Newton-Cartan trace anomaly and non-relativistic RG flow

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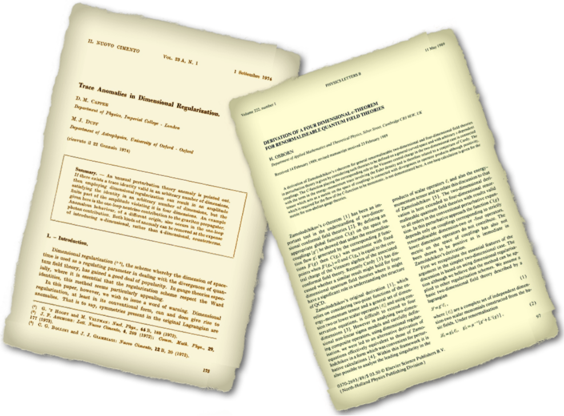
R.Auzzi, S.Baiguera, G.N., JHEP 1602 (2016) 003; *Erratum*: JHEP 1602 (2016) 177

R.Auzzi, G.N., JHEP 1607 (2016) 047

R.Auzzi, S.Baiguera, F.Filippini, G.N., JHEP 1611 (2016) 163

Main Purpose

Two “milestones” in the study of anomalies and local RG equations (relativistic case)



What’s the analog in the non relativistic case?

Weyl Invariance and Weyl Anomaly

$$\text{Let} \quad e^{iW[g]} = \int \mathcal{D}\phi \, e^{iS[g,\phi]}$$

CLASSICAL CASE: Weyl Invariance $S[g,\phi] = S[g',\phi']$, under

$g'_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x), \quad \phi'(x) = e^{\alpha\sigma(x)}\phi(x)$ implies

$$T^{\mu}_{\ \mu} = -\frac{2g^{\mu\nu}}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0$$

QUANTUM CASE: The corresponding vev does not vanish anymore (anomaly)

$$\langle T^{\mu}_{\ \mu} \rangle = -\frac{2g^{\mu\nu}}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}} \neq 0$$

Weyl anomaly in 2 dimensions

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{c}{24\pi} R$$

Zamolodchikov's c-theorem (1986):

There exists a function $C(g_i)$ of the couplings g_i such that

- It decreases monotonically along the RG flow
- At the fixed points g_i^*, $C(g_i^*) = c$ equals the central charge of the CFT (the same number entering the trace anomaly)

Originally derived from CFT correlators, the same C function appears also as *entanglement entropy* and as *euclidean partition function on the sphere*.

Road to $D = 4$: Trace Anomaly

Trace Anomaly was not immediately recognized from the scientific community
Capper and Duff (1974) had some troubles at first.

The reaction of physicists went along the *canonical* three steps

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- **It's wrong!**
- **It's trivial!**
- **I thought of it first!**

Trace Anomaly in $D = 4$

$$\langle T^{\mu}{}_{\mu} \rangle = -aE_4 + bW^2 + eR^2 + a' \nabla^2 R \,,$$

$$W^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$$

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

- eR^2 is zero at the conformal fixed point
- $\nabla^2 R$ terms can be eliminated by the addition of suitable local counterterm in the action (*e.g.* $\sqrt{g}R^2$)
- In the free theory, $a = \frac{1}{90(8\pi)^2}(n_S + 11n_F + 62n_V)$

In 1988 Cardy proposed the even dimensional generalization of the c-Theorem: the a-Theorem
In 1989 it was given a perturbative proof by Osborn
There exists two versions:
Assuming two fixed points in the UV and IR,

- Strong: $a(g_i)$ decreases along the RG flow
- Weak: $a_{UV} > a_{IR}$

Such a function (initially identified as the coefficient of the E_4 term in the trace anomaly), was found in several different contexts, revealing therefore its *universal character*.

Tools to study irreversibility properties of RG flows: Timeline

- Conjecture of *a*-Theorem (Cardy 1988)
- Perturbative proof (Osborn 1989)
- In SUSY it is related to *R*-symmetry (Anselmi, Freedman, Grisaru and Johansen, 1998; *a*-max, Intriligator and Wecht, 2003)
- Holographic version (Freedman, Gupser, Pilch and Warner 1999)
- Related to entanglement entropy (Solodukhin 2008, Casini, Huerta, Myers, 2011)
- Related to the euclidean partition function on the sphere (Jefferis, Klebanov, Pufu and Safdi 2011)
- Non perturbative proof of the weak version of the theorem (Komargodski and Schwimmer 2011)

$a = n_S + 11 n_F + 62 n_V$ counts the number massless degrees of freedom

Example: QCD with N_c colors and N_f flavours

- $a_{UV} = 11 N_c N_f + 62(N_c^2 - 1) = \#$ massless d.o.f. in UV
- after CSB, in IR are expected $a_{IR} = N_f^2 - 1$ Goldstone bosons
- *a*-Theorem constraints $N_f - \frac{11}{2} N_c < \sqrt{(11 N_c/2)^2 + 62(N_c^2 - 1) + 1}$
- But for asymptotic freedom $N_f - \frac{11}{2} N_c < 0$

Cohomology (WZCC)

Anomalies must satisfy Wess-Zumino consistency conditions:

$$\Delta_\sigma^{\mathcal{W}} W \sim \langle T^\mu{}_\mu \rangle = -aE_4 + bW^2 + eR^2 + a'\nabla^2 R$$

$$[\Delta_\sigma^{\mathcal{W}}, \Delta_{\sigma'}^{\mathcal{W}}] W = 0 \quad \text{(WZCC)}$$

This eliminates R^2 term at conformal fixed point,

Besides...

Cohomology (WZCC)

WZCC permits a natural classification of anomalies

$$\langle T^{\mu}{}_{\mu}\rangle = -aE_4 + bW^2 + a'\nabla^2 R$$

- **type A** anomaly: non-trivial solution of WZCC and **cannot** be eliminated by local counterterm (like E_4))
- **type A'** anomaly: non-trivial solution of WZCC but **can** be eliminated by a local counterterm (like $\nabla^2 R$)
- **type B** anomaly: trivial solution of WZCC (like W^2)

Is there a non-relativistic version?

In order to inspect these issues, we should couple the non-relativistic theory to a non-relativistic version of gravity

Gravity here is just a source for energy-momentum tensor:

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}}$$

The kind of non-relativistic gravity depends on the symmetry of the non-relativistic theory: **Lifshitz** or **Schrödinger**

With or without boost

Dynamical exponent: Time and Space scale in a different way:

$$t \to \lambda^z t \,, \qquad x \to \lambda x \,.$$

- Without boost: studied in detail for various d, z (Arav, Chapman and Oz, 2014). All anomalies are type B
- With boost: anomalies are type B, plus at least one type A anomaly, for $d = z = 2$ (Jensen 2014)

Newton-Cartan gravity

Introduced in 1923 by Cartan to formulate Newtonian gravitation in a frame independent way

- introduce a degenerate "spatial metric" $h^{\mu\nu}$
- 1-form $n = n_\mu dx^\mu$ (local time direction), $n_\mu h^{\mu\nu} = 0$
- vector field v^μ, such that $v^\mu n_\mu = 1$
- this defines $h_{\mu\nu}$ such that $v^\mu h_{\mu\nu} = 0$, $h^{\mu\alpha} h_{\alpha\nu} = \delta^\mu_\nu - v^\mu n_\nu$
- background gauge field for particle number symmetry A_μ

  (arising from the ambiguity $v^\mu \to v^\mu + h^{\mu\nu} A_\nu$)

Milne boost invariance

Invariance under local non-relativistic boost transforms v^μ, $h_{\mu\nu}$, A_μ non-trivially (n_μ and $h^{\mu\nu}$ are left invariant)

$$v'^\mu = v^\mu + h^{\mu\nu}\psi_\nu$$

$$h'_{\mu\nu} = h_{\mu\nu} - (n_\mu P^\rho_\nu + n_\nu P^\rho_\mu)\psi_\rho + n_\mu n_\nu h^{\rho\sigma}\psi_\rho\psi_\sigma$$

$$A'_\mu = A_\mu + P^\rho_\mu\psi_\rho - \frac{1}{2}n_\mu h^{\alpha\beta}\psi_\alpha\psi_\beta$$

$$P^\mu_\nu = \delta^\mu_\nu - v^\mu n_\nu = h^{\mu\alpha}h_{\alpha\nu}$$

Symmetries of the NC geometry

All together, NC gravity encodes the following symmetries:

- local coordinate transformations (parametrized by ξ^α)
- Milne boosts (parametrized by ψ_ρ)
- $U(1)$ gauge transformations (parametrized by Λ)

$$\delta h^{\mu\nu} = \mathcal{L}_{\xi^\alpha} h^{\mu\nu}$$

$$\delta n_\mu = \mathcal{L}_{\xi^\alpha} n_\mu$$

$$\delta h_{\mu\nu} = \mathcal{L}_{\xi^\alpha} h_{\mu\nu} - 2n_{(\mu} P^\rho_{\nu)} \psi_\rho$$

$$\delta v^\mu = \mathcal{L}_{\xi^\alpha} v^\mu + h^{\mu\nu} \psi_\nu$$

$$\delta A_\mu = \mathcal{L}_{\xi^\alpha} A_\mu + P^\rho_\mu \psi_\rho + \partial_\mu \Lambda$$

Sources of the energy momentum tensor multiplet

The independent components of NC metric provide the sources for stress tensor, number current, energy current, momentum density

$$\delta W = \int d^3x \sqrt{-g} \left(\frac{1}{2} T_{ij} \delta \tilde{h}_{ij} + j^\mu \delta A_\mu - \epsilon^\mu \delta n_\mu - p_i \delta u_i \right) ,$$

Null reduction is a formidable instrument to deal with these symmetries: introducing the metric,

$$G_{MN} = \begin{pmatrix} 0 & n_\mu \\ n_\nu & n_\mu A_\nu + n_\nu A_\mu + h_{\mu\nu} \end{pmatrix}$$

$$G^{MN} = \begin{pmatrix} A^2 - 2v \cdot A & v^\mu - h^{\mu\sigma} A_\sigma \\ v^\nu - h^{\nu\sigma} A_\sigma & h^{\mu\nu} \end{pmatrix}$$

and the null vector,

$$n^M = (1, 0, \ldots) \,, \quad n_M = (0, n_\mu)$$

any NC scalar can be written in terms of the scalars of the extra dimensional space that can be build with G and n (Duval, Kunzle 1984).

Causality

The form $n = n_\mu dx^\mu$ gives the local time direction

Frobenius condition

$$dn \wedge n = 0\,,$$

it is equivalent integrability of time slices

If it not satisfied,
no absolute notion of future and past,
causality is lost

(whether it is needed, depends on the circumstances)

No Frobenius, the anomaly has infinite number of terms

$d = 2$, $z = 2$ case, (Jensen, 2014):

$$\mathcal{A} = \langle T^i_i - 2\epsilon^0 \rangle = -aE_4 + bW^2 + eR^2 + a'\nabla^2 R + \dots$$

the same expression as for the relativistic $3 + 1$ anomaly, once null reduction has been performed

$+ \dots$ stands for an **infinite number** of higher derivative terms, such as:

$$\dots + W_{ABCP} W^{ABCQ} W^P{}_{MQN} n^M n^N + \dots,$$

(for each n^A, we buy 1 more derivative)

Classification of anomaly terms

The infinite terms entering the anomaly can be conveniently classified [Auzzi, Baiguera, Filippini and N., 2016]

- they can be arranged into (infinite) distinct sectors closed under WT (labelled by an integer N_t)
- each sector contains a *finite* number of terms, and an iterative procedure identifies the independent ones
- WZCC can be studied separately in each sector
- $N_t = 0$. Usual terms, $-aE_4 + bW^2 + a'\nabla^2 R$
- $N_t = 1$. In principle 86 terms are possible, reduced to 3 independent by the iterative procedure, and reduced to 0 by the WZCC: **no anomaly in the $N_t = 1$ sector**

With Frobenius, type B

$$dn \wedge n = 0\,,$$

The terms in the anomaly collapse from ∞ to 6

$$\mathcal{A} = \langle T^i_i - 2\epsilon^0 \rangle = bW^2 + \text{ local counterterms}$$

only one type B anomaly

(Auzzi, Baiguera, N., 2015; Arav, Chapman, Oz, 2016)

A free scalar in curved NC space

Explicit evaluation for a scalar (Auzzi and N., 2016)

$$\int d^{d+1}x\sqrt{g}\left\{imv^{\mu}\left(\phi^{\dagger}D_{\mu}\phi - D_{\mu}\phi^{\dagger}\phi\right) - h^{\mu\nu}D_{\mu}\phi^{\dagger}D_{\nu}\phi - \xi R\phi^{\dagger}\phi\right\} ,$$

$$D_{\mu}\phi = \partial_{\mu}\phi - imA_{\mu}\phi\,, \quad g = det(h_{\mu\nu} + n_{\mu}n_{\nu})$$

Conformal coupling:

$$\xi = \frac{1}{6}$$

Heat kernel method

Heat kernel method: $\triangle = \triangle_t + \partial_i^2 = -2m\sqrt{-\partial_t^2} + \partial_i^2$

$$e^{\triangle_t} = e^{-2m\sqrt{-\partial_t^2}} = \int_0^\infty \frac{d\sigma}{\sigma} \frac{m}{\sqrt{\pi\sigma}} e^{-\frac{m^2}{\sigma}} e^{-\sigma(-\partial_t^2)}$$

In flat spacetime:

$$\langle x, t | e^{s\triangle} | y, t' \rangle = \frac{1}{2\pi} \frac{ms}{m^2 s^2 + \frac{(t-t')^2}{4}} \frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{(x-y)^2}{4s}\right)$$

A relativistic (euclidean) scalar in D dimension would give:

$$\langle x | e^{s\Box} | y \rangle = \frac{1}{(4\pi s)^{D/2}} \exp\left(-\frac{(x-y)^2}{4s}\right)$$

$$\tilde{K}_\triangle(s) = \langle x, t | e^{s\triangle} | x, t \rangle = \frac{2}{m(4\pi s)^{1+d/2}}\,.$$

The same spectral dimension as relativistic scalar in $d+2$ dimensions !!

Heat kernel in curved spacetime

Seeley-DeWitt expansion:

$$\tilde{K}_M(s) = \frac{1}{s^{d/2+1}} \left(a_0 + a_2 s + a_4 s^2 + \dots \right) .$$

a_4 is proportional to the Weyl anomaly

Result:

$$\langle T^i_i - 2\epsilon^0 \rangle = \frac{1}{8m\pi^2} \left(-\frac{1}{360} E_4 + \frac{3}{360} W^2 + \right.$$

$$\left. + \frac{1}{2} \left(\xi - \frac{1}{6} \right)^2 R^2 + \frac{1 - 5\xi}{30} \nabla^2 R \right) + \dots ,$$

A Non-Relativistic anomaly can be obtained as IR limit of a Relativistic one?

Relativistic vs. Non-Relativistic

A Non-Relativistic anomaly can be obtained as IR limit of a Relativistic one?

NO

- The conformal symmetries are indeed different in the two cases ($z = 2$ and $z = 1$). The relativistic trace anomaly coefficient counts the number of massless degrees of freedom, which is zero for a $z \neq 1$ fixed point
- The relativistic scale symmetry is *broken* by the mass gap necessary to have a non relativistic limit
- The non-relativistic scale symmetry is an IR emergent phenomenon which does not exist in the far relativistic UV

A non-relativistic a-theorem ?

Consequence: the meaning of the non relativistic *a*-theorem (if it exists) is certainly different from the relativistic one

A free scalar contributes to the non-relativistic *a*-anomaly with

$$\frac{1}{8m\pi^2}\,\frac{1}{360}$$

If the UV and IR fixed points contain just scalars, the non relativistic a-theorem would tell that:

$$\sum_k^{UV} \frac{1}{m_k} \geq \sum_k^{IR} \frac{1}{m_k}$$

Future directions

- Try to check the conjecture in some examples
- Heat kernel calculation for fermions
- Is there any interesting SUSY story, as in a-maximization?
- Osborn: positivity of the metric is needed for a perturbative proof
- Non perturbative proof: what is the counterpart of the dispersion relations a la Komargodski and Schwimmer?
- What about anyons?

Thank you!