How Well can we Find Out Whether a Wave Function has Collapsed?

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Question

Setting:

- Hilbert space $\mathscr{H}$ of dimension $d \in \mathbb{N}$
- $\mathbb{S}(\mathscr{H}) = \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ unit sphere
- $B = \{b_1, \ldots, b_d\}$ an orthonormal basis of $\mathscr{H}$
- initial wave function $\psi \in \mathbb{S}(\mathscr{H})$
- with prob $0 < p < 1$ a "collapse" occurs (as in a "quantum-mechanical measurement of an observable" with eigenbasis B). Afterwards: wave function is the $\mathbb{S}(\mathscr{H})$-valued random variable

$$\psi' = \begin{cases} \psi & \text{with prob } 1-p \\ \frac{\langle b_k|\psi\rangle}{|\langle b_k|\psi\rangle|} b_k & \text{with prob } p\left|\langle b_k|\psi\rangle\right|^2 \text{ for } k = 1, \ldots, d\,. \end{cases}$$

Question

How well can we find out experimentally whether a collapse has occurred?

Motivation: GRW theory

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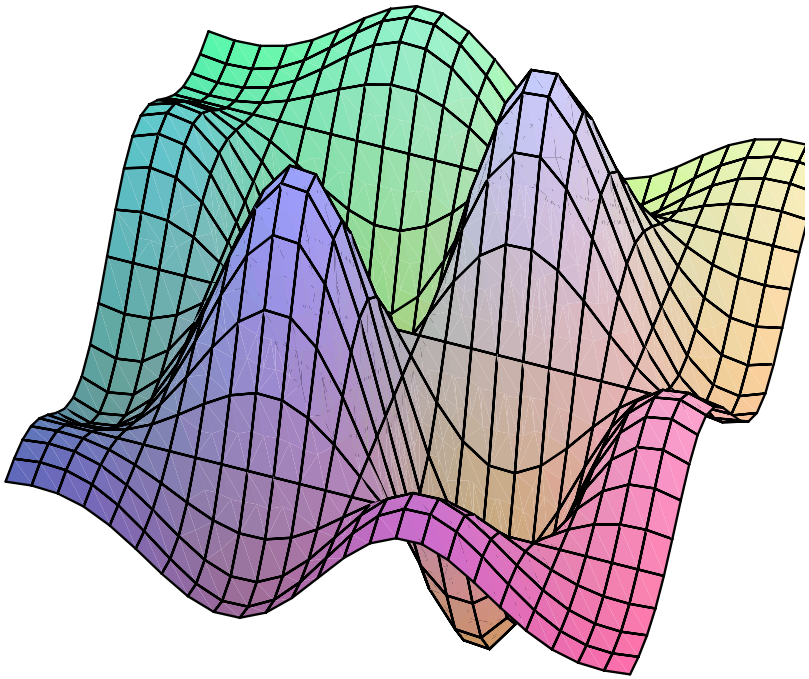
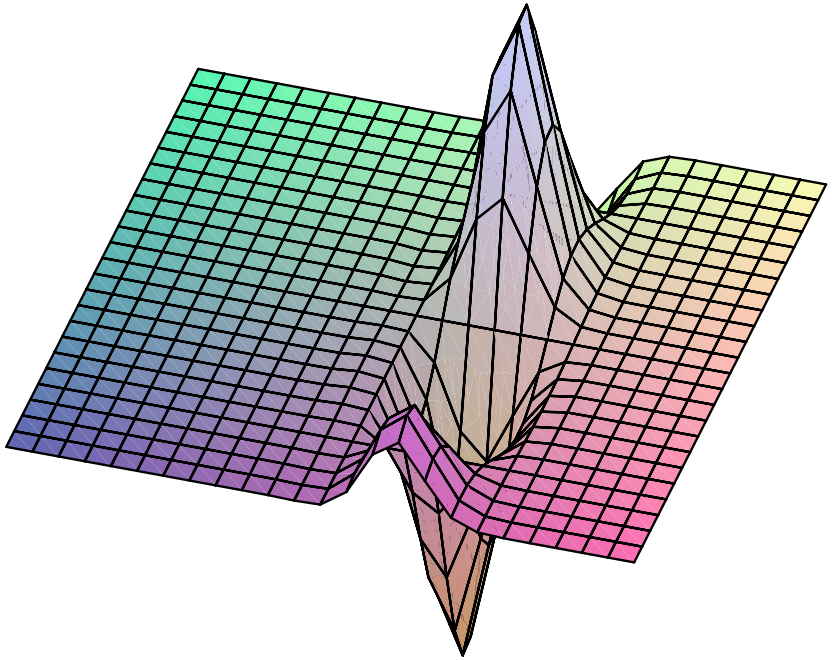


after:



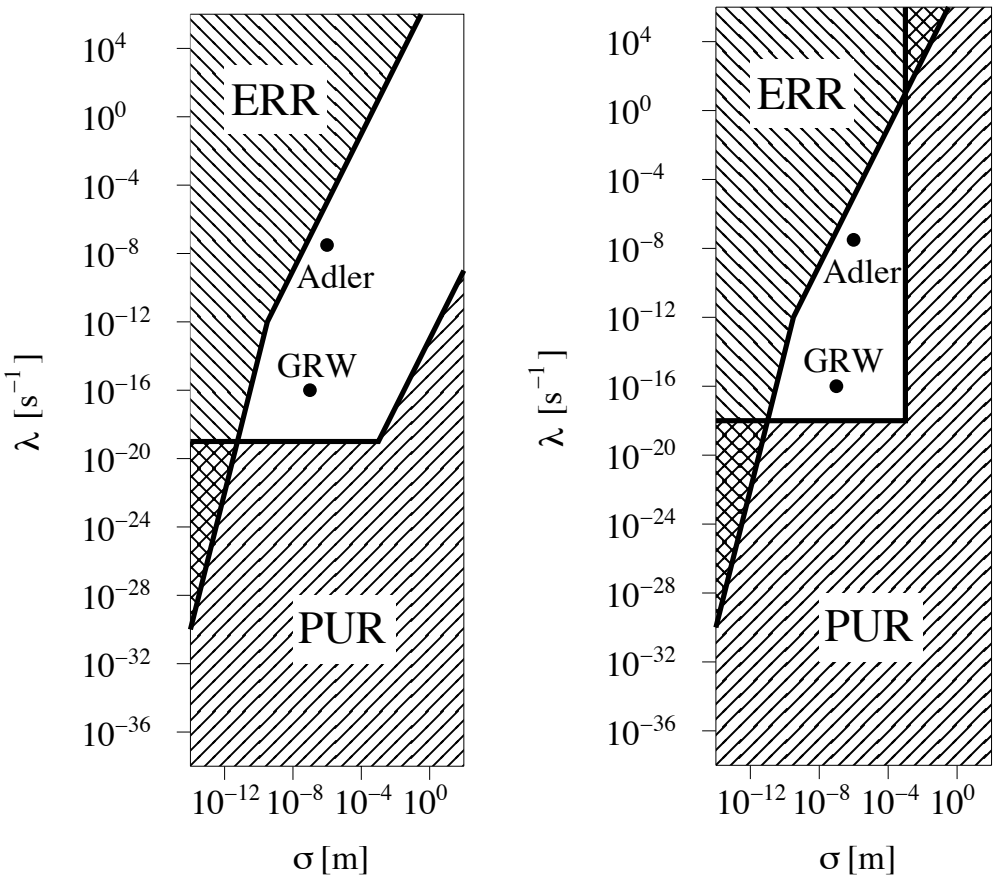
GRW's stochastic time evolution for ψ

- for non-relativistic quantum mechanics of N particles
- intended to replace the Schrödinger eq as a fundamental law of nature
- involves two new constants of nature:
  - $\lambda \approx 10^{-16}\,\mathrm{sec}^{-1}$, called collapse rate per particle.
  - $\sigma \approx 10^{-7}\,\mathrm{m}$, called collapse width.
- Def: ψ evolves as if an observer outside the universe made, at random times with rate $N\lambda$, a quantum measurement of the position observable of a randomly chosen particle with inaccuracy σ.
- "Rate $N\lambda$" means $\mathbb{P}(\text{event in the next } dt \text{ seconds}) = N\lambda\, dt$.
- explicitly: Schrödinger evolution interrupted by jumps of the form

$$\psi_{T+} = e^{-\frac{(\mathbf{q}_k - \mathbf{q})^2}{4\sigma^2}} \psi_{T-}\,,$$

  i.e., multiplication by a Gaussian fct with random index k, center $\mathbf{q}$, and time T.

The empirical predictions of GRW theory deviate very slightly from those of quantum mechanics. Experiments yield bounds on parameters (σ, λ).



Parameter diagram (log-log scale). Left: with flash ontology, Right: with matter density ontology. ERR = empirically refuted region (equal), PUR = philosophically unsatisfactory region. From [Feldmann, RT: *J. Phys. A* 2012]

Limitations to knowledge in GRW theory

GRW theory describes one possibility of what reality can be like so that the inhabitants find the rules of QM confirmed.

GRW theory thus explains the quantum mechanical "measurement" of "observables." Of course, this is not a measurement in the literal sense (discovering a pre-existing value).

How then about measurements in the literal sense: Can we measure, e.g., the number of collapses in a system (e.g., water droplet with 10^{15} molecules) during the time interval $[t_1, t_2]$? Is there a "Geiger counter for collapses"?

Short answer: no

There are limitations to knowledge: Nature knows the exact number of collapses, but inhabitants can find it out only with macroscopic inaccuracy (say, $\pm 10^9$ per second).

Examples: limitations to knowledge (1)

- In QM: Alice prepares an ensemble of systems in pure states with distribution μ_1 on $\mathbb{S}(\mathscr{H})$. $\mu_2 \neq \mu_1$ has the same density matrix, $\rho_{\mu_1} = \rho_{\mu_2}$, $\rho_\mu = \int_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, |\psi\rangle\langle\psi|$. Bob cannot decide empirically whether Alice used μ_1 or μ_2. Yet, there is a fact about whether the distribution is really μ_1 or μ_2 (after all, Alice knows the pure state of every system).
- QM: We cannot measure the wave function ψ. Example: Alice chooses a direction in $\mathbb{R}^3$ and prepares an electron with spinor in this direction. Empirically, Bob can obtain no more than one bit of information about this direction.
- Every-day: We cannot look into the future, not directly into the (timelike) past, and not into a spacelike separated region.
- GR: black holes
- In Bohmian mechanics, we cannot measure the instantaneous velocity of a particle unless we have information about its wave function.

Examples: limitations to knowledge (2)

- Heisenberg uncertainty is a *limitation to knowledge* in Bohmian mechanics (we cannot know the position and the asymptotic velocity of a particle exactly at the same time), but not in GRW theory (position and momentum are not even defined if no experiment is performed).
- Limitations to knowledge conflict with the (exaggerated) "positivistic" idea that quantities that cannot be measured are unreal and only operational statements (if we do X, the result will be Y with probability p) are meaningful/scientific.

"To admit things not visible to the gross creatures that we are is, in my opinion, to show a decent humility, and not just a lamentable addiction to metaphysics."

John S. Bell

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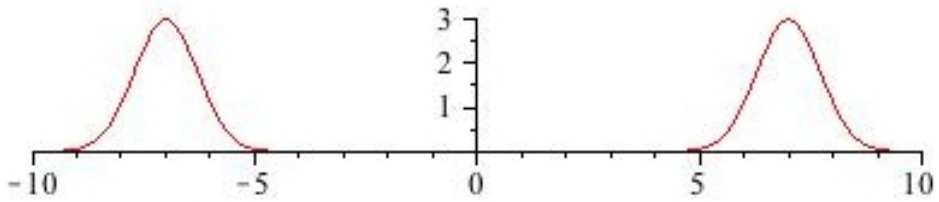
Question

How well can we find out experimentally whether a collapse has occurred?

Example

Let $d = 2$, $\psi(t_1) = \frac{1}{\sqrt{2}} b_1 + \frac{1}{\sqrt{2}} b_2$, (1)



$C \in \{0, 1\}$ number of collapses

Experiment $\mathscr{E}_1 :=$ quantum-mechanical measurement of $I - |\psi\rangle\langle\psi|$.
Provides probabilistic information: If outcome $Z = 1$ then $C = 1$; if $Z = 0$ then $\mathbb{P}(C = 0|Z = 1) = p/(2 - p)$.

Def: reliability of a yes-no experiment

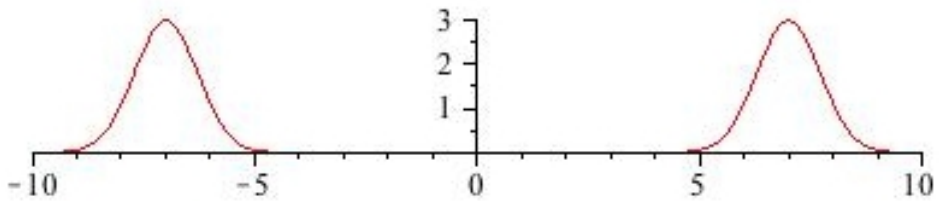
$R(\mathscr{E}) = \mathbb{P}$(correct answer to whether a collapse occurred)

$$= \mathbb{P}(Z = 0|C = 0)\mathbb{P}(C = 0) + \mathbb{P}(Z = 1|C = 1)\mathbb{P}(C = 1)$$

Here, $R(\mathscr{E}_1) = 1 - p/2$.

Bounds on reliability

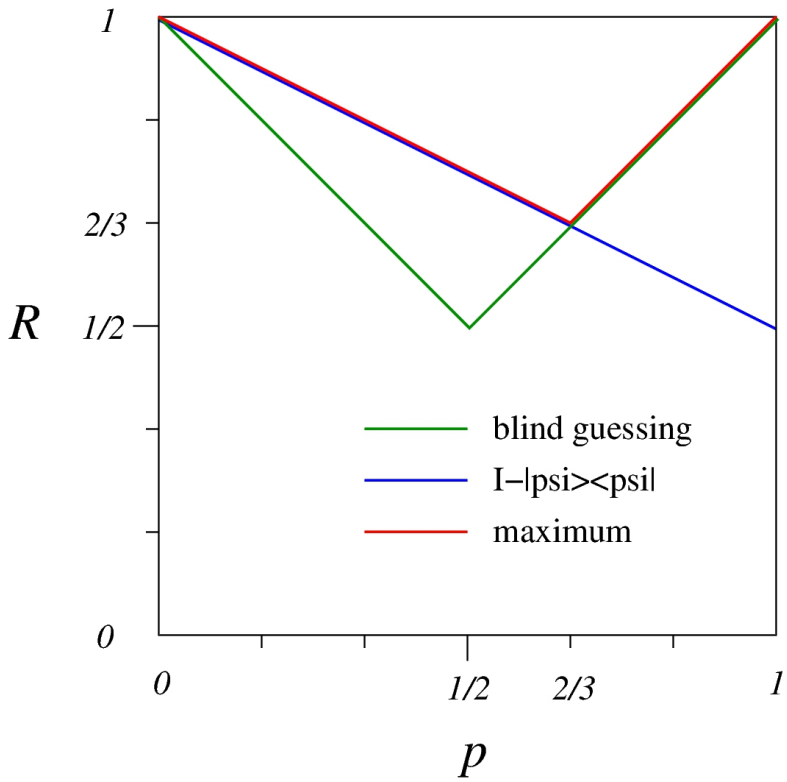
$$\psi(t_1) = \frac{1}{\sqrt{2}} b_1 + \frac{1}{\sqrt{2}} b_2, \tag{1}$$



Theorem 1 [CWC, RT 2014]

For ψ, B as in example (1) it is impossible to decide reliably whether a collapse occurred. For $0 \leq p \leq \frac{2}{3}$, no experiment is more reliable than $\mathscr{E}_1$. For $\frac{2}{3} \leq p \leq 1$, no experiment is more reliable than blind guessing.

$\mathscr{E}_0$:= blind guessing: always $Z = 1$ if $p > \frac{1}{2}$ ($Z = 0$ if $p \leq \frac{1}{2}$).
$R(\mathscr{E}_0) = \max\{p, 1-p\}$



POVM = positive-operator-valued measure

Definition

A POVM is a collection of positive operators E_z such that $\sum_z E_z = I$.

Note: $\langle\psi|E_z|\psi\rangle$ is a probability distribution over z.

Main theorem about POVMs

For every experiment $\mathscr{E}$ there is a POVM (E_z) so that the outcome Z of $\mathscr{E}$, if carried out on a system with wave function $\psi \in \mathbb{S}(\mathscr{H})$, has distribution

$$\mathbb{P}(Z = z) = \langle\psi|E_z|\psi\rangle\,.$$

Bohmian mechanics: [Dürr, Goldstein, Zanghì 2004]

GRW: [Goldstein, RT, Zanghì 2007]

QM: proof not possible

Yes-no experiment $\mathscr{E} \leftrightarrow$ positive operator E

$R(\mathscr{E}) = R(E)$

Distinguish between two density matrices

More general problem:

- Bob gives Alice a system that has density matrix ρ_1 with prob. p and ρ_2 with prob. $1-p$.
- Alice performs an experiment with possible outcomes $Z = 1, 2$ (POVM $= E_1$, $E_2 = I - E_1$).
- She guesses $\rho = \rho_1$ if $Z = 1$ and $\rho = \rho_2$ if $Z = 2$.
- What is the optimal choice of E_1 that maximizes reliability? (p, ρ_1, ρ_2 are known)

Our problem corresponds to $\rho_2 = |\psi\rangle\langle\psi|$ and

$$\rho_1 = \mathrm{diag}\rho_2 = \sum_{k=1}^{d} |b_k\rangle\langle b_k|\rho_2|b_k\rangle\langle b_k| \,.$$

Helstrom's Theorem

$$\begin{aligned} R_{\rho_1,\rho_2,p}(\mathscr{E}) &= p\mathbb{P}(Z=1|\rho_1) + (1-p)\mathbb{P}(Z=2|\rho_2) \\ &= p\mathrm{tr}\left[\rho_1 E_1\right] + (1-p)\mathrm{tr}\left[\rho_2(I-E_1)\right] \\ &= 1-p+\mathrm{tr}\left[AE_1\right] \end{aligned}$$

with

$$A = p\rho_1 - (1-p)\rho_2\,.$$

Helstrom's Theorem (1976)

For any ρ_1, ρ_2, an optimal operator E_1^{opt} that maximizes R is

$$E_1^{\mathrm{opt}} = P_A^+\,,$$

the projection onto the positive spectral subspace of A, i.e, onto the sum of all eigenspaces of A with positive eigenvalues. The maximal reliability is

$$R_p^{\max}(\rho_1,\rho_2) = (1-p) + \lambda^+\,,$$

where λ^+ is the sum of the positive eigenvalues (with multiplicities) of A.

(Think of A as diagonal.)

Detecting collapse for any ψ, also in higher dimension

Theorem 2 [CWC, RT: *J. Phys. A* 2014]

Arbitrary $\mathscr{H}$, $\dim \mathscr{H} = d < \infty$, ONB B, $\psi \in \mathbb{S}(\mathscr{H})$, $p \in [0,1]$. For all yes-no experiments $\mathscr{E}$,

$$R_\psi(\mathscr{E}) \leq \begin{cases} 1 - \frac{p}{d} & \text{for } 0 \leq p \leq \frac{d}{d+1} \\ p & \text{for } \frac{d}{d+1} \leq p \leq 1\,. \end{cases}$$

That is, for $\frac{d}{d+1} \leq p$, no experiment is more reliable than blind guessing.

- For the optimal experiment, by the way, E is usually *not* $I - |\psi\rangle\langle\psi|$, though still a projection.
- The optimal experiment perturbs the wave function ψ' very much. If we may not perturb ψ' then no experiment is more reliable than blind guessing.

ψ-dependent bounds (1)

Theorem 3 [CWC, RT: *J. Phys. A* 2014]

Let $0 < p < 1$ and $\psi \in \mathbb{S}$ with $\psi_k := \langle b_k|\psi\rangle \neq 0$ for all $k = 1, \ldots, d$. Then

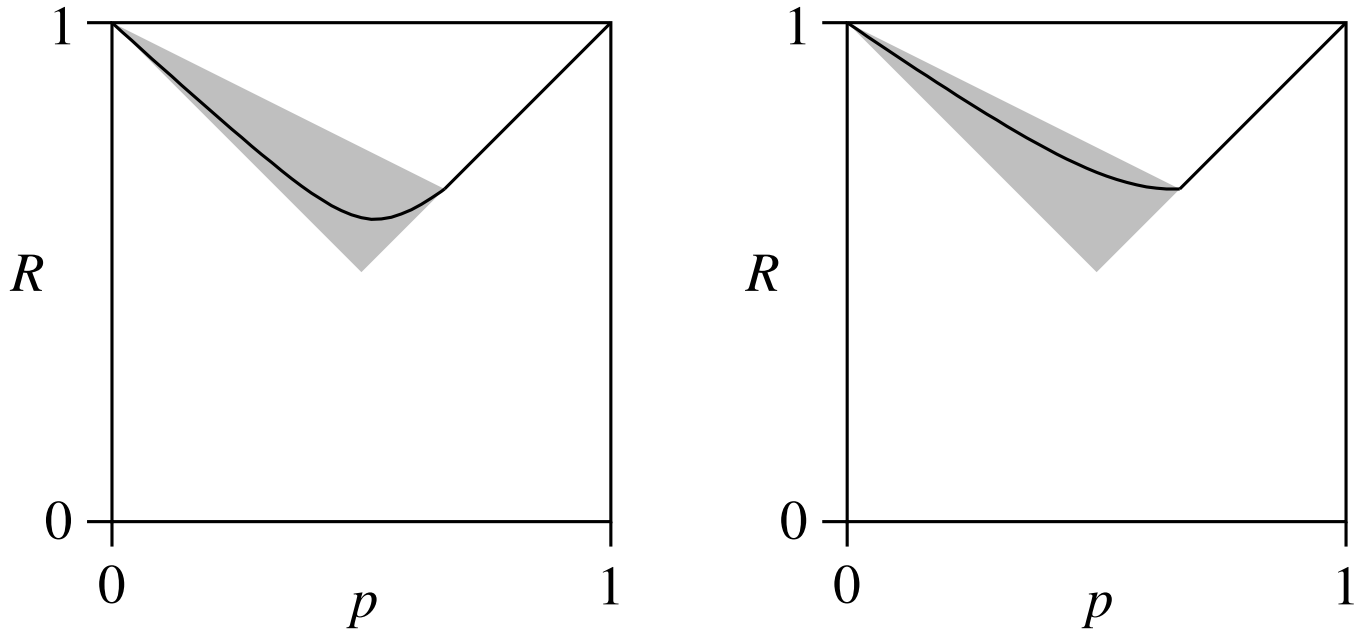
$$R_p^{\max}(\psi) = \begin{cases} p & \text{if } p \geq d/(d+1) \\ p(1 + f_\psi^{-1}(\frac{p}{1-p})) & \text{if } p < d/(d+1) \end{cases},$$

where $f_\psi : [0, \infty) \to (0, d]$ is the bijection given by

$$f_\psi(z) = \sum_{k=1}^{d} \frac{|\psi_k|^2}{z + |\psi_k|^2}.$$

The optimal operators E_1^{opt} are projections that depend on ψ.

ψ-dependent bounds (2)



Examples of graphs of $R_p^{\max}(\psi)$ as a function of p for (LEFT) $\psi = \sqrt{0.05}\, b_1 + \sqrt{0.95}\, b_2$ and (RIGHT) $\psi = \sqrt{0.2}\, b_1 + \sqrt{0.8}\, b_2$. For any ψ, the bound lies in the shaded region according to Theorem 2.

Random ψ

- So far, ψ was known. New challenge: Choose experiment $\mathscr{E}$ without knowing ψ (only $\mathscr{H}$, B, and p).
- First variant: ψ is random with know distribution μ on $\mathbb{S}(\mathscr{H})$.
- Reliability $R = \mathbb{P}(\text{correct answer})$
  $R_\mu(\mathscr{E}) = \int_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, R_\psi(\mathscr{E})$
- Thm 2 $\Rightarrow$ $R_\mu(\mathscr{E}) \leq \max\{1 - \frac{p}{d}, p\}$
- $R_\mu(\mathscr{E}) = R_\rho(\mathscr{E})$

Extreme case $\mu = u$ uniform

No experiment is more reliable than blind guessing,

$$R_u(\mathscr{E}) \leq \max\{p, 1-p\}\,.$$

In fact, no experiment $\mathscr{E}$ (also with more than two possible outcomes z) provides any probabilistic information at all about C: The conditional distribution of C given Z coincides with the a priori distribution,

$$\mathbb{P}(C=1|Z=z) = p\,, \quad \mathbb{P}(C=0|Z=z) = 1-p\,.$$

Unknown ψ (1)

What does it mean for a yes-no experiment $\mathscr{E}$ to be reliable for unknown ψ? Two approaches:

1. "Bayesian" approach: Regard ψ as uniformly distributed on $\mathbb{S}(\mathscr{H})$. See above.
2. without hypothesis on the distribution of ψ: For any given $\mathscr{E}$ consider

$$S_{\mathscr{E}} = \left\{ \psi \in \mathbb{S}(\mathscr{H}) : R_\psi(\mathscr{E}) > R_\psi(\mathscr{E}_0) \right\}.$$

   Maximize $u(S_{\mathscr{E}})$.

We know already that $R_\psi(\mathscr{E})$ is on average no better than blind guessing.

Question

Can we obtain $u(S_{\mathscr{E}}) > \frac{1}{2}$?

Unknown ψ (2)

Theorem 4 [CWC, RT: *J. Math. Phys.* 2015]

- For $d = 2$, $u(S_{\mathscr{E}}) \leq \frac{1}{2}$ for all yes-no experiments $\mathscr{E}$ and all $0 \leq p \leq 1$.
- For $p < \frac{1}{2} - \frac{1}{\sqrt{8}} \approx 0.146$ and $p > \frac{1}{2} + \frac{1}{\sqrt{8}} \approx 0.854$, $u(S_{\mathscr{E}}) \leq \frac{1}{2}$ for all yes-no experiments $\mathscr{E}$ and all $d \in \mathbb{N}$.
- For every $d \geq 3$ and some p, there is a yes-no experiment $\mathscr{E}$ with $u(S_{\mathscr{E}}) > \frac{1}{2}$.

Conjecture [CWC, RT: *J. Math. Phys.* 2015]

For every d, p, and $\mathscr{E}$, $u(S_{\mathscr{E}}) \leq 1 - \frac{1}{e} \approx 0.632$.

That is, we can do better than blind guessing for more than 50% of all wave functions, but not for more than 64%.

Thank you for your attention