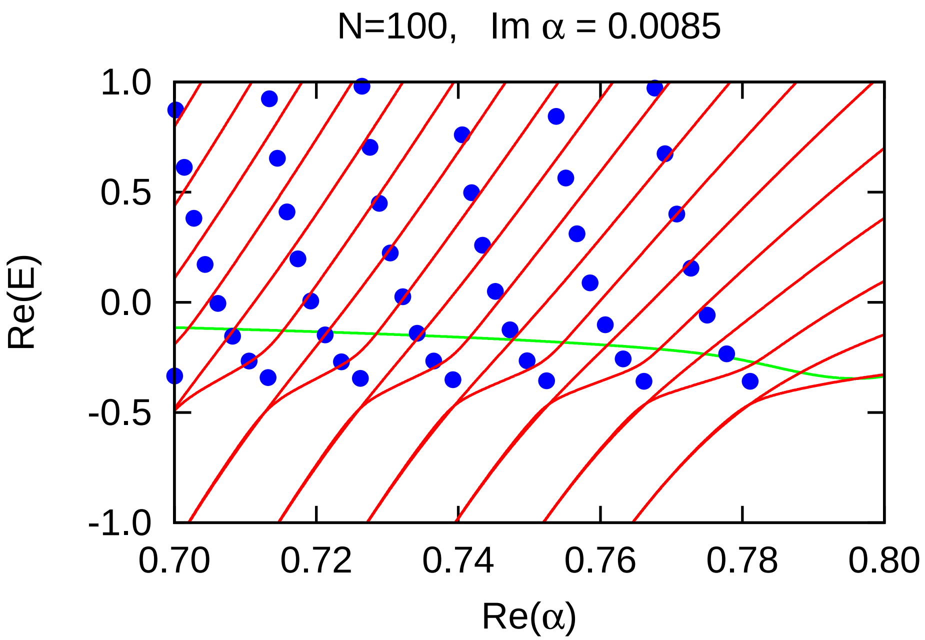
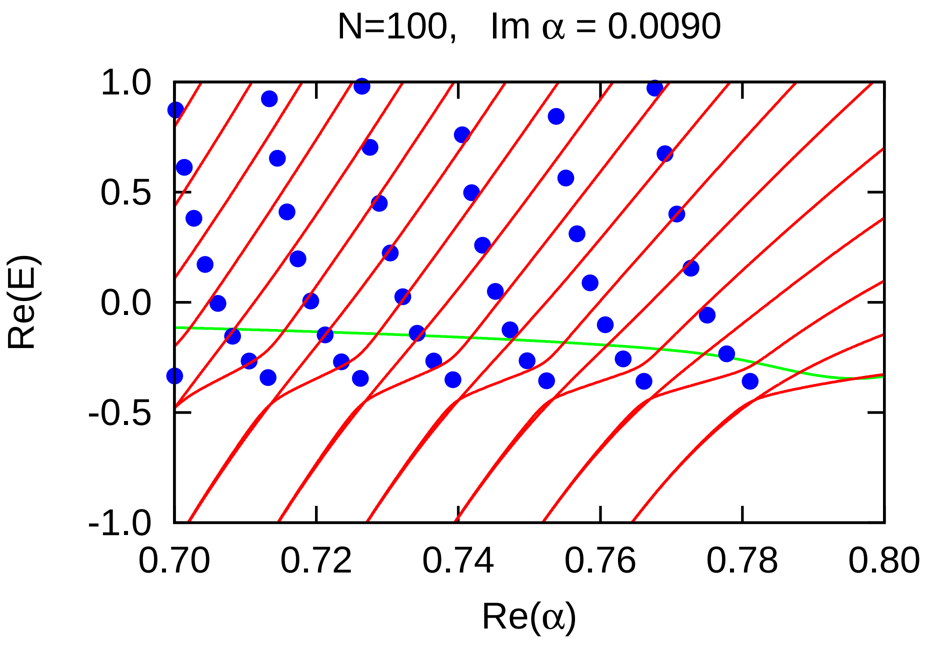
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0075
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0080
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0085
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0090
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0.5
0.0
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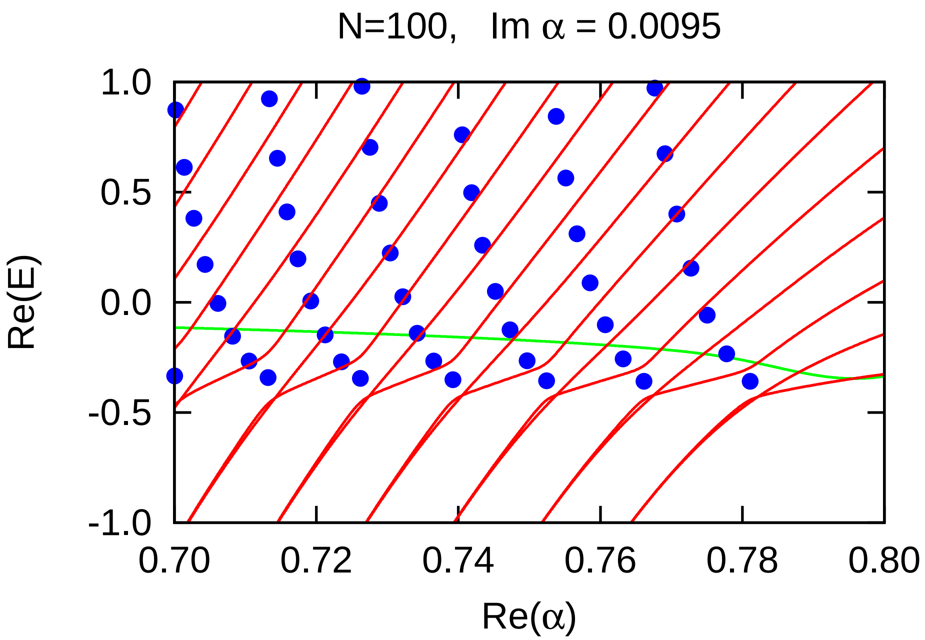
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)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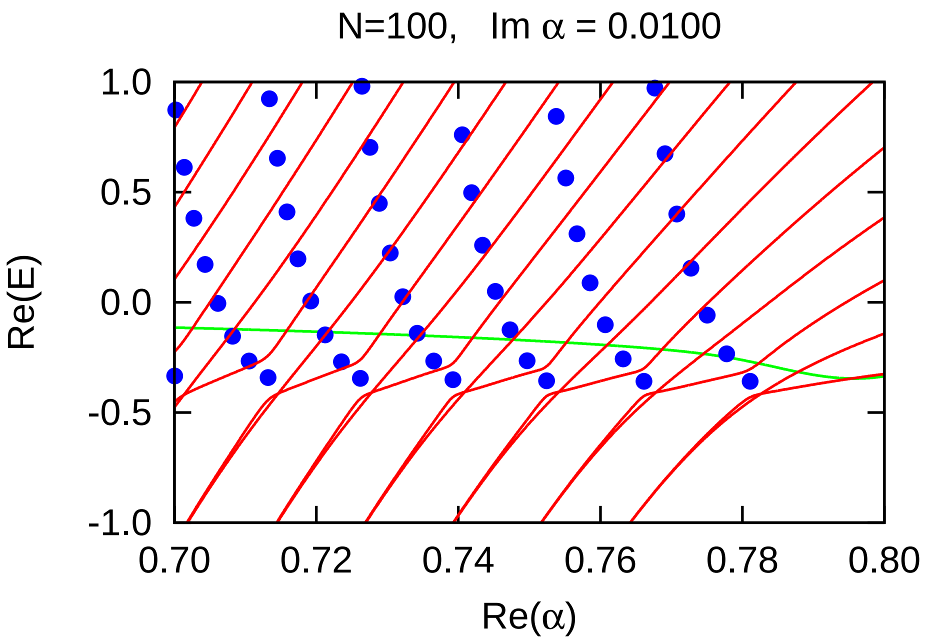
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)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1.0
0.5
0.0
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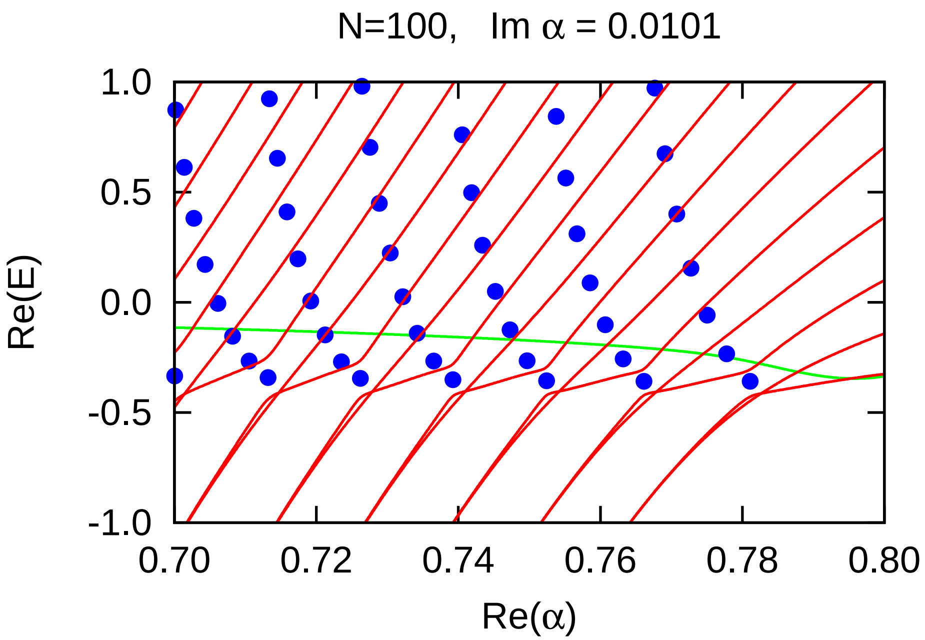
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)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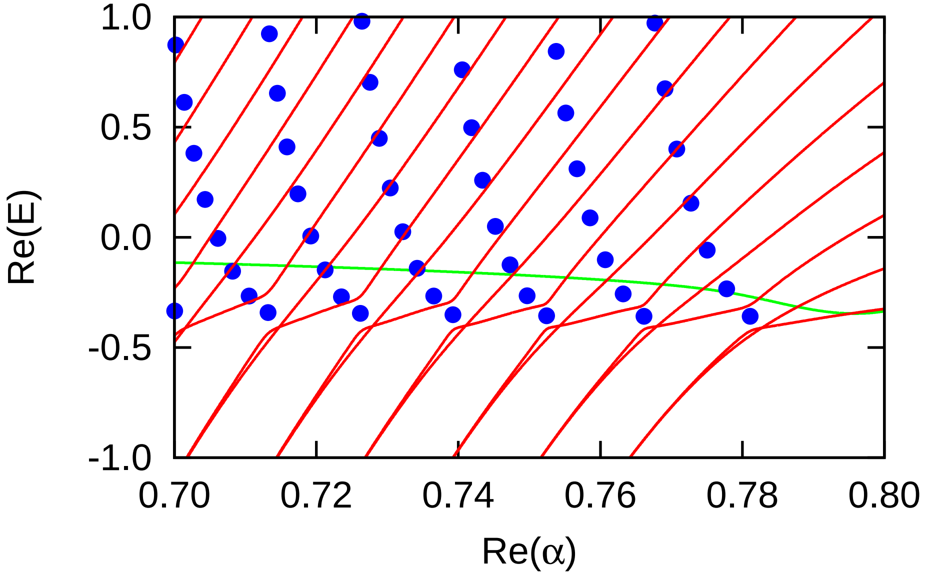
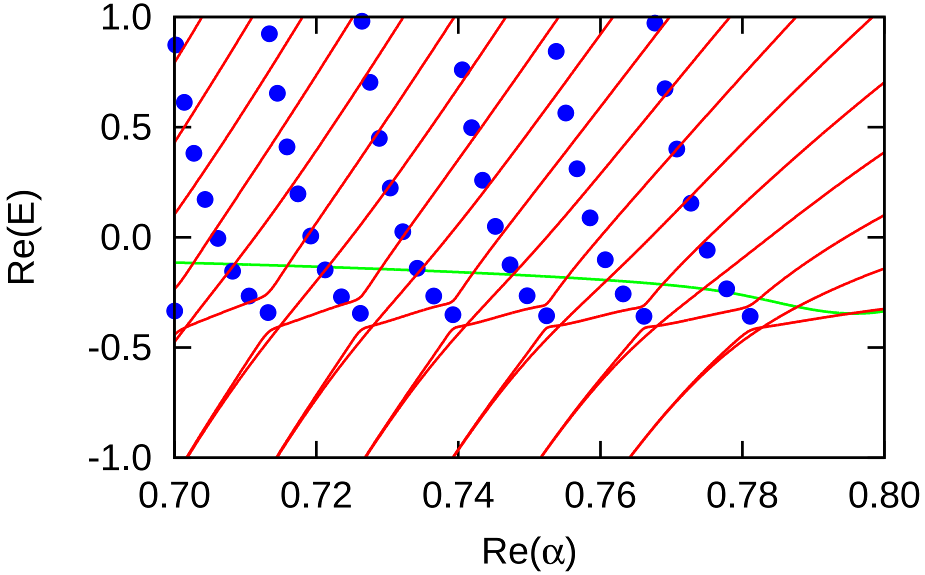
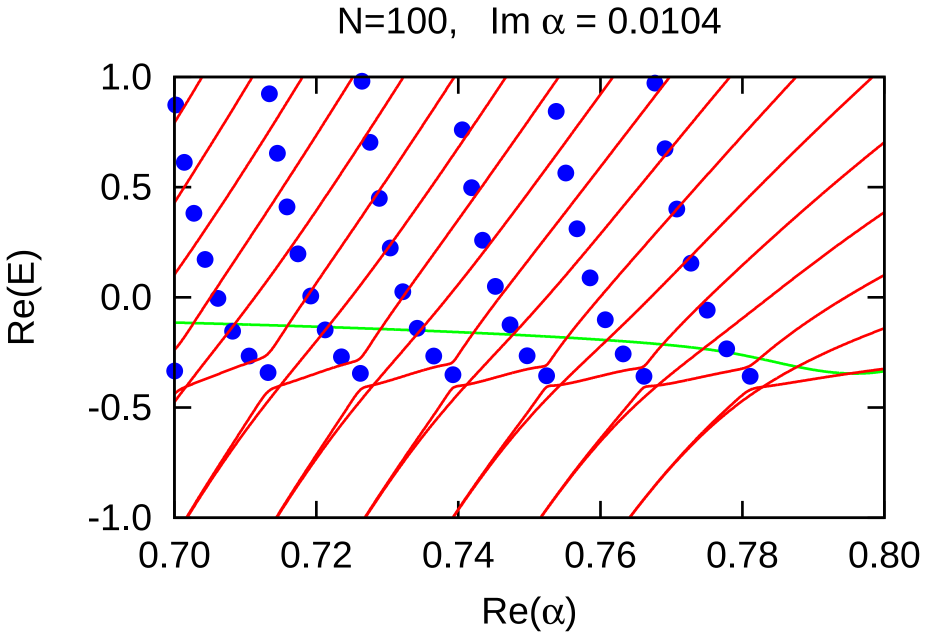
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

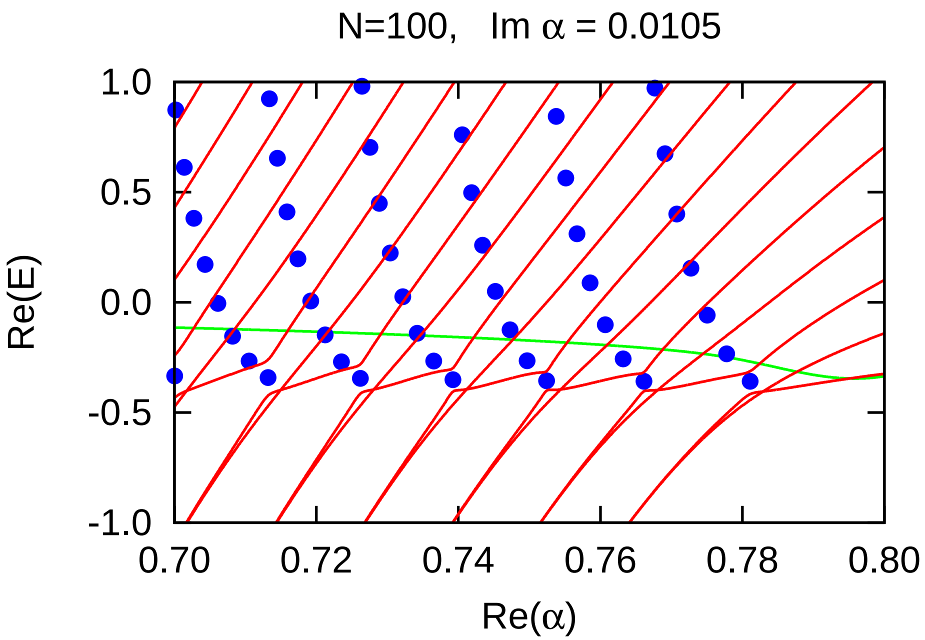
N=100, Im α = 0.0102

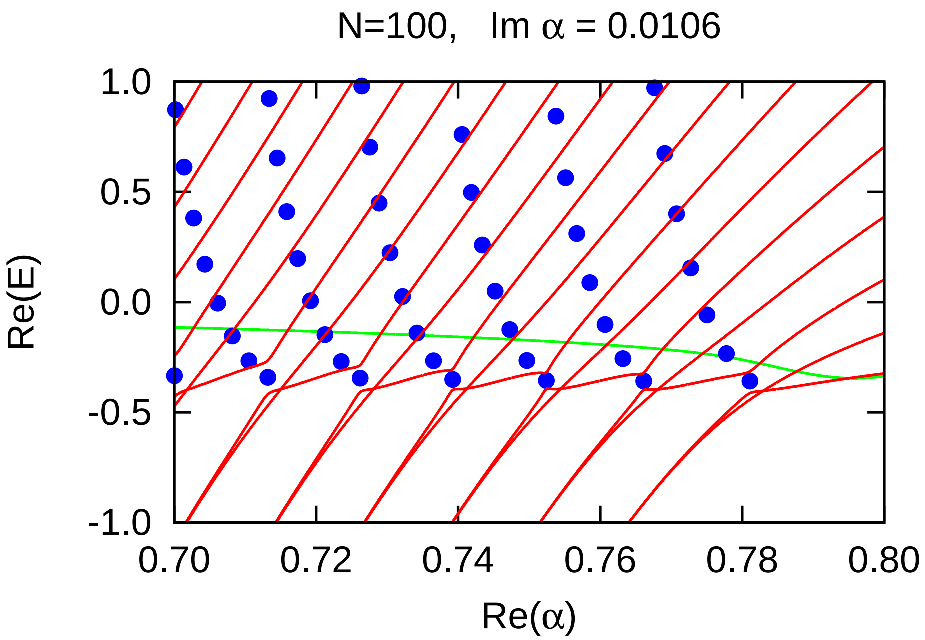


N=100, Im α = 0.0103

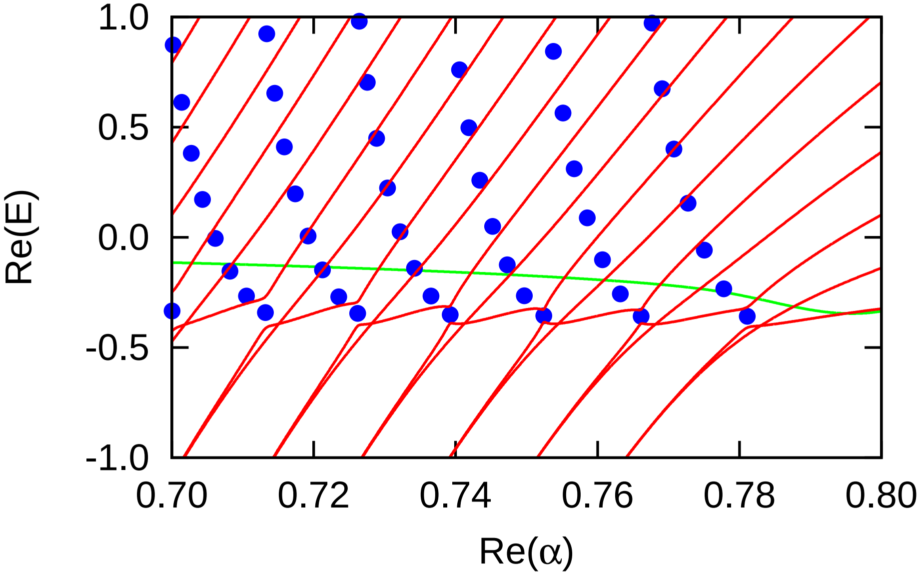


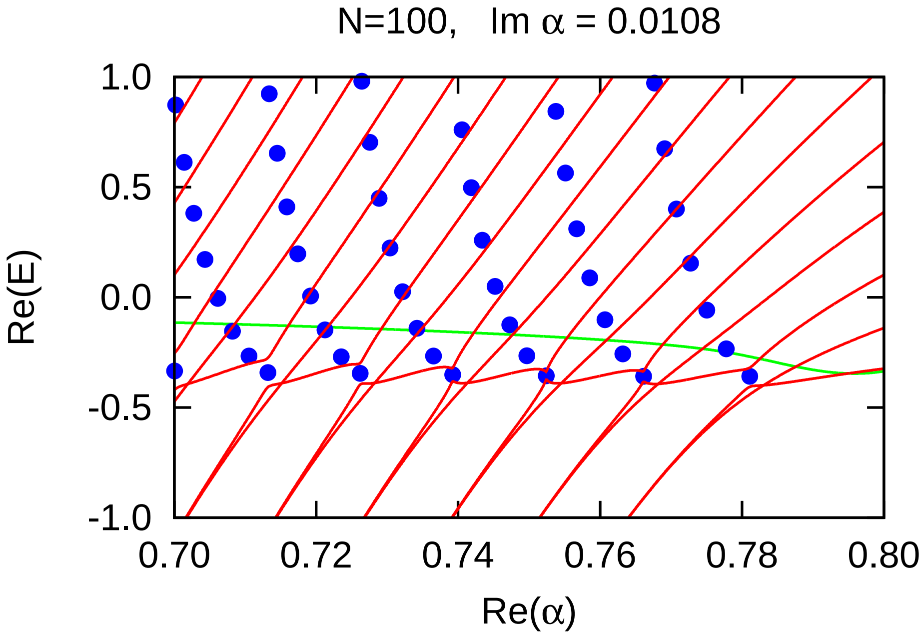

N=100, Im α = 0.0104
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0105
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

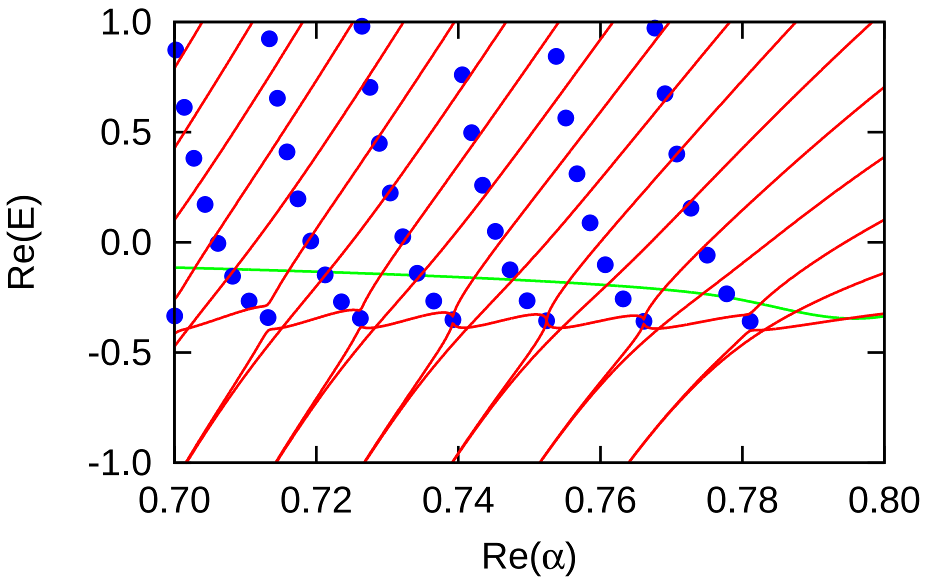

N=100, Im α = 0.0106
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

N=100, Im α = 0.0107

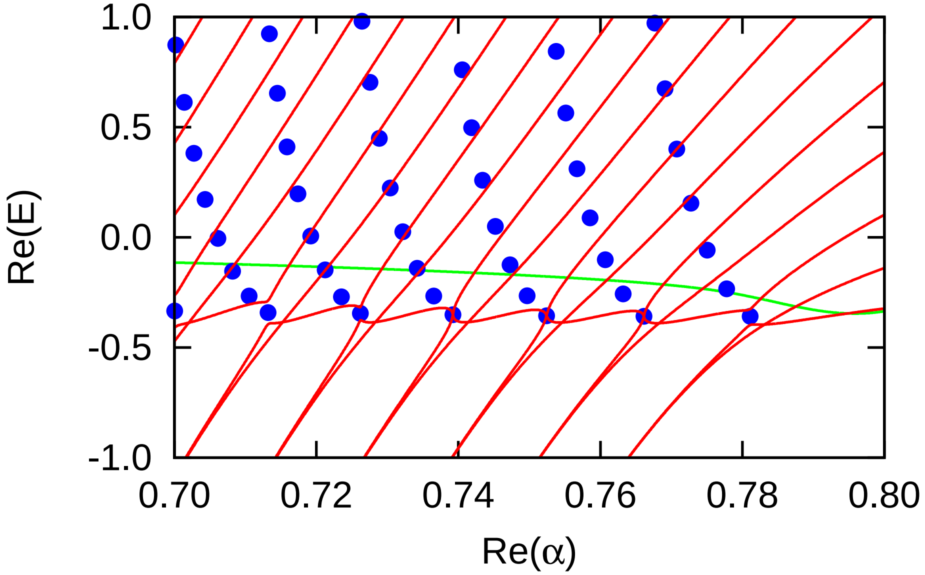


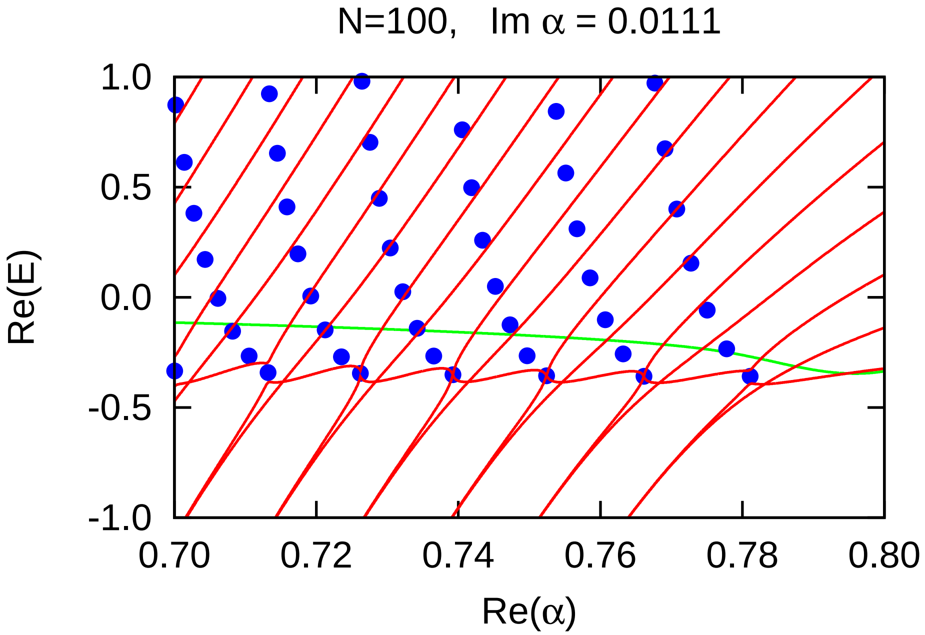

N=100, Im α = 0.0108
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

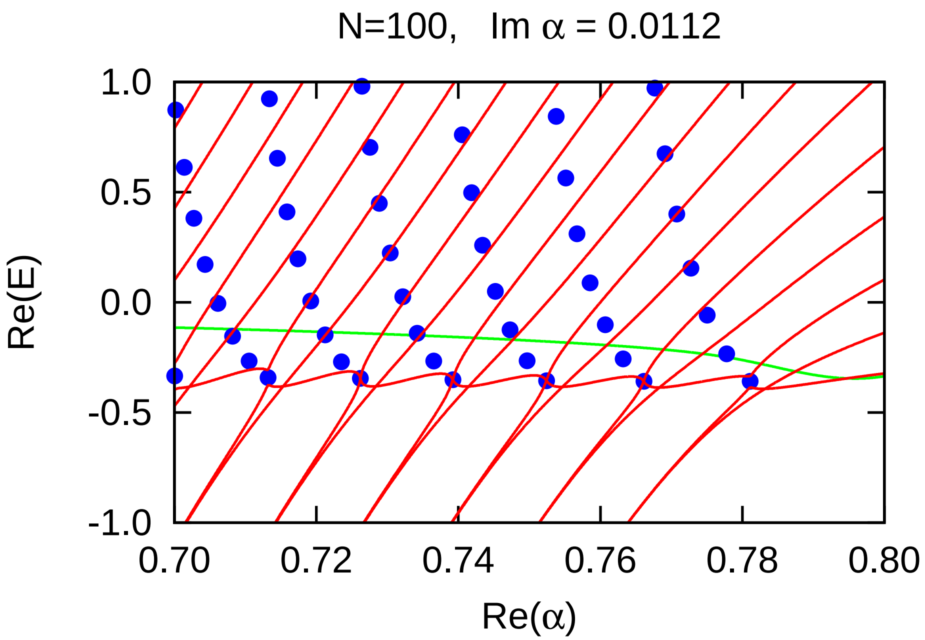
N=100, Im α = 0.0109

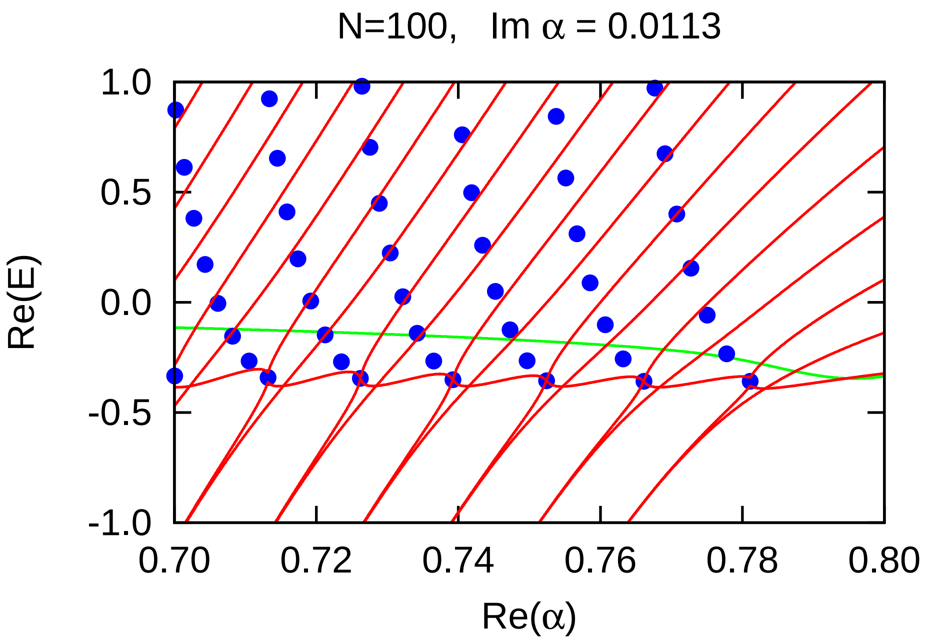


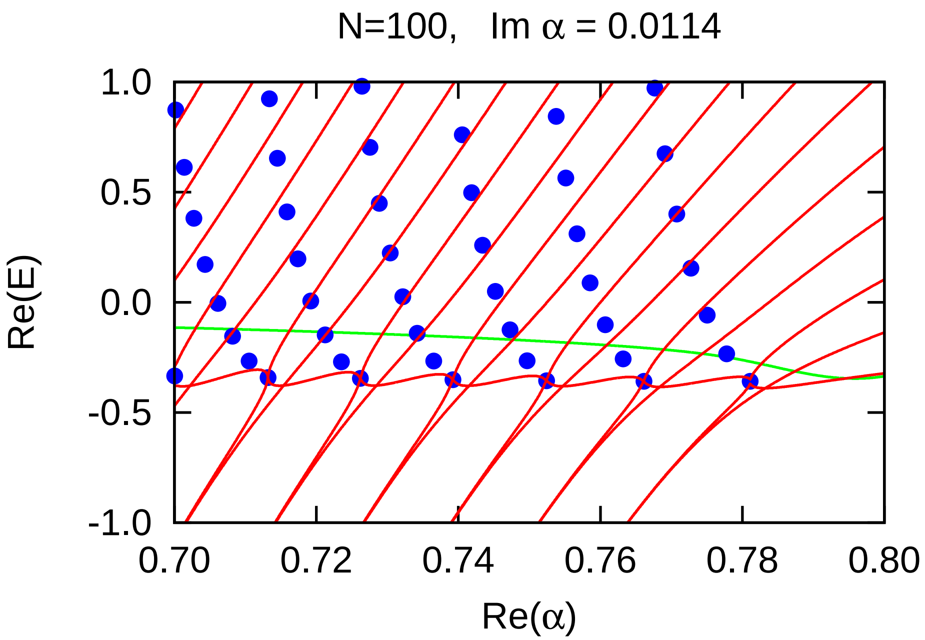
N=100, Im α = 0.0110

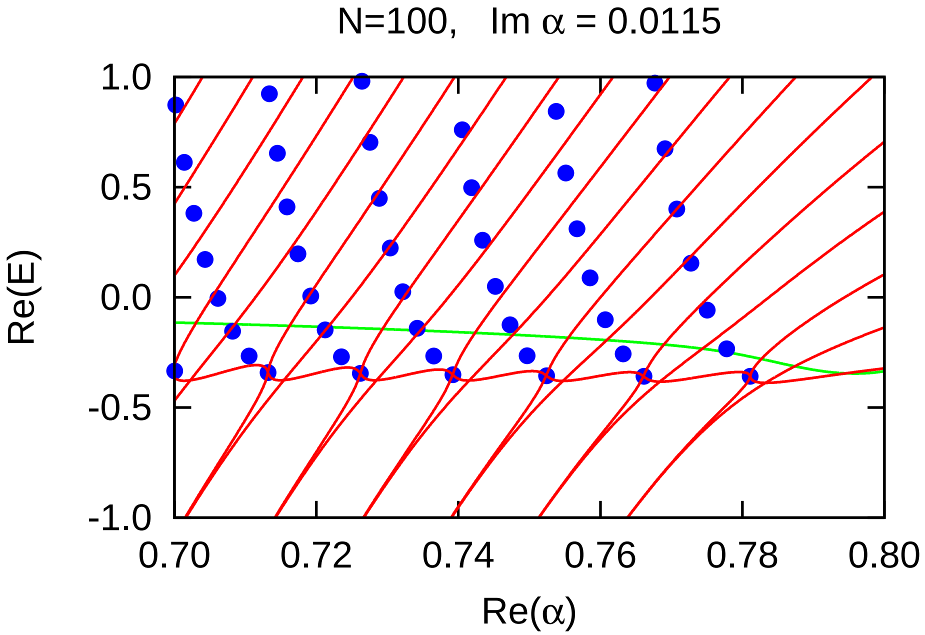


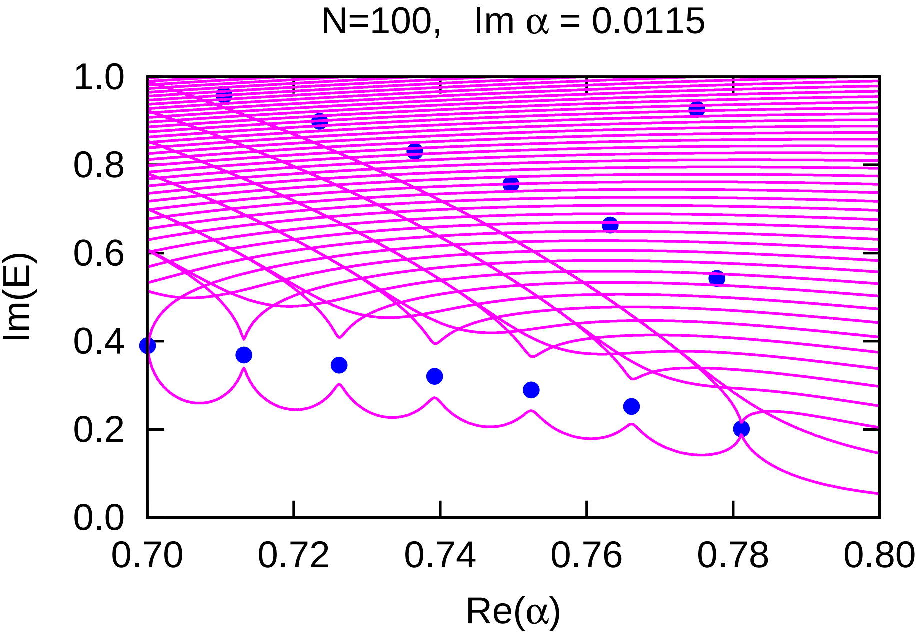

N=100, Im α = 0.0111
Re(E)
Re(α)
1.0
0.5
0.0
-0.5
-1.0
0.70
0.72
0.74
0.76
0.78
0.80

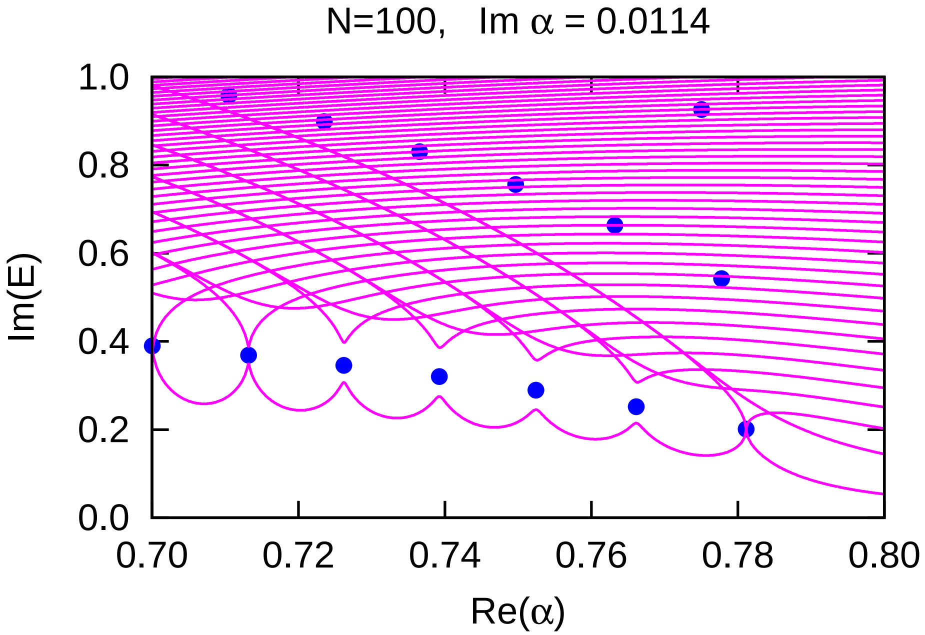

N=100, Im α = 0.0112
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

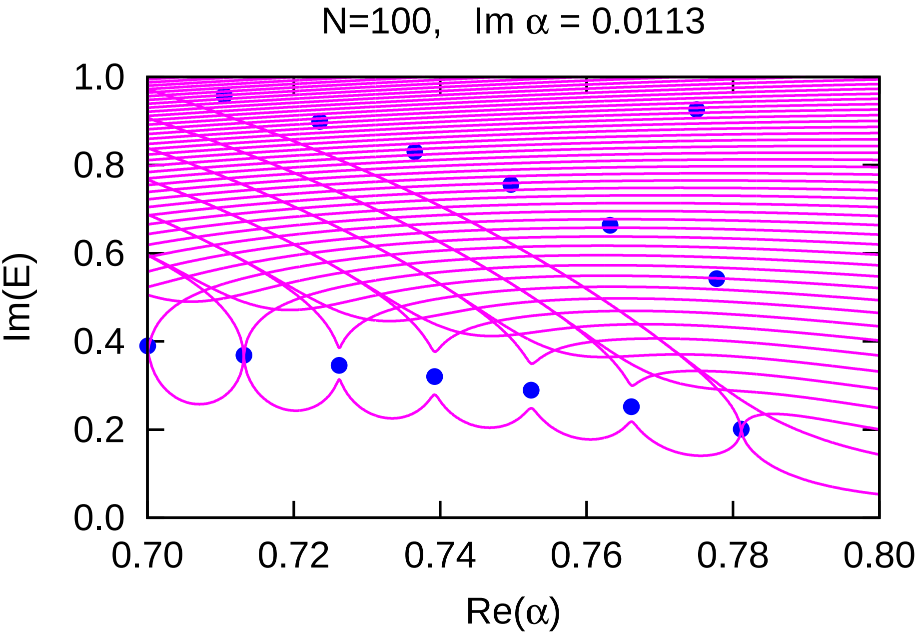

N=100, Im α = 0.0113
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0114
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

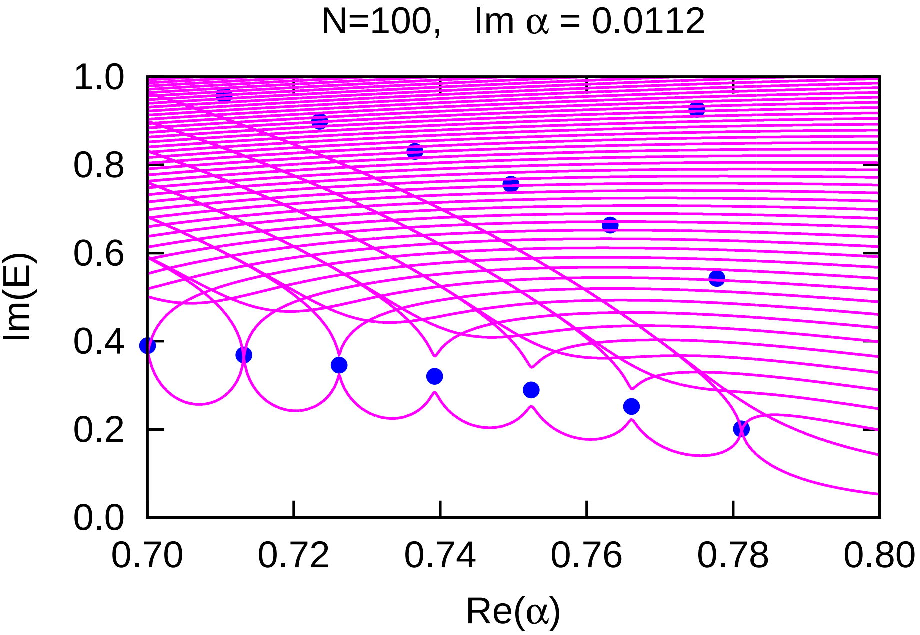

N=100, Im α = 0.0115
1.0
0.5
0.0
-0.5
-1.0
Re(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0115
Im(E)
Re(α)
1.0
0.8
0.6
0.4
0.2
0.0
0.70
0.72
0.74
0.76
0.78
0.80

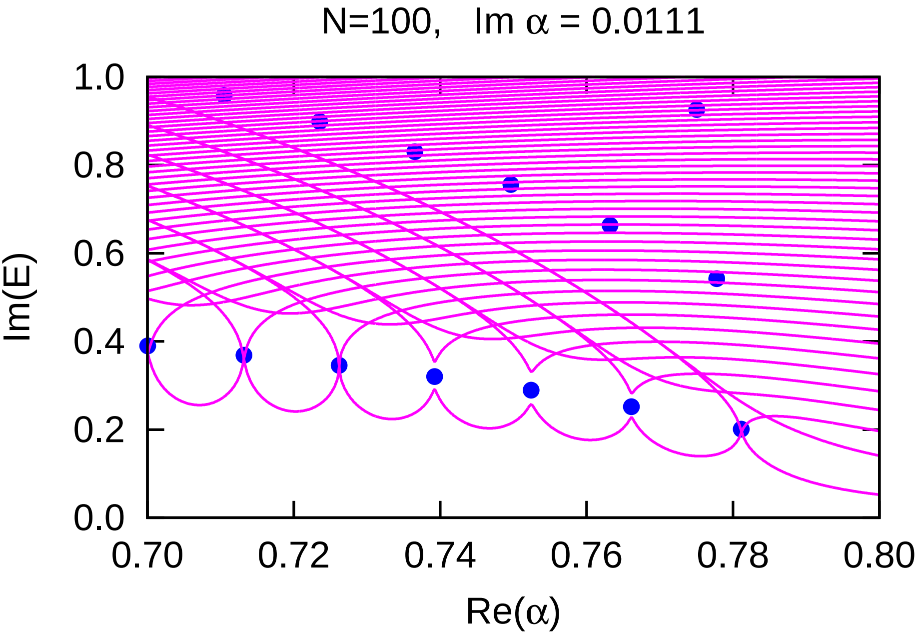

N=100, Im α = 0.0114
1.0
0.8
0.6
0.4
0.2
0.0
Im(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

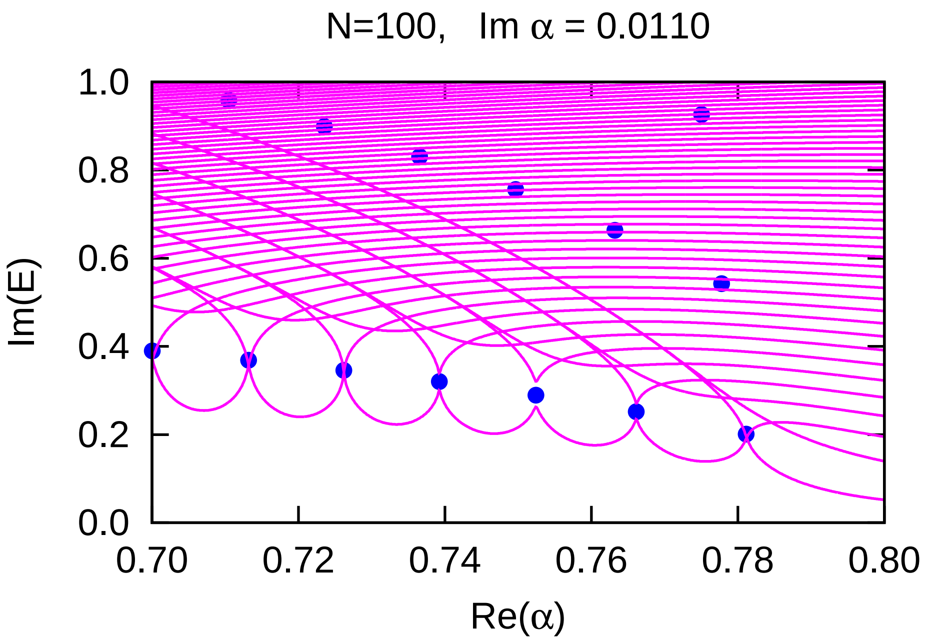


N=100, Im α = 0.0113
1.0
0.8
0.6
0.4
0.2
0.0
Im(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

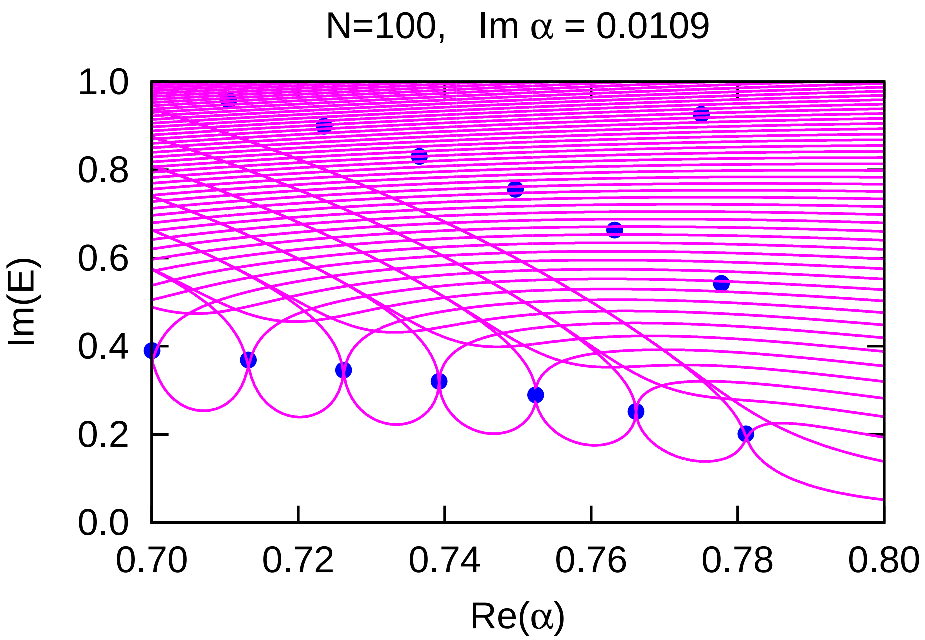


N=100,   Im α = 0.0112
Im(E)
1.0
0.8
0.6
0.4
0.2
0.0
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

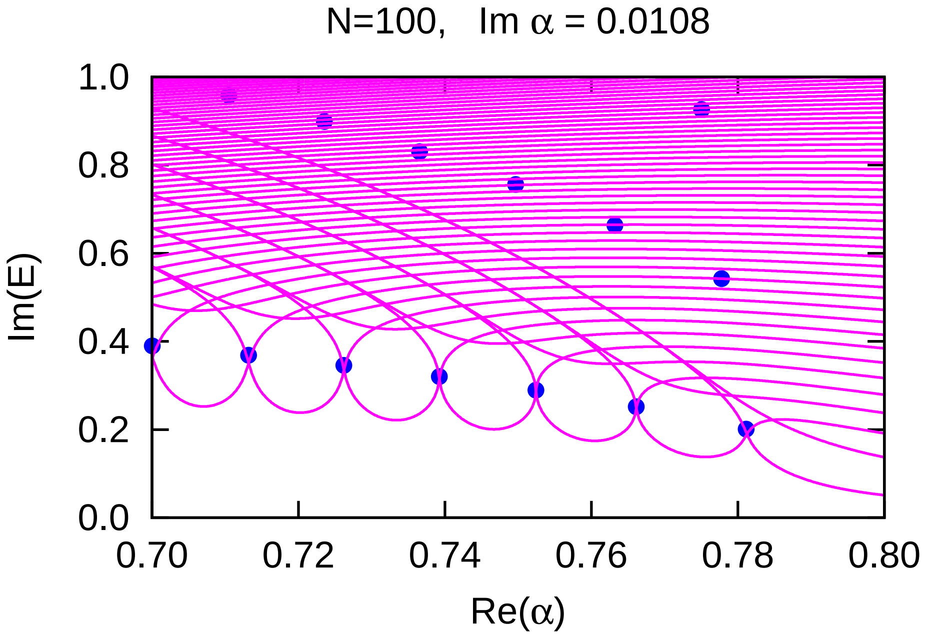

N=100,   Im α = 0.0111
Im(E)
1.0
0.8
0.6
0.4
0.2
0.0
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)



N=100, Im α = 0.0110
1.0
0.8
0.6
0.4
0.2
0.0
Im(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)



N=100, Im α = 0.0109
1.0
0.8
0.6
0.4
0.2
0.0
Im(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)


N=100, Im α = 0.0108
1.0
0.8
0.6
0.4
0.2
0.0
Im(E)
0.70
0.72
0.74
0.76
0.78
0.80
Re(α)

How does the dependence of $E_n(\alpha)$ look for $\alpha \in \mathbb{C}$?

The eigenvalues $E_n(\alpha)$ coalesce at each EP
as $\mathrm{Im}\,\alpha$ is gradually increased.

Any insights related to ESQPT ?

The lowest progression of EPs appears to approach the separatrix
at which the ESQPT takes place.

Šindelka, Santos, Moiseyev, Phys. Rev. A, **95**, *010103(R) (2017)*

Summary

- The LMG model does contain EPs in the complex α-plane.
- These EPs are closely related both to QPT and to ESQPT.
- QPT: Each EP moves towards $\alpha_{\mathrm{QPT}} = 0.8$ for $N \to +\infty$.
- The separatrix of ESQPT is also the borderline
  beyond which the EPs start to appear.
  The complex valued surfaces of $E_n(\alpha)$
  become topologically nontrivial
  once one crosses the ESQPT separatrix.

Šindelka, Santos, Moiseyev, Phys. Rev. A, **95**, *010103(R) (2017)*

Thanks for your attention!