The hadronic vacuum polarisation contribution to the muon $g - 2$

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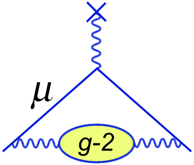
(in collaboration with Daisuke Nomura & Thomas Teubner [KNT17])

LFC17: Old and New Strong Interactions from LHC to Future Colliders

ECT, Trento, Italy

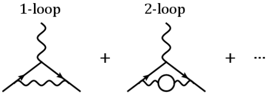


14^{th} September 2017

The Standard Model (SM) contributions to a_μ

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had, VP}} + a_\mu^{\text{had, LbL}} + a_\mu^{\text{BSM??}}$$

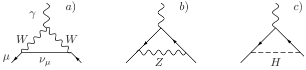
QED



Known to five-loop (12,672 diagrams)

99.99% of a_μ^{SM} 0.001% of δa_μ^{SM}

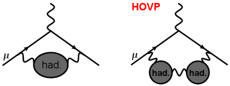
EW



Known to two-loop (with m_H known)

0.0001% of a_μ^{SM} 0.2% of δa_μ^{SM}

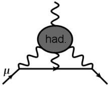
HVP



Non-perturbative (experimental input)

0.006% of a_μ^{SM} **73% of δa_μ^{SM}**

HLbL



Non-perturbative (data input + model/lattice)

0.0001% of a_μ^{SM} 27% of δa_μ^{SM}

BSM

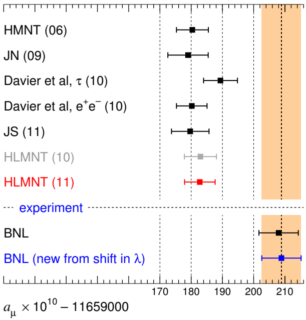
? ? ? ? ? ? ?

$\Rightarrow$ 0.0002% of a_μ^{exp}

The previous analysis... [HLMNT(11), J. Phys. G38 (2011), 085003]

QED contribution	11 658 471.808 (0.015) $\times 10^{-10}$	Kinoshita & Nio, Aoyama et al
EW contribution	15.4 (0.2) $\times 10^{-10}$	Czarnecki et al
Hadronic contribution		
LO hadronic	**694.9 (4.3)** $\times 10^{-10}$	HLMNT11
NLO hadronic	−9.8 (0.1) $\times 10^{-10}$	HLMNT11
light-by-light	10.5 (2.6) $\times 10^{-10}$	Prades, de Rafael & Vainshtein
Theory TOTAL	**11 659 182.8 (4.9)** $\times 10^{-10}$	
Experiment	**11 659 208.9 (6.3)** $\times 10^{-10}$	world avg
Exp − Theory	26.1 (8.0) $\times 10^{-10}$	**3.3 σ discrepancy**

(Numbers taken from HLMNT11, arXiv:1105.3149)



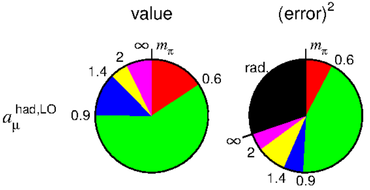
→ $a_\mu^{\text{had, LOVP}}$ still dominated uncertainty

→ Potential for improvement from experimental data and data combination

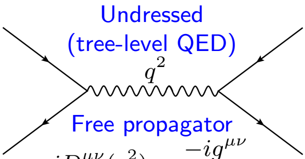
→ New x4 accuracy measurements planned from Fermilab and J-PARC

$\Longrightarrow$ If a_μ^{SM} & a_μ^{EXP} improve as planned...

g-2 discrepancy $> 7\sigma$!

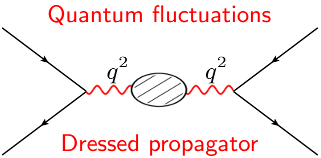


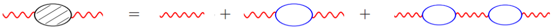
The vacuum polarisation (VP)



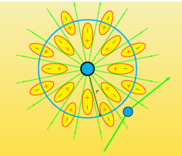
$$iD_\gamma^{\mu\nu}(q^2) = \frac{-ig^{\mu\nu}}{q^2}$$

$\Rightarrow$



$iD_\gamma^{\mu\nu}(q^2) =$  $+\cdots = \dfrac{-ig^{\mu\nu}}{q^2\left(1 - \Pi_\gamma(q^2)\right)}$

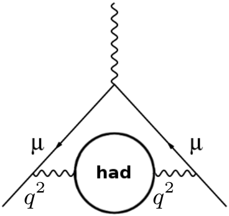
(sum of all 1PI hadronic 'blobs'...)

Virtual loop(s) = VP = charge screening = 

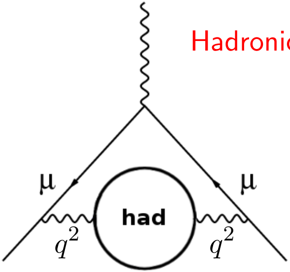
$\Rightarrow$ Running QED coupling:

$$\alpha \to \alpha(q^2) = \frac{\alpha}{1 + 4\pi\alpha \mathrm{Re}\Pi_\gamma(q^2)} = \frac{\alpha}{1 - \Delta\alpha(q^2)}$$

where $\Delta\alpha(q^2) = \Delta\alpha_{\mathrm{lep}}(q^2) + \boldsymbol{\Delta\alpha^{(5)}_{\mathrm{had}}(q^2)} + \Delta\alpha(q^2)_{\mathrm{top}}$



The hadronic vacuum polarisation (HVP)

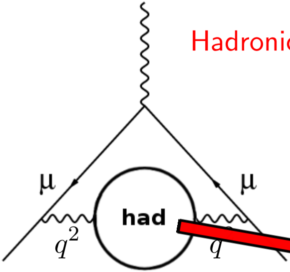


$\gamma* \to \pi^0\gamma \to \gamma*$
$\gamma* \to \pi^+\pi^- \to \gamma*$
$\gamma* \to K^+K^- \to \gamma*$
.
.
.
.

Hadronic 'blob' contains sum of all hadronic states

The hadronic vacuum polarisation (HVP)

Hadronic ‘blob’ contains sum of all hadronic states



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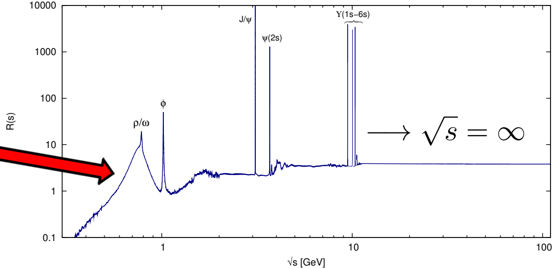


$\longrightarrow \sqrt{s} = \infty$

$\rightarrow$ Requires knowing entire hadronic spectrum

$(s_{\rm th} \leq \sqrt{s} \leq \infty)$

$\Rightarrow$ Then, how do we calculate $a_\mu^{\rm had,VP}$?

Dispersion relation (analyticity) and the optical theorem

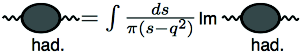
So far, we know we need to relate

- The hadronic vacuum polarisation tensor, $\Pi_{\rm had}(q^2)$
- The hadronic spectrum

$\Rightarrow$ We do this in two steps...

1) Analyticity

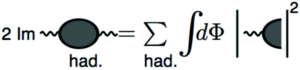
Relate real part of $\Pi(s)$ to its imaginary part



$$\Pi_{\rm had}(q^2) = \frac{q^2}{\pi}\int_{s_{\rm th}}^{\infty} {\rm d}s \frac{{\rm Im}\Pi_{\rm had}(s)}{s(s-q^2-i\varepsilon)}$$

2) Optical theorem

Relate imaginary part of $\Pi(s)$ to total $e^+e^- \to$ hadrons cross section



$${\rm Im}\Pi_{\rm had}(s) = \left(\frac{s}{4\pi\alpha}\right)\sigma_{\rm had}(s)$$

$\rightarrow$ ...convolute with one loop contribution from coupling of virtual photon to muon...

$$a_\mu^{\rm had,LOVP} = \frac{1}{4\pi^3}\int_{s_{\rm th}}^{\infty} {\rm d}s\ \sigma^0_{{\rm had},(\gamma)}(s) K(s)$$

HVP dispersion integral input

$$a_\mu^{\mathrm{had,LOVP}} = \frac{1}{4\pi^3}\int_{s_{\mathrm{th}}}^{\infty} \mathrm{d}s\ \sigma^0_{\mathrm{had},(\gamma)}(s) K(s)$$

Important points to note:

⇒ Can also define dispersive integral for NLOHVP with identical data input

⇒ Integral has 1/s dependence $\left(K(s) = \frac{m_\mu^2}{3s}\tilde{K}(s)\right)$

⇒ Must undress cross section of VP effects: $\sigma_{\mathrm{had}} \to \sigma^0_{\mathrm{had}}$

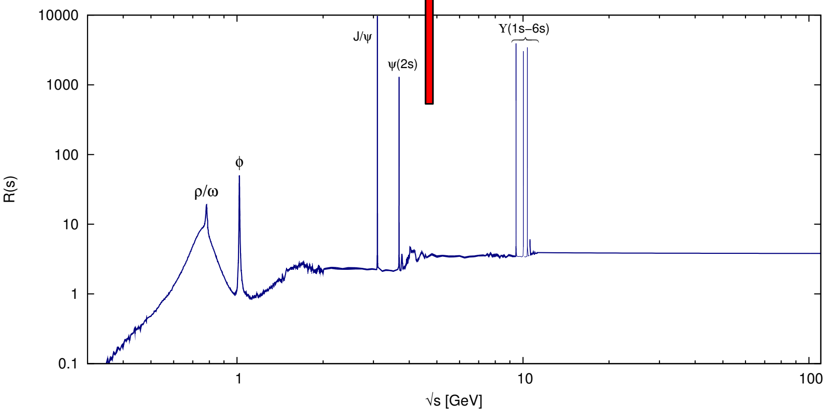
⇒ Cross section must include FSR effects: $\sigma^0_{\mathrm{had}} \to \sigma^0_{\mathrm{had},(\gamma)}$

⇒ Can conventionally define hadronic R-ratio: $R(s) = \frac{\sigma^0_{\mathrm{had},\gamma}(s)}{\sigma_{\mathrm{pt}}(s)} \equiv \frac{\sigma^0_{\mathrm{had},\gamma}(s)}{4\pi\alpha^2/3s}$

$$a_\mu^{\mathrm{had,LOVP}} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty} \frac{\mathrm{d}s}{s} R(s) K(s)$$

Hadronic cross section input

$$a_\mu^{\rm had,LOVP} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty}\frac{{\rm d}s}{s}R(s)K(s), \text{ where } K(s) = \frac{m_\mu^2}{3s}\tilde{K}(s)$$



Hadronic cross section input

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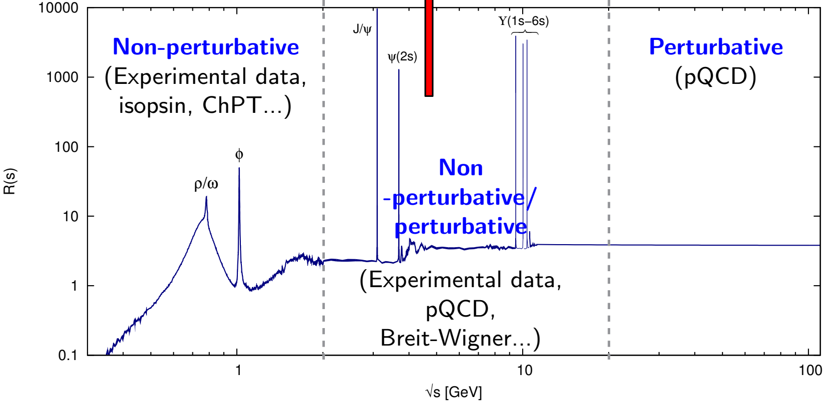

Must build full hadronic cross section/R-ratio...

Building the hadronic R-ratio

$m_\pi \leq \sqrt{s} \leq 2$ GeV

- Input experimental hadronic cross section data*
- Combine all available data in exclusive hadronic final states $(\pi^+\pi^-, K^+K^-, ...)$
- Sum ~ 35 exclusive channels
- Robust treatment of experimental errors
- Estimate missing data input (isospin relations, ChPT...)

$2 \leq \sqrt{s} \leq 11.2$ GeV

- Can use experimental inclusive R data* or pQCD
- Must use data at quark flavour thresholds
- Combine all available R data
- Robust treatment of experimental errors
- Include narrow resonances

$11.2 \leq \sqrt{s} < \infty$ GeV

- Calculate R using pQCD (`rhad`)

*$\sigma_{\rm had}$ experiments

- KLOE
- BaBar
- SND
- CMD-(2/3)
- KEDR
- BESIII
- More old and new...

Combining experimental data

Question:

What are the main points of concern when correcting, combining and integrating experimental data to evaluate $a_\mu^{\text{had, VP}}$?

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⇒ When combining data...

→ ...how to best amalgamate large amounts of data from different experiments

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- → ...finding a solution that is free from bias

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$\Rightarrow$ The reliability of the integral and error estimate

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⇒ When combining data...

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→ ...finding a solution that is free from bias

⇒ The reliability of the integral and error estimate

⇒ The choices when estimating unmeasured hadronic final states

Radiative corrections (for data that is not $\sigma^0_{\mathrm{had},(\gamma)}$)

Vacuum polarisation corrections

⇒ Fully updated, self-consistent VP routine: [vp_knt_v3_0]

→ Cross sections undressed with full photon propagator (must include imaginary part), $\sigma^0_{\mathrm{had}}(s) = \sigma_{\mathrm{had}}(s)|1 - \Pi(s)|^2$

⇒ If correcting data, apply corresponding radiative correction uncertainty

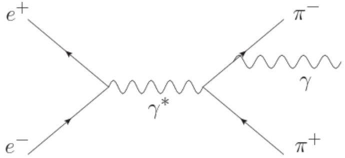
→ Take $\frac{1}{3}$ of total correction per channel as conservative extra uncertainty

Final state radiation corrections

⇒ For $\pi^+\pi^-$, include through sQED approximation
[Eur. Phys. J. C 24 (2002) 51, Eur. Phys. J. C 28 (2003) 261]

⇒ For higher multiplicity states, difficult to estimate correction

∴ Apply conservative uncertainty



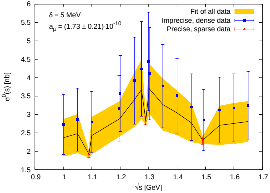
Need new, more developed tools to increase precision here

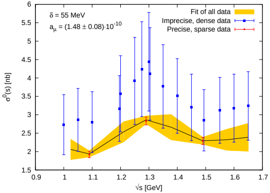
(e.g. - CARLOMAT 3.1 [Eur.Phys.J. C77 (2017) no.4, 254]?)

Clustering data

⇒ Re-bin data into *clusters*

Better representation of data combination through adaptive clustering algorithm





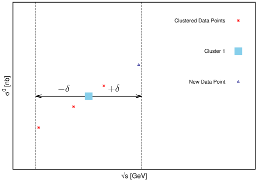
→ More and more data ⇒ risk of over clustering

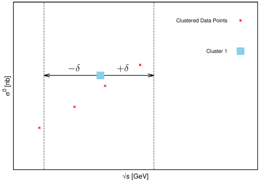
⇒ loss of information on resonance

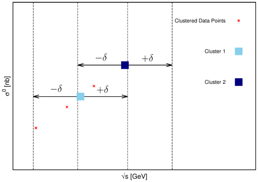
→ Scan cluster sizes for optimum solution (error, χ^2, check by sight...)

⇒ Scanning/sampling by varying bin widths

→ Clustering algorithm now adaptive to points at cluster boundaries







Correlation and covariance matrices

⇒ Correlated data beginning to dominate full data compilation...

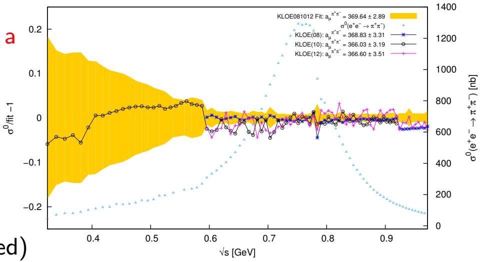
→ Non-trivial, energy dependent influence on both mean value and error estimate

KNT17 prescription

- Construct full covariance matrices for each channel & entire compilation
  ⇒ Framework available for inclusion of any and all inter-experimental correlations
- If experiment does not provide matrices...
  → Statistics occupy diagonal elements only
  → Systematics are 100% correlated
- If experiment does provide matrices...
  → Matrices **must** satisfy properties of a covariance matrix

e.g. - KLOE $\pi^+\pi^-\gamma(\gamma)$ combination covariance matrices update

⟹ Originally, NOT a positive semi-definite matrix:



(Corrected in separate work to be published)

Systematic bias and use of the data/covariance matrix

⇒ Data is re-binned using an adaptive clustering algorithm

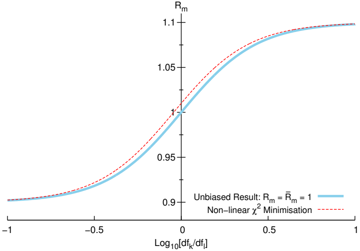
⇒ Iterative fit of covariance matrix as defined by data → D'Agostini bias

[Nucl.Instrum.Meth. A346 (1994) 306-311]

HLMNT11

⇒ Non-linear χ^2 minimisation fitting nuisance parameters

→ Penalty trick bias

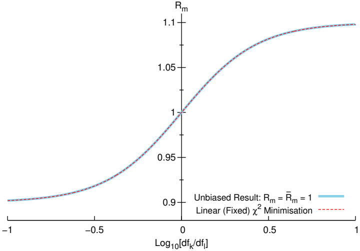


⟹

KNT17

⇒ Fix the covariance matrix in an iterative χ^2 minimisation

→ Free from bias



Allows for increased fit flexibility and full use of energy dependent, correlated uncertainties

Linear χ^2 minimisation

⇒ Redefine clusters to have linear cross section

→ Fix covariance matrix with linear interpolants at each iteration (extrapolate at boundary)

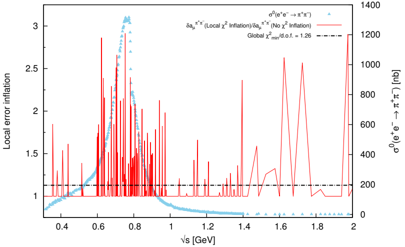
$$\chi^2 = \sum_{i=1}^{N_{\rm tot}} \sum_{j=1}^{N_{\rm tot}} \left(R_i^{(m)} - \mathcal{R}_m^i\right) \mathbf{C}^{-1}\left(i^{(m)}, j^{(n)}\right) \left(R_j^{(n)} - \mathcal{R}_n^j\right)$$

⇒ Through correlations and linearisation, result is the minimised solution of all neighbouring clusters

→ ...and solution is the product of the influence of all correlated uncertainties

⇒ The flexibly of the fit to vary due to the energy dependent, correlated uncertainties benefits the combination

→ ...and any data tensions are reflected in a local χ^2 error inflation

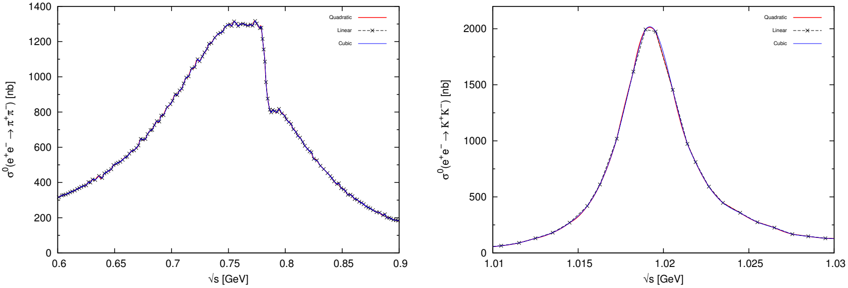


Integration

⇒ Trapezoidal rule integral

→ Consistency with linear cluster definition

→ High data population ∴ Accurate estimate from linear integral



→ Higher order polynomial integrals give (at maximum) differences of $\sim 10\%$ of error

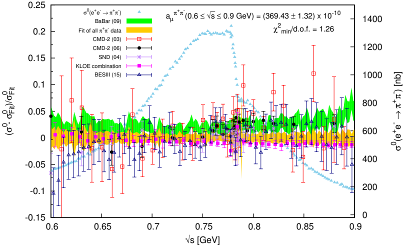
⇒ Estimates of error non-trivial at integral borders

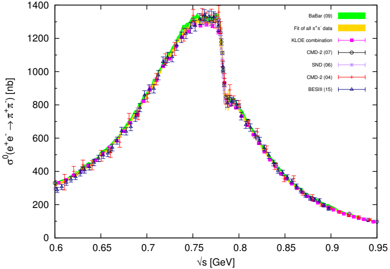
⟶ Extrapolate/interpolate covariance matrices

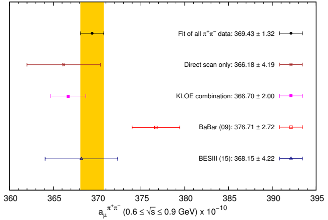
$\pi^+\pi^-$ channel [preliminary]

⇒ Large improvement for 2π estimate

→ BESIII [Phys.Lett. B753 (2016) 629-638] and KLOE combination provide downward influence to mean value







⇒ Correlated & experimentally corrected $\sigma^0_{\pi\pi(\gamma)}$ data now entirely dominant

$a_\mu^{\pi^+\pi^-}(0.305 \le \sqrt{s} \le 2.00$ GeV):

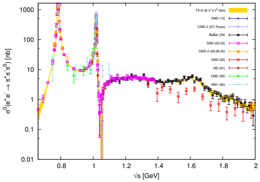
HLMNT11: 505.77 ± 3.09

KNT17: 502.85 ± 1.93

(no radiative correction uncertainties)

Other notable exclusive channels

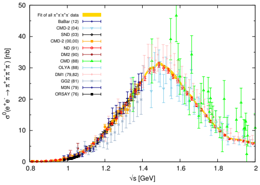
$\pi^+\pi^-\pi^0$



HLMNT11: 47.51 ± 0.99

KNT17: 47.92 ± 0.70

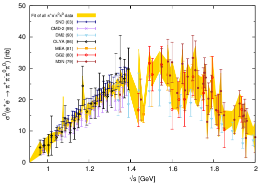
$\pi^+\pi^-\pi^+\pi^-$



HLMNT11: 14.65 ± 0.47

KNT17: 15.18 ± 0.14

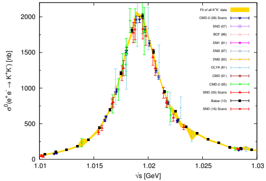
$\pi^+\pi^-\pi^0\pi^0$



HLMNT11: 20.37 ± 1.26

KNT17: 20.07 ± 1.19

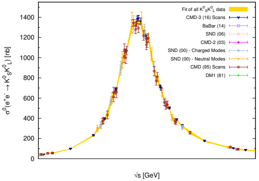
K^+K^-



HLMNT11: 22.15 ± 0.46

KNT17: 22.79 ± 0.25

$K^0_S K^0_L$



HLMNT11: 13.33 ± 0.16

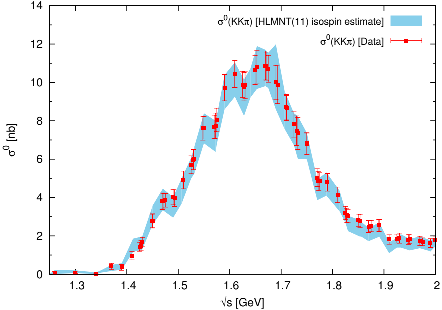
KNT17: 13.04 ± 0.12

$KK\pi$, $KK\pi\pi$ and isospin

⇒ New BaBar data for $KK\pi$ and $KK\pi\pi$ removes reliance on isospin (only $K^0_S = K^0_L$)

$KK\pi$

$K^0_S K^0_L \pi^0$ [Phys.Rev. D95 (2017), 052001]

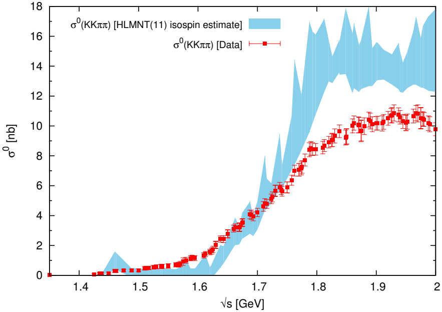


HLMNT11: 2.77 ± 0.15

KNT17: 2.82 ± 0.14

$KK\pi\pi$

$K^0_S K^0_L \pi^+\pi^-$ [Phys.Rev. D80 (2014), 092002]
$K^0_S K^0_S \pi^+\pi^-$ [Phys.Rev. D80 (2014), 092002],
$K^0_S K^0_L \pi^0\pi^0$ [Phys.Rev. D95 (2017), 052001]
$K^0_S K^\pm \pi^\mp \pi^0$ [arXiv:1704.05009]



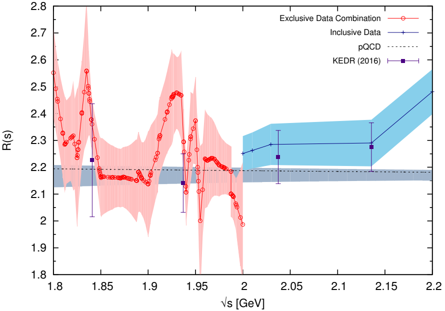
HLMNT11: 3.31 ± 0.58

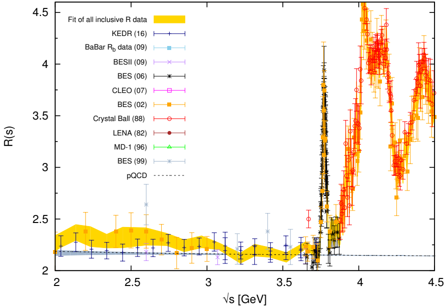
KNT17: 2.42 ± 0.09

⇒ **But**, still reliant on isospin estimates for $\pi^+\pi^-3\pi^0$, $\pi^+\pi^-4\pi^0$, $KK3\pi$...

Inclusive

⇒ New KEDR inclusive R data ranging $1.84 \leq \sqrt{s} \leq 3.05$ GeV [Phys.Lett. B770 (2017) 174-181] and $3.12 \leq \sqrt{s} \leq 3.72$ GeV [Phys.Lett. B753 (2016) 533-541]





$a_\mu^{\text{had, LOVP}}(1.84 \leq \sqrt{s} \leq 2.00 \text{ GeV})$:

pQCD : 6.42 ± 0.03

Data : 6.88 ± 0.25

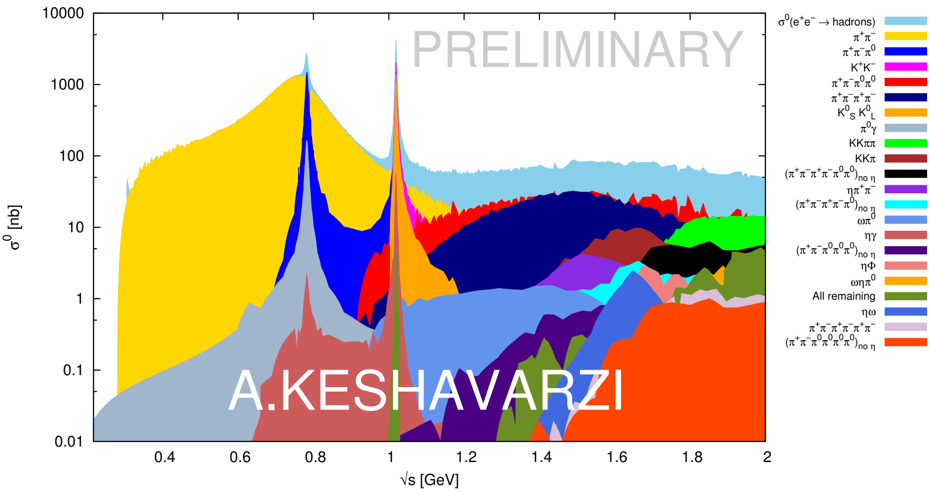
$a_\mu^{\text{had, LOVP}}(2.60 \leq \sqrt{s} \leq 3.73 \text{ GeV})$:

pQCD (inflated errors) : 10.82 ± 0.38

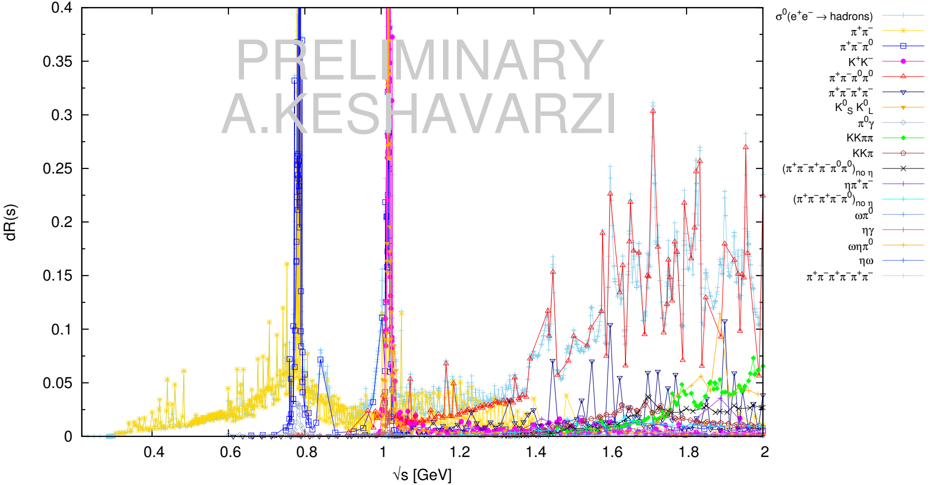
Data : 11.20 ± 0.14

$\Longrightarrow$ **Choose to adopt entirely data driven estimate from threshold to 11.2 GeV**

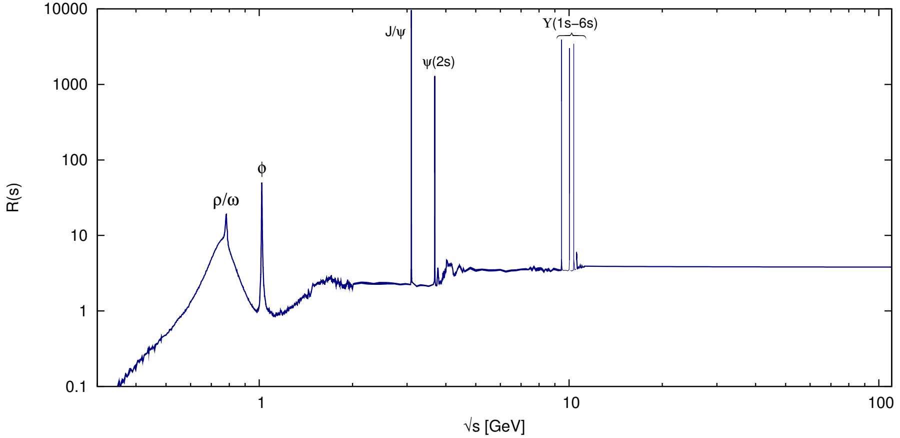
Contributions to mean value below 2GeV



Contributions to uncertainty below 2GeV



$R(s)$ for $m_\pi \leq \sqrt{s} < \infty$



$\Rightarrow$ Full compilation data set for hadronic R-ratio to be made available soon...

$\Longrightarrow$ **...complete with full covariance matrix**

KNT17 $a_\mu^{\text{had, VP}}$ and $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ update [preliminary]

$(g-2)_\mu$

HLMNT(11): 694.91 ± 4.27

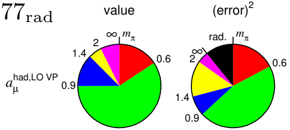
$\downarrow$

This work: $a_\mu^{\text{had, LOVP}} = 692.23 \pm 1.26_{\text{stat}} \pm 2.02_{\text{sys}} \pm 0.31_{\text{vp}} \pm 0.70_{\text{fsr}}$

$= 692.23 \pm 2.42_{\text{exp}} \pm 0.77_{\text{rad}}$

$= 692.23 \pm 2.54_{\text{tot}}$

$a_\mu^{\text{had, NLOVP}} = -9.83 \pm 0.04_{\text{tot}}$



$\Delta\alpha(M_Z^2)$

HLMNT11: $(276.26 \pm 1.38_{\text{tot}}) \times 10^{-4}$

$\downarrow$

This work: $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = (276.06 \pm 0.39_{\text{stat}} \pm 0.64_{\text{sys}} \pm 0.08_{\text{vp}} \pm 0.82_{\text{fsr}}) \times 10^{-4}$

$= (276.06 \pm 0.76_{\text{exp}} \pm 0.83_{\text{rad}}) \times 10^{-4}$

$= (276.06 \pm 1.13_{\text{tot}}) \times 10^{-4}$

$\Rightarrow$ Accuracy better then 0.4%
(uncertainties include all available correlations)

Full KNT17 VP package
[vp_knt_v3_0.f]
available for use

KNT17 vs. DHMZ17 vs. FJ17 [preliminary]

⇒ Different data treatment/methods produce **very** different results

Channel $\sqrt{s} \leq 1.8$ GeV	KNT17	DHMZ17	FJ17
$\pi^+\pi^-$	502.73 ± 1.94	507.14 ± 2.58	
$\pi^+\pi^-2\pi^0$	17.82 ± 0.99	18.03 ± 0.54	
$2\pi^+2\pi^-$	14.00 ± 0.20	13.68 ± 0.31	
K^+K^-	22.75 ± 0.26	22.81 ± 0.41	
$K_S^0K_L^0$	13.03 ± 0.20	12.82 ± 0.24	
Total HVP $\sqrt{s} < \infty$ GeV	692.23 ± 2.54	693.1 ± 3.4	689.43 ± 3.25

⇒ Between $1.8 \leq \sqrt{s} \leq 2$ GeV, KNT use data, DHMZ use pQCD

BUT, pQCD $= 8.30 \pm 0.09$, KNT data $= 8.42 \pm 0.29$, DHMZ data $= 7.71 \pm 0.32$

⇒ DHMZ17 use correlated systematics differently in determination of the mean value

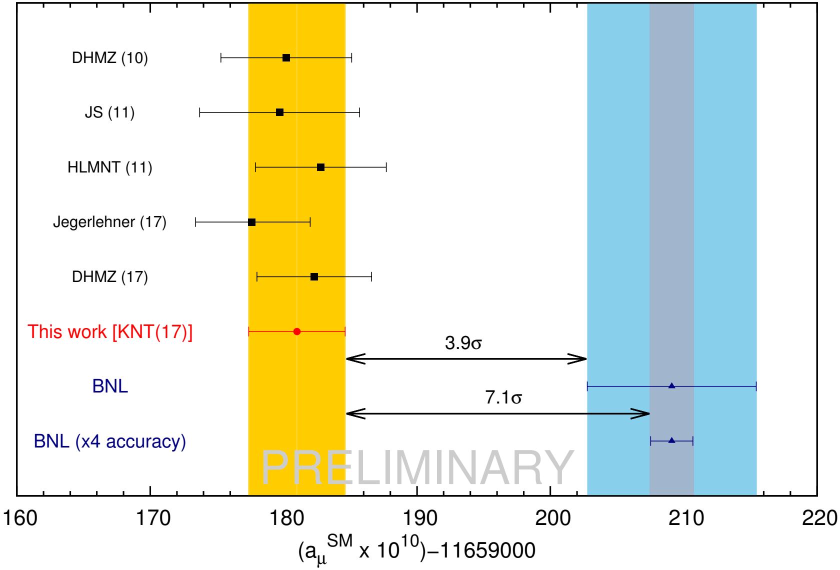
→ Determining $\pi^+\pi^-$ using only local weighted average gives 508.91 ± 2.84

⟶ Much better agreement when neglecting the effect of correlated uncertainties on the mean value

KNT17 a_μ^{SM} update [preliminary]

	2011		2017	
QED	11658471.81 (0.02)	$\longrightarrow$	11658471.90 (0.01)	[Phys. Rev. Lett. 109 (2012) 111808]
EW	15.40 (0.20)	$\longrightarrow$	15.36 (0.10)	[Phys. Rev. D 88 (2013) 053005]
LO HLbL	10.50 (2.60)	$\longrightarrow$	9.80 **(2.60)**	[EPJ Web Conf. 118 (2016) 01016]
NLO HLbL			0.30 (0.20)	[Phys. Lett. B 735 (2014) 90]
	HLMNT11		**KNT17**	
LO HVP	694.91 (4.27)	$\longrightarrow$	692.23 **(2.54)**	this work
NLO HVP	-9.84 (0.07)	$\longrightarrow$	-9.83 (0.04)	this work
NNLO HVP			1.24 (0.01)	[Phys. Lett. B 734 (2014) 144]
Theory total	11659182.80 (4.94)	$\longrightarrow$	11659181.00 (3.62)	this work
Experiment			11659209.10 (6.33)	world avg
Exp - Theory	26.1 (8.0)	$\longrightarrow$	28.1 (7.3)	this work
Δa_μ	3.3σ	$\longrightarrow$	3.9σ	this work

KNT17 a_μ^{SM} update [preliminary]



Conclusions

- ✓ Can use dispersive techniques to calculate HVP and HLbL contributions to a_μ
- ✓ Many necessary changes made in order to improve data combination

⇒ When combining data...

  - ✓ ...all covariance matrices are correctly constructed with a framework that can accommodate any available correlations
  - ✓ ...employ a linear χ^2 minimisation that has been shown to be free from bias

- ✓ New method shows improvements in all channels due to increased fit flexibility
- ✓ Less reliance on isospin for estimated states with more measured final states
- ✓ $a_\mu^{\mathrm{had,LOVP}}$ accuracy better than 0.4%
- ✓ $(g-2)_\mu$ theory initiative to collaborate to produce single estimate for a_μ^{SM}

Eagerly awaiting new FNAL result!

Extra Slides

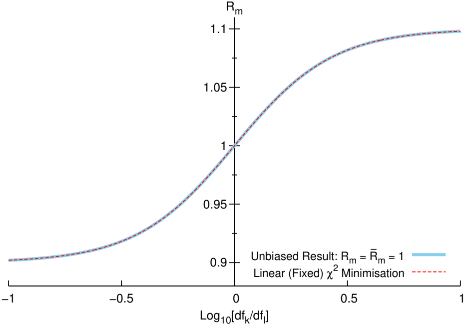
Fixing the covariance matrix [JHEP 1005 (2010) 075, Eur.Phys.J. C75 (2015), 613]

⇒ Apply a procedure to fix the covariance matrix

$$\mathbf{C}_I\left(i^{(m)}, j^{(n)}\right) = \mathsf{C}^{\mathsf{stat}}\left(i^{(m)}, j^{(n)}\right) + \frac{\mathsf{C}^{\mathsf{sys}}\left(i^{(m)}, j^{,n)}\right)}{R_i^{(m)} R_j^{(n)}} R_m R_n \quad ,$$

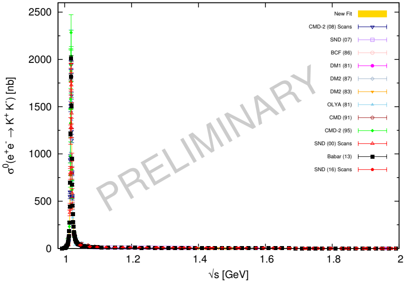
in an iterative χ^2 minimisation method that, to our best knowledge, is free from bias

- ⇒ Fixing with theory value regulates influence
- ⇒ Can be shown from toy models to be free from bias
- ⇒ Swift convergence
- ⇒ Comparison with past results shows HLMNT11 estimates are largely unaffected



Allows for increased fit flexibility and full use of energy dependent, correlated uncertainties

Kaon FSR study



Inclusive FSR correction was previously applied to K^+K^- cross section

KLN theorem requires all virtual and soft corrections necessarily included in given cross section

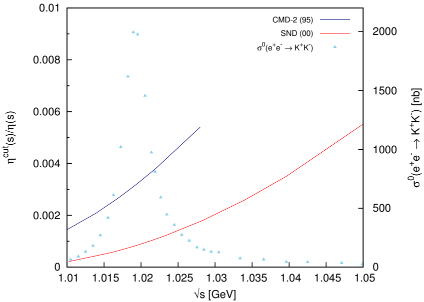
$\therefore$ Only hard real radiation is left to be corrected for

BUT K^+K^- cross section is totally dominated by ϕ resonance

$\Rightarrow$ No phase space for creation of hard real photons at ϕ

Inclusive FSR correction is large over-correction $\longrightarrow$

$\therefore$ No longer apply FSR correction



Properties of a covariance matrix

Any covariance matrix, $\mathcal{C}_{ij}$, of dimension $n \times n$ must satisfy the following requirements:

- As the diagonal elements of any covariance matrix are populated by the corresponding variances, all the diagonal elements of the matrix are positive. Therefore, the trace of the covariance matrix must also be positive

$$\text{Trace}(\mathcal{C}_{ij}) = \sum_{i=1}^{n} \sigma_{ii} = \sum_{i=1}^{n} \text{Var}_i > 0$$

- It is a symmetric matrix, $\mathcal{C}_{ij} = \mathcal{C}_{ji}$, and is, therefore, equal to its transpose, $\mathcal{C}_{ij} = \mathcal{C}_{ij}^T$
- The covariance matrix is a positive, semi-definite matrix,

$$\mathbf{a}^T \mathcal{C}\ \mathbf{a} \geq 0\ ;\ \mathbf{a} \in \mathbf{R^n},$$

  where $\mathbf{a}$ is an eigenvector of the covariance matrix $\mathcal{C}$
- Therefore, the corresponding eigenvalues $\lambda_{\mathbf{a}}$ of the covariance matrix must be real and positive and the distinct eigenvectors are orthogonal

$$\mathbf{b}\ \mathcal{C}\ \mathbf{a} = \lambda_{\mathbf{a}}(\mathbf{b} \cdot \mathbf{a}) = \mathbf{a}\ \mathcal{C}\ \mathbf{b} = \lambda_{\mathbf{b}}(\mathbf{a} \cdot \mathbf{b})$$

$$\therefore \text{if } \lambda_{\mathbf{a}} \neq \lambda_{\mathbf{b}} \Rightarrow (\mathbf{a} \cdot \mathbf{b}) = 0$$

- The determinant of the covariance matrix is positive: $\text{Det}(\mathcal{C}_{ij}) \geq 0$

Tests of reliability of f_k method

Did the f_k method incur a bias?

Compare f_k method and fixed matrix method with only multiplicative normalisation uncertainties.

$\rightarrow$ If we see **differences** in mean value, then bias previously influenced the fit.
$\longrightarrow$ Previous results unreliable

$\rightarrow$ If we see **no differences** in mean value, then bias did not influence fit (any change comes from improved treatment of systematics)
$\longrightarrow$ Previous results reliable

Example - $\pi^+\pi^-$
Set 1 - CMD-2(06) (0.7% Systematic Uncertainty), Set 2 - CMD-2(06) (0.8% Systematic Uncertainty), Set 3 - SND(04) (1.3% Systematic Uncertainty)

From $0.37 \rightarrow 0.97$ GeV

Fit Method:	f_k method		Fixed matrix method		
Channel	a_μ	χ^2_{min}/d.o.f.	a_μ	χ^2_{min}/d.o.f.	Difference
$\pi^+\pi^-$	481.42 ± 4.26	1.10	481.42 ± 4.05	1.02	0.00