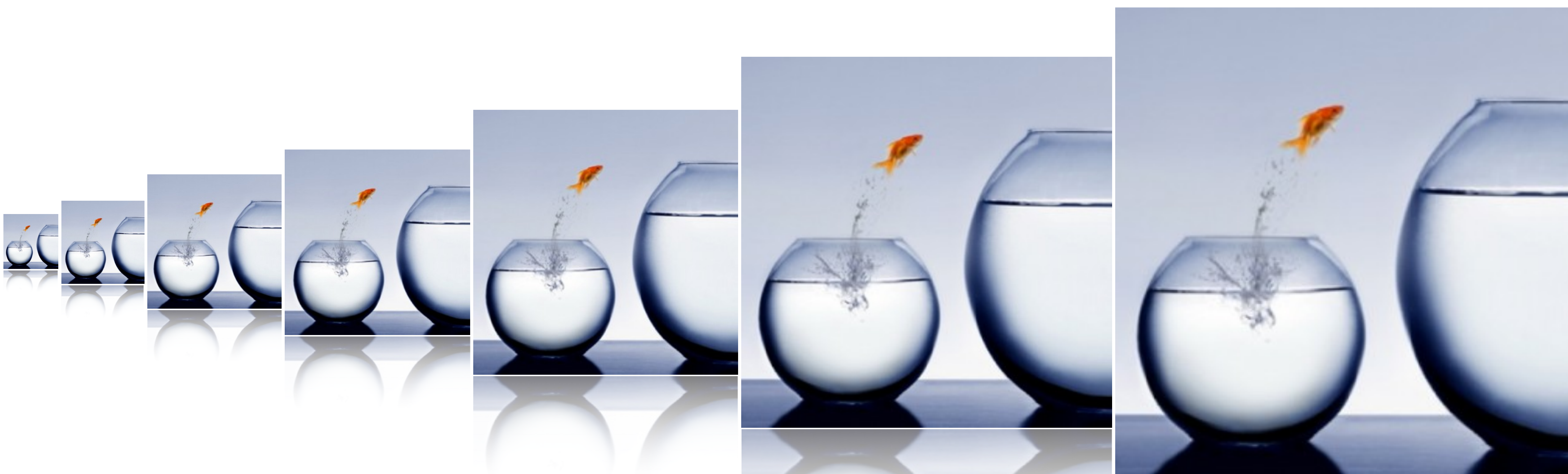
Non-commutative kinematics for a point-like particle



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Plan of the talk

Models of probability

Classical probability

The classical model of probability described by the triple $\{(\Omega, \mathcal{E}, P); \mathcal{R}; P(\cdot|\cdot)\}$.

The conditional probability is defined by means of the Bayes formula

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

Two features indicate “classicality” of probabilities:

1) $P(A \cap B) = P(B \cap A)$ $\longrightarrow$ $\boxed{P(A|B)P(B) = P(B|A)P(A)}$

2) Bayes theorem (law of total probability) $\longrightarrow$

$$P(B) = \sum_{i \in I} P(B|A_i)P(A_i)$$

when $\cup_{i \in I} A_i = \Omega$.

Quantum probability

The quantum model of probability is described by $\{\mathcal{B}(\mathcal{H}); Tr(\hat{\rho}\,\cdot\,)\}$.

Random variables are described with operators $\hat{A} \in \mathcal{B}(\mathcal{H})$.

Probabilities are obtained via expectation of projectors $P(A=\alpha) = Tr(\hat{\rho}|\alpha\rangle\langle\alpha|)$

Conditional probabilities are given by $P(\beta_j|\alpha_i) = Tr(|\alpha_i\rangle\langle\alpha_i|\beta_j\rangle\langle\beta_j|) = |\langle\beta_j|\alpha_i\rangle|^2$

It is a different model of probability: take $\hat{A} = \sum_{i=1}^{2} \alpha_i |\alpha_i\rangle\langle\alpha_i|$, $\hat{B} = \sum_{i=1}^{2} \beta_i |\beta_i\rangle\langle\beta_i|$, and $\hat{\rho} = |\psi\rangle\langle\psi|$. Suppose

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$$

then, the law of total probability is

$$P(B=\beta_1) = |\langle\beta_1|\psi\rangle|^2 = \sum_{i=1}^{2} P(\beta_1|\alpha_i)|\langle\alpha_i|\psi\rangle|^2 + \underline{2Re\big[c_1 c_2^* \langle\alpha_1|\beta_1\rangle\langle\beta_1|\alpha_2\rangle\big]}$$

The presence of interference indicates that **Bayes theorem do not hold**.

An interesting example

Consider a dice.

i) It can be described in the classical model of probability by

$$\Omega = \{1, \cdots, 6\} \qquad \mathcal{E} = \mathcal{P}(\Omega) \qquad P[D = i] = \frac{1}{6}$$

$$D(\omega) = \omega$$

ii) It can be described in the quantum model of probability using

$$\mathcal{B}(\mathcal{H}) = Mat(\mathbb{C}, 6) \qquad \hat{D} = \sum_{i=1}^{6} i|i\rangle\langle i| \qquad \hat{\rho} = \sum_{i=1}^{6} \frac{1}{6}|i\rangle\langle i|$$

Note that only operators **diagonal in** $\{|i\rangle\}$ make sense for the description of a dice. Clearly this prevent to have interference.

This example shows that we can use quantum model to describe classical random phenomena.

There is a way to "violate the Bayes theorem in classical probability"?

Yes. Take 3 random variables $A, B, C \in \mathcal{R}$ on a classical probability space which are dependent. Hence

$$P(A=\alpha, B=\beta, C=\gamma) \neq P(A=\alpha)P(B=\beta)P(C=\gamma)$$

The Bayes theorem holds

$$P(B=\beta) = \sum_{\alpha} P(\beta|\alpha)P(A=\alpha)$$

But…what happens after that we condition on $\{C=\gamma\}$?
Using $P(A=\alpha) = \sum_{\gamma} P(A=\alpha, C=\gamma)$ and $P(B=\beta) = \sum_{\gamma} P(B=\beta, C=\gamma)$ in the Bayes theorem, we get

$$P^B_\gamma(\beta) = \sum_\alpha P(\beta|\alpha)P^A_\gamma(\alpha) + \underline{\delta(\beta|A,\gamma)} \qquad P^X_\gamma(x) := P(X=x|C=\gamma)$$

$$\delta(\beta|A,\gamma) = \frac{1}{P[C=\gamma]} \sum_{\gamma\neq\gamma'} \left[\sum_\alpha P(\beta|\alpha)P[A=\alpha, C=\gamma'] - P[B=\beta, C=\gamma']\right]$$

Consider the law of total probability obtained before

$$P_\gamma^B(\beta) = \sum_\alpha P(\beta|\alpha)P_\gamma^A(\alpha) + \delta(\beta|A,\gamma)$$

Since it is similar to the law obtained in the quantum model of probability, **can we describe A and B with two non-commuting operators on a suitable Hilbert space?**

Two methods to answer to this question:

1) Using **Quantum-Like Representation Algorithm** (QLRA);

2) Using **Entropic uncertainty relations** (EUR);

QLRA has big advantages but also practical limitations. EUR are more useful in the model we are going to present.

Entropic uncertainty relations and non-commutativity

Given a discrete random variable X with probability distribution $\{p_X(x)\}$ the Shannon entropy is defined as

$$H(X) = -\sum_x p_X(x) \log p_X(x)$$

It works also for an operator $\hat{X}$ using $p_X(x) := |\langle x|\psi\rangle|^2$, where $|x\rangle$ is an eigenvector of $\hat{X}$.

Theorem. Let $\hat{X}$ and $\hat{Y}$ be two operators on a Hilbert space. Then $[\hat{X},\hat{Y}] \neq 0$ if and only if

$$H_\psi(\hat{X}) + H_\psi(\hat{Y}) \geq D > 0$$

for all ψ.

Inequalities like this are called EUR

Entropic ... ativity

Given a disc... $\ldots_k(x)\}$ the Shannon ent...

where

$$c = \max_{j,k} |\langle a_j | b_k \rangle|. \tag{5}$$

More recently, Kraus[5] has conjectured that this relation may be improved to

$$H(p) + H(q) \geqq -2 \ln c. \tag{6}$$

The advantage of these relations over (2) is that they have a right-hand side which is independent of the state ψ. Thus, they yield nontrivial information on the probability distributions ...

Generalized entropic uncertainty relations - H. Maassen and J. B. M. Uffink - Phys. Rev. Lett. **60**, 1103 – Published 21 March 1988

It works also ... an eigenvector ...

Theorem ... hen

$$H_\psi(\hat{X}) + H_\psi(\hat{Y}) \geq D > 0$$

for all ψ.

Inequalities like this are called EUR

Model A

What we want to obtain with the Model A?

We want construct a toy-model for a point-like particle whose random variables, describing the position and the velocity, do not commute when described as operators.

Description of the Model A

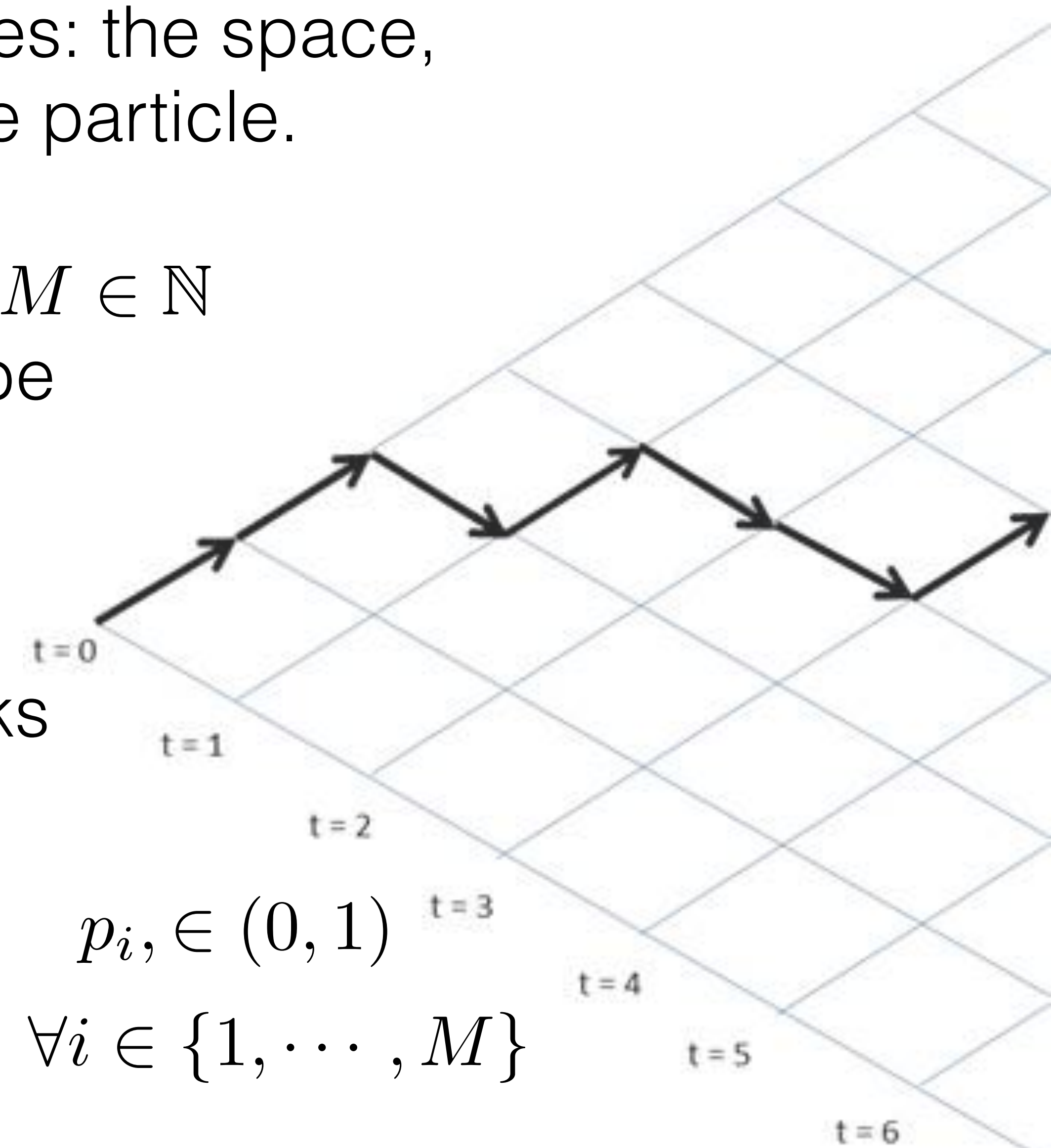
Model A is constructed using 3 random processes: the space, the position of the particle, and the velocity of the particle.

1) **<u>The space</u>:** It is described by a collection of $M \in \mathbb{N}$ independent random walks. At time N it can be thought as an M-vector

$$\mathbf{S}_N := (S_N^{(1)}, \cdots, S_N^{(M)})$$

The transition probabilities of the random walks are fixed

$$P[S_{N+1}^{(i)} = x+1 | S_N^{(i)} = x] = p_i \qquad p_i, \in (0,1)$$
$$P[S_{N+1}^{(i)} = x-1 | S_N^{(i)} = x] = 1 - p_i \qquad \forall i \in \{1, \cdots, M\}$$



2) **The position of the particle:** the position random variable select a point of the space and takes its value.

Selection random variable, $I_N \in \{1, \cdots, M\}$

$$X_N := \pi^{(I_N)}(\mathbf{S}_N)$$

Projector on the I_N-th component of the vector $\mathbf{S}_N$

This random variable is described by the probabilities $\{P[X_N = a]\}$.

3) **The velocity of the particle:** the velocity random variable is defined as

$$V_N = X_{N+1} - X_N$$

Introducing the transition probabilities $\alpha(b,a) := P[X_{N+1} = b | X_N = a]$ and observing that $P[V_N = c | X_N = a] = \alpha(a+c, a)$, we can conclude that

$$P[V_N = c] = \sum_a \alpha(a+c,a) P[X_N = a]$$

Thus given $\{P[X_N = a]\}$ and the transition probabilities, we can describe the velocity random variable.

The entropic uncertainty relation of the Model A

The basic assumption of the model is

Assumption: the physical space coincides with the space of the Model A.

Consequences:

1) the only probabilities that make sense are the one conditioned to a given configuration of the space $\mathbf{S}_N$, i.e.

$$\{P_{\mathbf{S}_N}[X_N = a]\} \qquad \{P_{\mathbf{S}_N}[V_N = c]\}$$

2) Bayes theorem do not hold,

$$P_{\mathbf{S}_N}[V_N = c] = \sum_a \alpha(a + c, a) P_{\mathbf{S}_N}[X_N = a] + \delta(c|X_N, \mathbf{S}_N)$$

Since space is random, we can always write the following

$$P[X_N = a] = \frac{1}{M!} \sum_{\sigma \in P_M} \sum_{i=1}^{M} \gamma(N, \sigma(i), a) P[S_N^{(\sigma(i))} = a]$$

$$P[X_N = S_N^{(\sigma(i))} | S_N^{(\sigma(i))} = a]$$

Acting on the particle only, we can change only $\gamma(N, \sigma(i), a)$, while the probability of the space are fixed. This decomposition hols also for $\alpha(b, a)$.

The probabilities of the model are $P_{\mathbf{S}_N}[X_N = a]$, $P_{\mathbf{S}_N}[V_N = c]$ and $\alpha(a + c, a)$.

Prop.

$$H_{\mathbf{S}_N}(V_N) \geq \sum_a P_{\mathbf{S}_N}[X_N = a] \Big(- \sum_c \alpha(a + c, a) \log \alpha(a + c, a) \Big)$$

This proposition is a property of the probability distributions related by the modified law of total probability seen before.

Th. In the Model A

$$H_{\mathbf{s}_N}(X_N) + H_{\mathbf{s}_N}(V_N) \geq D > 0$$

where D do not depend on $\{P_{\mathbf{s}_N}[X_N = a]\}$ and $\{P_{\mathbf{s}_N}[V_N = c]\}$.

Sketch of the proof

The proof is essentially based on the Jensen inequality for concave functions, i.e. $\mathbb{E}[\varphi(X)] \leq \varphi(\mathbb{E}[X])$, where φ is a concave function. Since

$$H_{\mathbf{s}_N}(V_N) \geq \sum_a P_{\mathbf{s}_N}[X_N = a]\Big(- \sum_c \alpha(a+c,a) \log \alpha(a+c,a)\Big)$$

and that

$$\alpha(a+c,a) = \frac{1}{M!} \sum_{\sigma \in P} \sum_{i=1}^{M} \gamma(N+1, \sigma(i), a+c | X_N = a) P[S_{N+1}^{\sigma(i)} = a+c]$$

Using the Jensen inequality one can write

$$- \sum_c \alpha(a+c,a) \log \alpha(a+c,a)$$

$$\geq - \sum_c \frac{1}{M!} \sum_{\sigma \in P} \sum_i \gamma(N+1, \sigma(i), a+c | X_N = a) \underline{P[S_{N+1}^{\sigma(i)} = a+c] \log P[S_{N+1}^{\sigma(i)} = a+c]}$$

$$\geq - \sum_c \min_i P[S_{N+1}^{\sigma(i)} = a+c] \log P[S_{N+1}^{\sigma(i)} = a+c]$$

Removing the dependence on a taking the minimum with respect to it we get

$$H_{\mathbf{S}_N}(X_N) + H_{\mathbf{S}_N}(V_N) \geq D > 0$$

where

$$D := \min_a \left[-\sum_c \min_i P[S_{N+1}^{(\sigma(i))} = a + c] \log P[S_{N+1}^{(\sigma(i))} = a + c] \right]$$

Starting from X_N we get the same bound since the transition probabilities are symmetric under the exchange of their arguments.

Can we write the bound in a better way? Yes, if we assume that all the probabilities of the random walks of the space are equal. In this case

$$D = H(S_{N+1})$$

and in addition, since conditioning reduces the entropy (i.e. $H(X) \geq H(X|Y)$) we can improve the bound. Indeed

$$P[S_{N+1}^{(i)} = x + 1 | S_N^{(i)} = x] = p_i$$

$$\begin{aligned} H(S_{N+1}) &\geq H(S_{N+1}|S_N) \\ &= \sum_a P[S_N = s] H(S_{N+1}|S_N = s) \\ &= -p \log p - (1-p) \log(1-p) \end{aligned}$$

Thus $\underline{H_{\mathbf{S}_N}(X_N) + H_{\mathbf{S}_N}(V_N) \geq -p \log p - (1-p) \log(1-p).}$

Conclusion

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Summarising, we proved the in the Model A

$$H_{\mathbf{s}_N}(X_N) + H_{\mathbf{s}_N}(V_N) \geq -p\log p - (1-p)\log(1-p)$$

Thus when we represent X_N and V_N with operators over the same Hilbert space, they do not commute:

$$[\hat{X}_N, \hat{V}_N] \neq 0$$

This is a consequence of the random nature of space in the model.

It is interesting to observe that in the Model A, everything is random and "quantumness" emerges when we consider part of it (e.g. the particle).

Thank you

Interesting readings and references

On entropic uncertainty relations:

[1] H.Maassen, J.Uffink - Generalised entropic uncertainty relations - *Phys. Rev. Lett.* 60(12): 1103, 1988.
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[3] Krishna, M., K. R. Parthasarathy - An entropic uncertainty principle for quantum measurements. - *Sankhyā: The Indian Journal of Statistics, Series A* (2002): 842-851.

On probability and QLRA:

[1] A. Khrennikov - Contextual probability approach to quantum formalism - *Springer*, 2009.
[2] A. Khrennikov - Ubiquitous quantum structure - *Springer*, 2010 .
[3] A. Khrennikov - Probability and randomness: quantum versa classical - *Imperial college pres*. , 2016.
[4] L. Accardi - Probabilita' quantistica (in Italian).
[5] D. Aerts - A possible explanation for probabilities of quantum mechanics - *Jour. Math. Phys.* 27(1) 202-210, 1986.