# 150 years of Maxwell's (other) equations and application to plasma acceleration

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# What this talk is about

Showing that an approach invented by Maxwell 150 years ago is until today important in science

- Introducing Mawell’s original “equations of matter”
- Using this moment approach to model hosing in PWFAs

# J. Clerk Maxwell (1831-1879)

Maxwell eqns of electromagnetism[1]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Maxwell moment eqns of matter for gases[2], Maxwellian equilibrium distribution function

Boltzmann kinetic eqn[3] for non-equil $f(\mathbf{r},\mathbf{v},t)$, *H*-theorem, Second Law

- Maxwell's *equations of matter* adapted ~100 years later to low temperature plasmas[4], and ~ 150 years later highly relativistic beams in <u>plasma accelerators</u>[5]

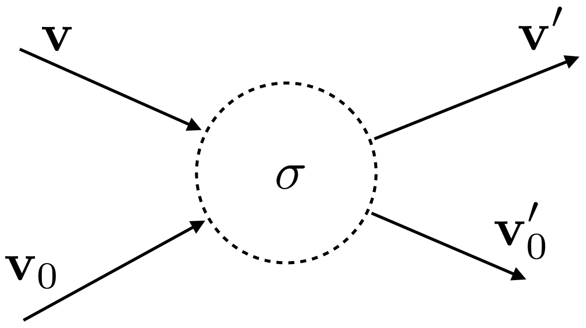

1. J.C. Maxwell, *Phil. Trans. Roy. Soc. London* 155, 459 (1865)
2. J.C. Maxwell, *Phil. Trans. Roy. Soc. London* 157, 49 (<u>1867</u>)
3. L. Boltzmann, Wien. Bericht 66, 275 (1872)
4. E.A. Mason and E.W. McDaniel, "Transport properties of ions in gases" (Wiley, New York, 1988)
5. R.E. Robson, T.J. Mehrling and J. Osterhoff, accepted for publication in Europ. J. Phys. (2017)

# Maxwell's original "equations of matter"

Velocity distribution $f\left(\boldsymbol{r},\mathbf{v},t\right)$

Moments $\quad \overline{Q}=\frac{1}{n}\int d^3v\,Q(\mathbf{v})\,f\left(\boldsymbol{r},\mathbf{v},t\right) \qquad Q(\mathbf{v})=1, m\mathbf{v}, \frac{1}{2}mv^2, \ldots$

Collisions interaction potential V(r)




$$\boxed{\frac{\partial\left(n\overline{Q}\right)}{\partial t}=-\nabla\cdot\Gamma_Q+\frac{n}{m}\overline{\boldsymbol{F}\cdot\nabla_{\mathrm{v}}Q(\mathbf{v})}+n\overline{\left.\frac{\partial Q}{\partial t}\right)}_{col}}$$

Collision term simplifies for V(r) ~ r$^{-4}$ (Maxwell model)

$$\textit{Number density}: \; n\left(\boldsymbol{r},t\right)=\int d^3v\,f\left(\boldsymbol{r},\mathbf{v},t\right) \qquad \textit{Flux of property } Q: \; \Gamma_Q=n\,\overline{\mathbf{v}Q}$$

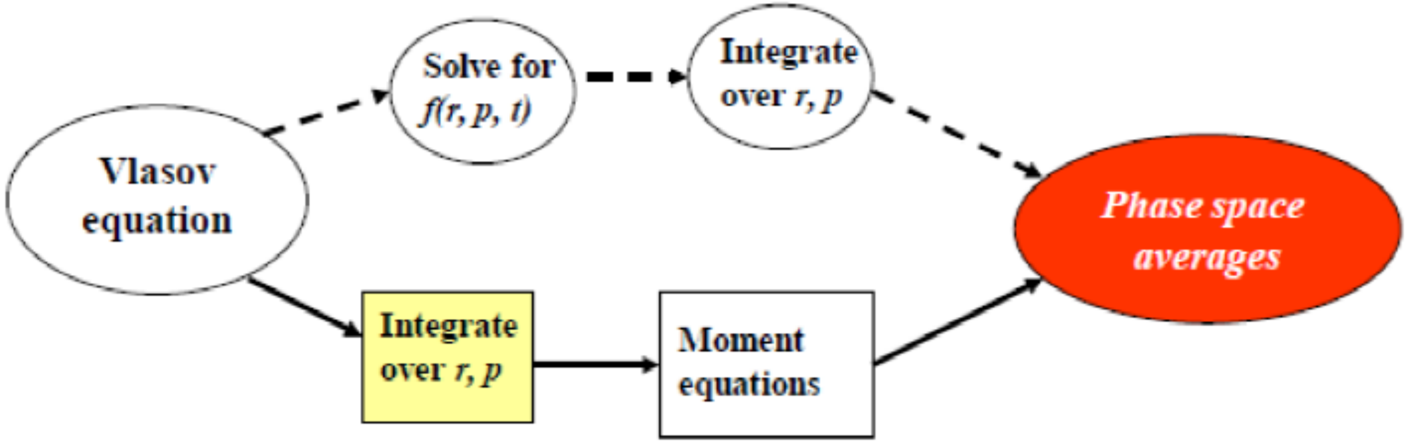
$$Q=1\rightarrow \qquad \frac{\partial n}{\partial t}+\nabla\cdot n\,\overline{\mathbf{v}}=0$$

$$Q=m\mathbf{v}\;\rightarrow\; \frac{\partial(nm\,\overline{\mathbf{v}})}{\partial t}=-\nabla\cdot(nm\,\overline{\mathbf{v}\mathbf{v}})+n\,\boldsymbol{F}-n\mu\nu_m\overline{\mathbf{v}}$$

Modern <u>fluid equations</u> are moment equations of the same *mathematical form* as Maxwell's original expressions: Robson et al, Rev. Mod. Phys. Rev Mod Phys 77, 1303-20 (2005)

# Maxwells moment approach for relativistic systems

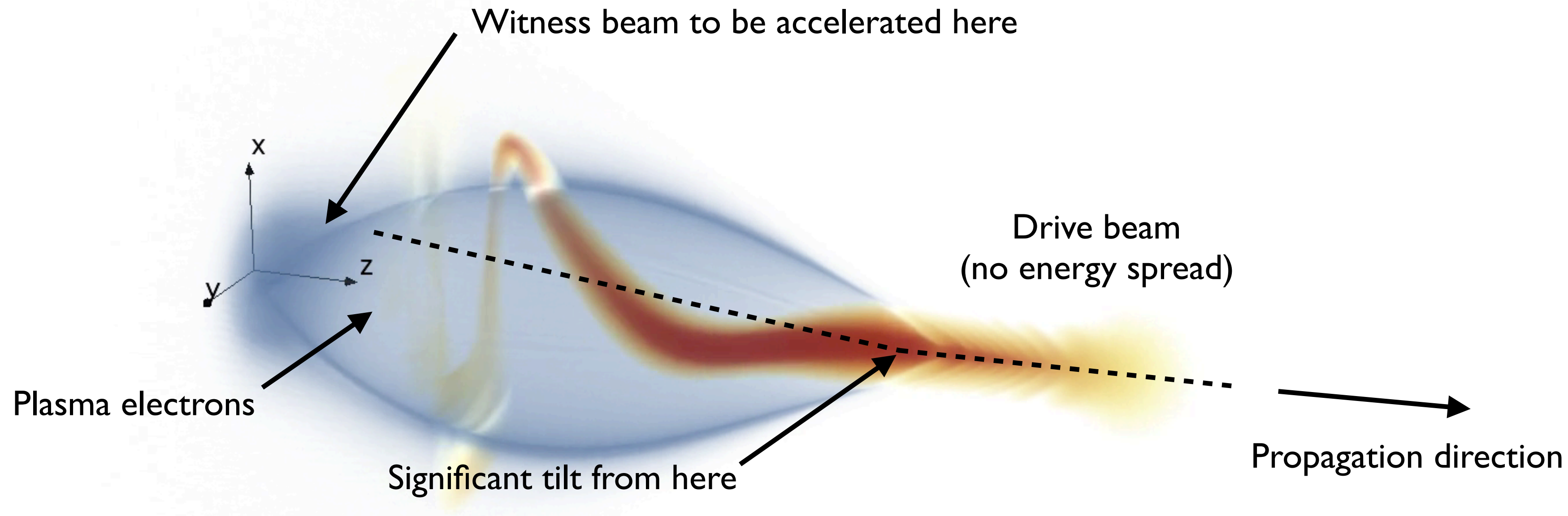
- $(\partial_t + \mathbf{v}.\nabla + \mathbf{F}.\,\partial_{\mathbf{p}})f = (\partial_t f)_{\text{collisions}} \approx 0$ (Vlasov kinetic equation, $\mathbf{v} = \mathbf{p}/m\gamma$) (1)
- Phase space averages $<\phi(\boldsymbol{r},\boldsymbol{p})> = \frac{1}{\mathcal{N}}\int d\,(\textit{phase space})\, f(\boldsymbol{r},\mathbf{v},t)\,\phi\,(\boldsymbol{r},\boldsymbol{p})$
- $\int d\,(\textit{phase space})\ \phi_i\,(\boldsymbol{r},\boldsymbol{p}) \times \textit{Vlasov eqn} \rightarrow$ moment equations for $<\phi_i(\boldsymbol{r},\boldsymbol{p})>$ $(i=1,2,3,\,..)$
- Traditional approach: First find $f(\boldsymbol{r},\mathbf{v},t)$ from (1) or PIC simulation and then form moments -more accurate, but less efficient




- Maxwell method: Solve for moments directly - requires closure Ansatz

# Example: Application of moment approach on hosing in PWFAs

**Hosing is a challenge and it is vital to study the connected dynamics**




HiPACE

**Plasma wakefield acceleration with a tilted beam**

3D PIC simulation using the code HiPACE

**Hosing is a challenge!**

- Initial beam asymmetry is amplified
- Is beam breakup inevitable?

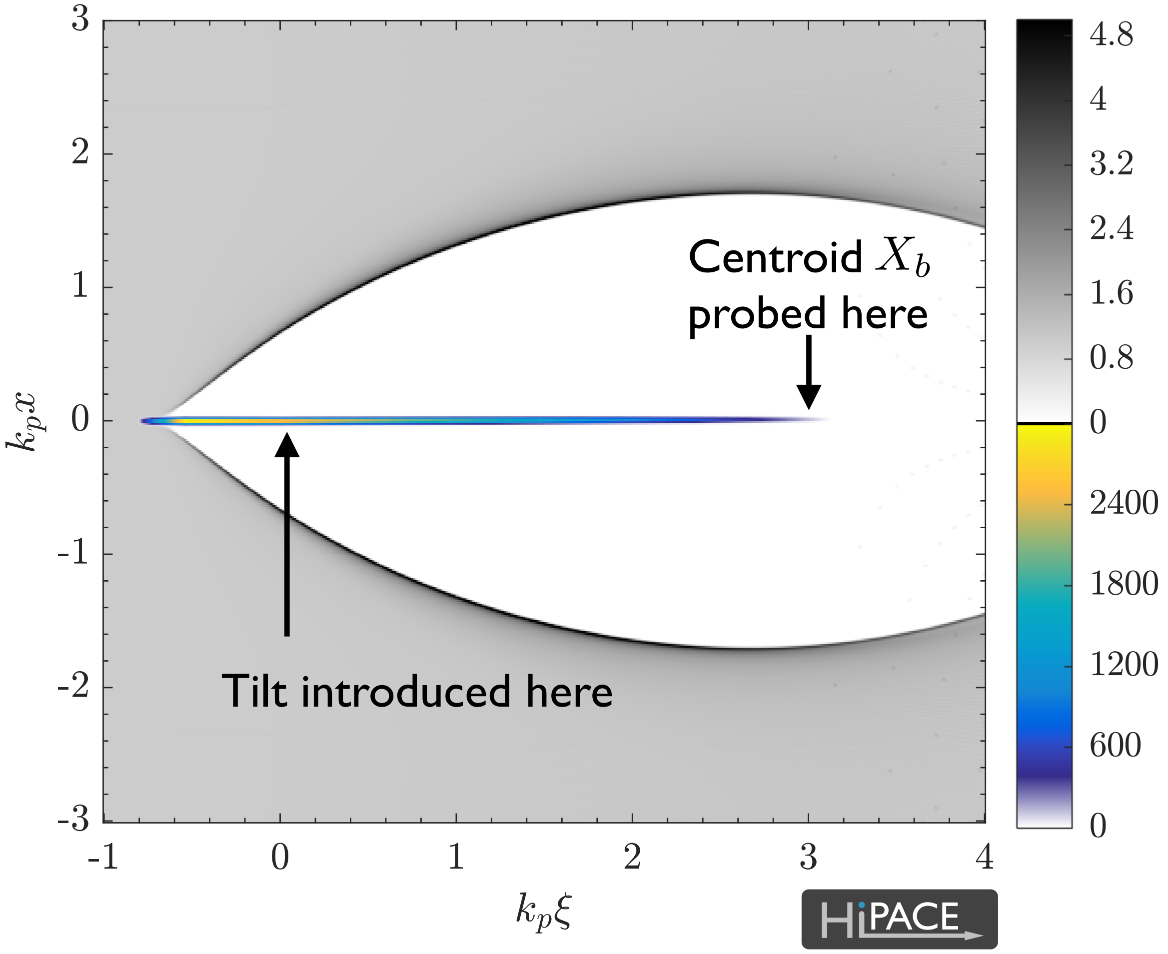
# Describing dynamics of plasma sheath electrons in hosing

Tilted beam with triangular current profile

Init. centroid: $X_{b,0}(\xi) = 4.17 \times 10^{-3} \times \xi\, \Theta(k_p \xi)$

Energy: 28.5 GeV; Peak current: 6.65 kA

Setup as in C. Huang, *et al.* PRL **99**, 255001 (2007).



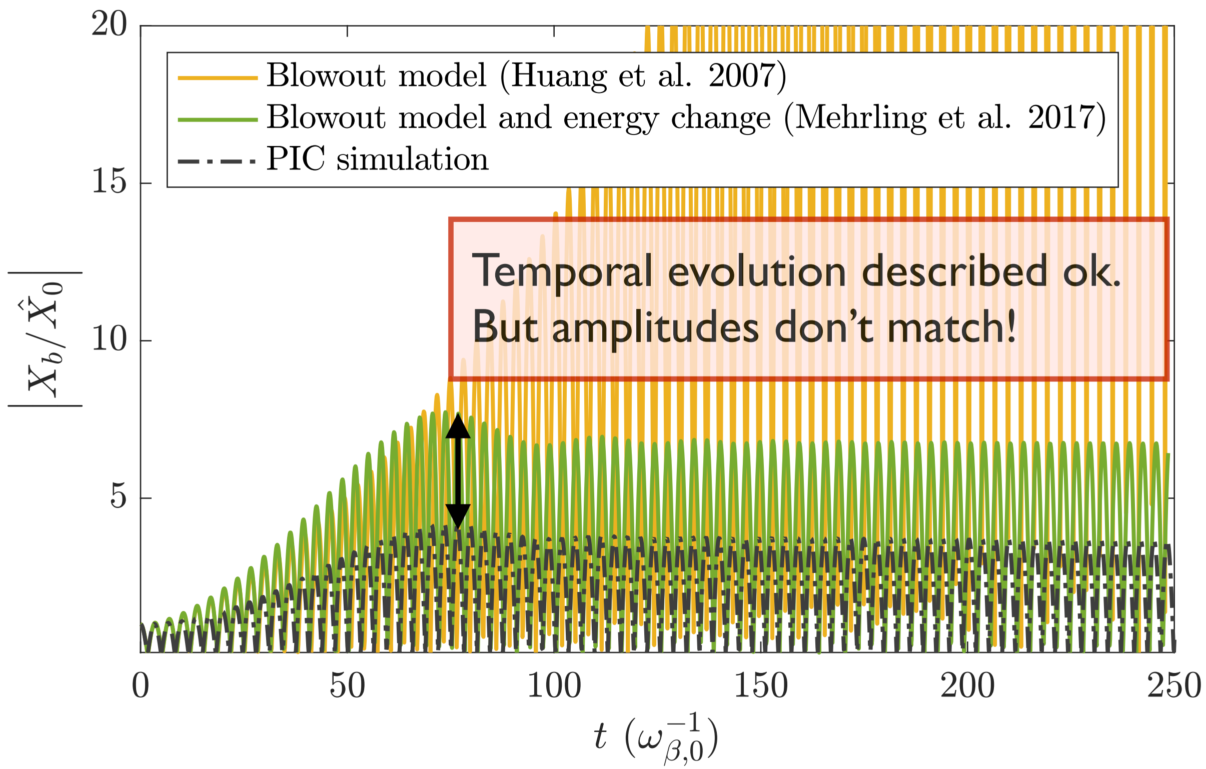

**Current description of hosing in the blowout**

$$\frac{\partial^2 X_c}{\partial \xi^2} + \frac{k_p^2 c_\psi(\xi) c_r(\xi)}{2}(X_c - X_b) = 0$$

*C. Huang et al., PRL 99, 255001 (2007).

$$\frac{\partial^2 X_b}{\partial t^2} + \lambda(\xi, t)\frac{\partial X_b}{\partial t} + \Omega^2(\xi, t)(X_b - X_c) = 0$$

**T. Mehrling et al., PRL 118, 174801 (2017).



# Describing dynamics of plasma sheath electrons in hosing

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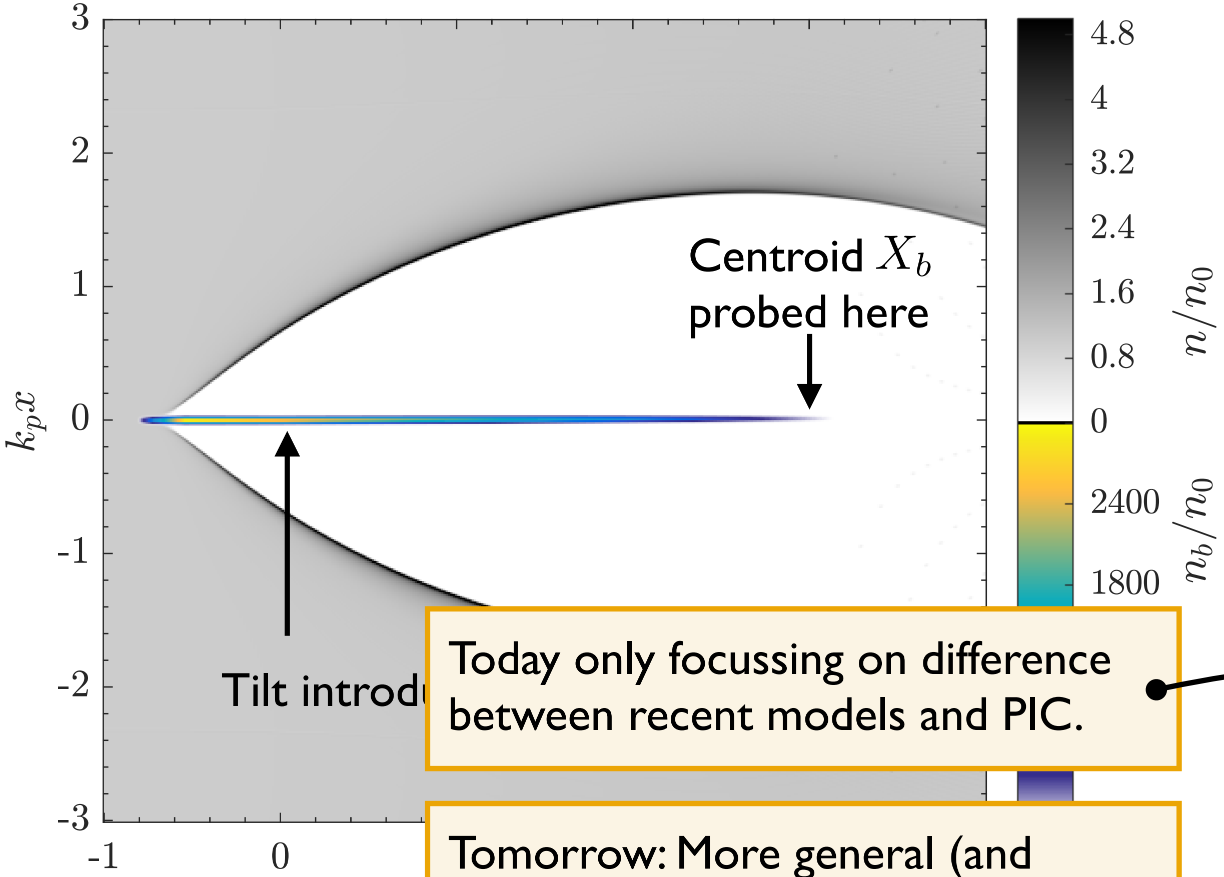


Today only focussing on difference between recent models and PIC.

Tomorrow: More general (and instructive) talk on hosing at 12:00

**Current description of hosing in the blowout**

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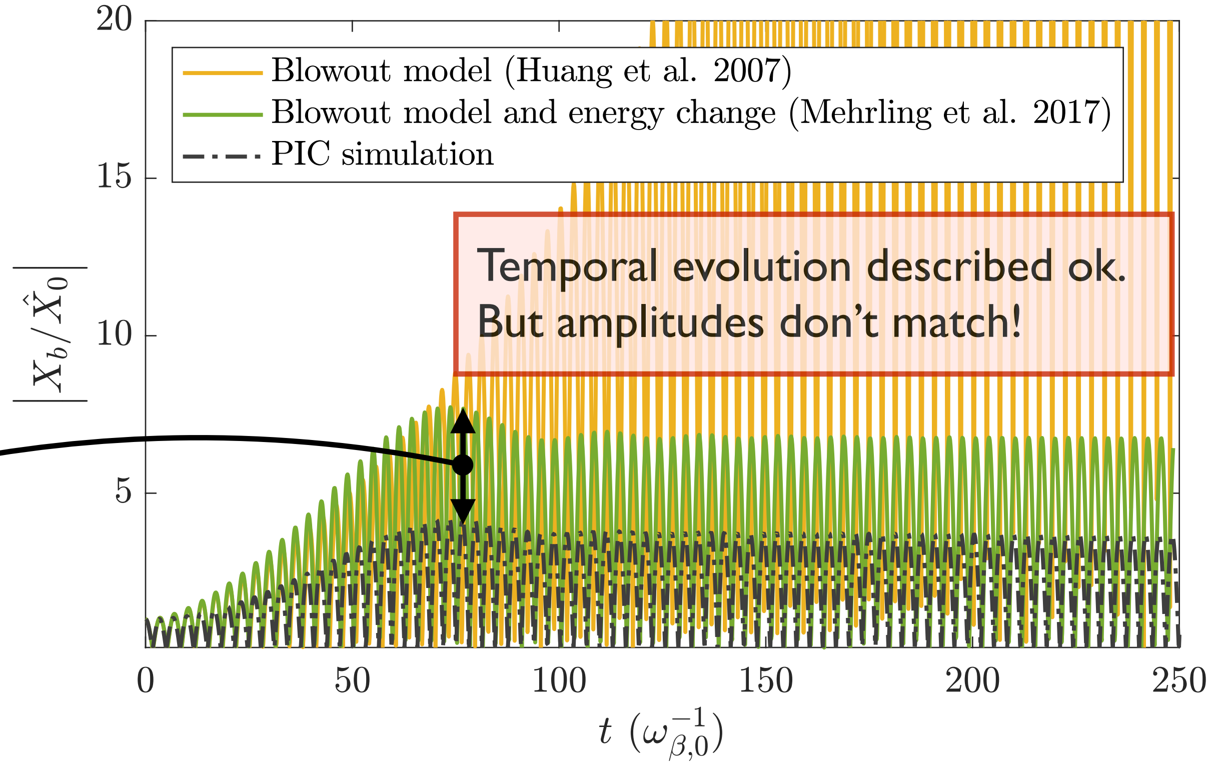
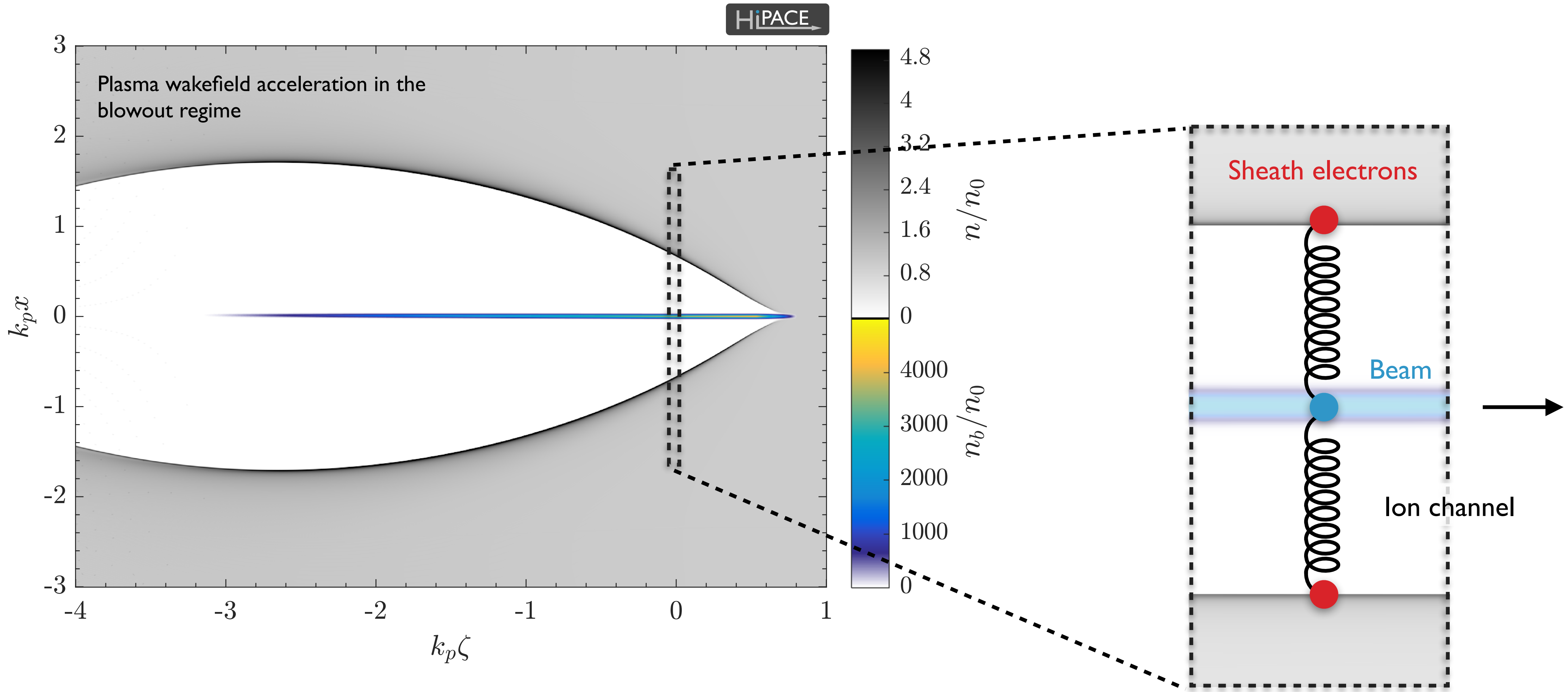


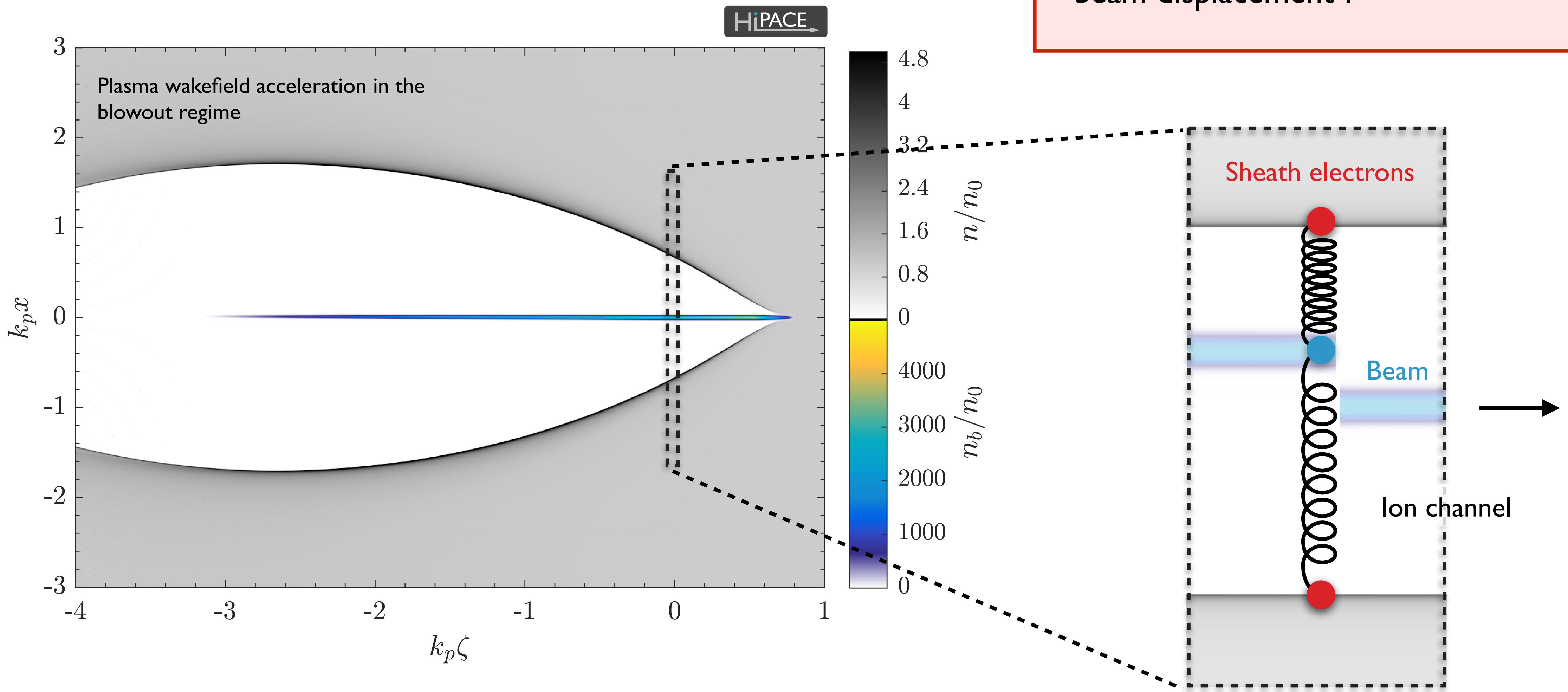

Temporal evolution described ok. But amplitudes don't match!

# Describing dynamics of plasma sheath electrons in hosing



# Describing dynamics of plasma sheath electrons in hosing

**Question:**
How does the sheath respond to the beam displacement ?



# Phase space density and Vlasov equation

## Symmetric blowout:

Phase space density of sheath electrons: $f_{p,0} = f_{p,0}(r, p_r, \psi; \xi, t)$ $\quad\quad \psi = \gamma - p_z - 1$

From $df_{p,0}/dt = 0$ obtain Vlasov equation in the QSA*: $\partial_\xi f_{p,0} = -\frac{\gamma}{1+\psi}\left(\frac{p_r}{\gamma}\partial_r + F_r\partial_{p_r} + F_\psi\partial_\psi\right) f_{p,0},$

With $F_\psi = d\psi/dt$ , $F_r = dp_r/dt$ and $\gamma = \dfrac{1+p_r^2+(1+\psi)^2}{2(1+\psi)}$

## Perturbation from symmetric blowout:

Mean transverse position $\quad\quad$ Mean transverse momentum

Phase space density of sheath electrons: $f_p = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\langle x\rangle^n \langle p_x\rangle^m}{n!\,m!}(-\cos\theta)^{n+m}\,\partial_r^n \partial_{p_r}^m f_{p,0}$

Perturbed phase space density of sheath electrons: $f_p = f_p(r, \theta, p_r, \psi; \xi, t)$

Vlasov equation: $\partial_\xi f_p = -\frac{\gamma}{1+\psi}\left(\frac{p_r}{\gamma}\partial_r + \dot{\theta}\partial_\theta + F_r\partial_{p_r} + F_\psi\partial_\psi\right) f_p$

*Quasi-Static Approximation (QSA): P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. A 41, 4463 (1990).

# Derivation of the moment equation

Vlasov equation: $\partial_\xi f_p = -\frac{\gamma}{1+\psi}\left(\frac{p_r}{\gamma}\partial_r + \dot{\theta}\partial_\theta + F_r\partial_{p_r} + F_\psi\partial_\psi\right) f_p$

Moments of perturbed phase space density

$$\langle \Phi(r,\theta,p_r,\psi)\rangle (\xi,t) = \frac{1}{N}\int_0^\infty r dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp_r \int_{-1}^{\infty} d\psi \; \Phi f_p \; , \qquad N = \int_0^\infty r dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp_r \int_{-1}^{\infty} d\psi \; f_p \; ,$$

(Vlasov equation) x ($\Phi$) and integration by parts over phase space yields <u>moment equation</u>:

$$\partial_\xi \langle\Phi\rangle = \left\langle \frac{p_r}{1+\psi}\partial_r\Phi \right\rangle + \left\langle \frac{\gamma\dot{\theta}}{1+\psi}\partial_\theta\Phi \right\rangle + \left\langle \frac{\gamma F_r}{1+\psi}\partial_{p_r}\Phi \right\rangle + \left\langle \frac{\gamma F_\psi}{1+\psi}\partial_\psi\Phi \right\rangle$$

# Example: Application of moment approach on hosing

Moment equation:

$$\partial_\xi \langle \Phi \rangle = \left\langle \frac{p_r}{1+\psi} \partial_r \Phi \right\rangle + \left\langle \frac{\gamma \dot{\theta}}{1+\psi} \partial_\theta \Phi \right\rangle + \left\langle \frac{\gamma F_r}{1+\psi} \partial_{p_r} \Phi \right\rangle + \left\langle \frac{\gamma F_\psi}{1+\psi} \partial_\psi \Phi \right\rangle$$

$$\boxed{\Phi = x}$$

$$\partial_\xi \langle x \rangle = \partial_\xi \langle r \cos\theta \rangle = \left\langle \frac{p_r \cos\theta}{1+\psi} \right\rangle$$

$$\boxed{\Phi = p_r \cos\theta}$$

$$\partial_\xi^2 \langle x \rangle = \partial_\xi \frac{\langle p_r \cos\theta \rangle}{\langle 1+\psi \rangle} = \ldots$$

Channel centroid equation

Approximation:

$$\left\langle \frac{p_r \cos\theta}{1+\psi} \right\rangle = \left\langle \frac{p_r \cos\theta}{\langle 1+\psi \rangle} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\psi - \langle \psi \rangle}{\langle 1+\psi \rangle} \right)^n \right\rangle \simeq \frac{\langle p_r \cos\theta \rangle}{\langle 1+\psi \rangle}$$

Expand force-terms as:

$$F_p = \sum_{n=0}^{\infty} \frac{\langle x \rangle^n}{n!} (-\cos\theta)^n \partial_r^n F_{p,0} \ ,$$

$$F_b = \sum_{n=0}^{\infty} \frac{X_b^n}{n!} (-\cos\theta)^n \partial_r^n F_{b,0} \ ,$$

# Example: Application of moment approach on hosing

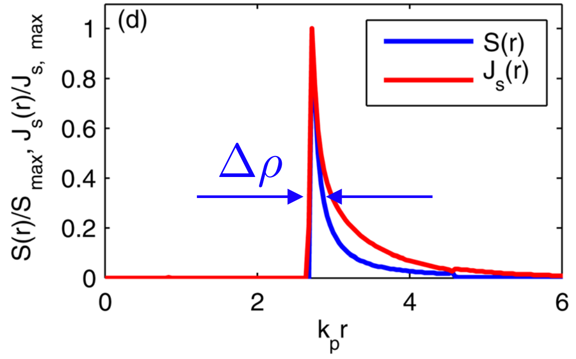
Channel centroid equation (narrow beam):

$$\partial_\xi^2 \langle x \rangle = \frac{X_b}{2\langle 1+\psi\rangle_0} \left\langle \frac{\partial_r A_{z,b}}{r} \right\rangle_0 + \frac{\langle x \rangle}{2\langle 1+\psi\rangle_0} \left\langle \frac{\gamma}{1+\psi}\frac{\partial_r \Psi}{r} + \frac{\partial_\xi A_{r,p} + \partial_r A_{z,p}}{r} \right\rangle_0$$

Using blowout model by Yi et al.* to obtain:

$$\frac{1}{\langle 1+\psi\rangle_0} \left\langle \frac{\partial_r A_{z,b}}{r} \right\rangle_0 = \frac{\Lambda}{R^2}\left[1 - \Delta\rho\left(\frac{2}{R} - \frac{R}{4}\right)\right] + \mathcal{O}\left(\Delta_\rho^2\right)$$

$$\frac{1}{\langle 1+\psi\rangle_0} \left\langle \frac{\gamma}{1+\psi}\frac{\partial_r \Psi}{r} + \frac{\partial_\xi A_{r,p} + \partial_r A_{z,p}}{r} \right\rangle_0 = \frac{R'^2 - 1}{4} + \frac{\Delta_\rho}{4}\left[\frac{1+\Lambda}{R} + \frac{R\left(R'^2 - R^2 - 1\right)}{4} - \frac{R^2 R'}{2}\right] + \mathcal{O}\left(\Delta_\rho^2\right)$$




*S.A. Yi, et al., Physics of Plasmas 20, 013108 (2013).

Norm. blowout radius: $R(\xi)$   Norm. Beam current: $\Lambda(\xi) = 4I_b(\xi)/I_A$   Sheath-thickness: $\Delta\rho$

# Example: Application of moment approach on hosing

New channel centroid equation:

$$\boxed{\frac{\partial^2 X_c}{\partial \xi^2} + \frac{k_p^2}{2}\left[c_c(\xi) X_c - c_b(\xi) X_b\right] = 0}$$

With the coefficients:

$$c_b(\xi) = \frac{\Lambda(\xi)}{R(\xi)^2}\left[1 - \Delta\rho\left(\frac{2}{R(\xi)} - \frac{R(\xi)}{4}\right)\right]$$

$$c_c(\xi) = \frac{R'(\xi)^2 - 1}{4} + \frac{\Delta_\rho}{4}\left[\frac{1 + \Lambda(\xi)}{R(\xi)} + \frac{R(\xi)\left(R'(\xi)^2 - R(\xi)^2 - 1\right)}{4} - \frac{R(\xi)^2 R'(\xi)}{2}\right]$$

# Excellent agreement between new model and PIC

**Tilted beam with triangular current profile**

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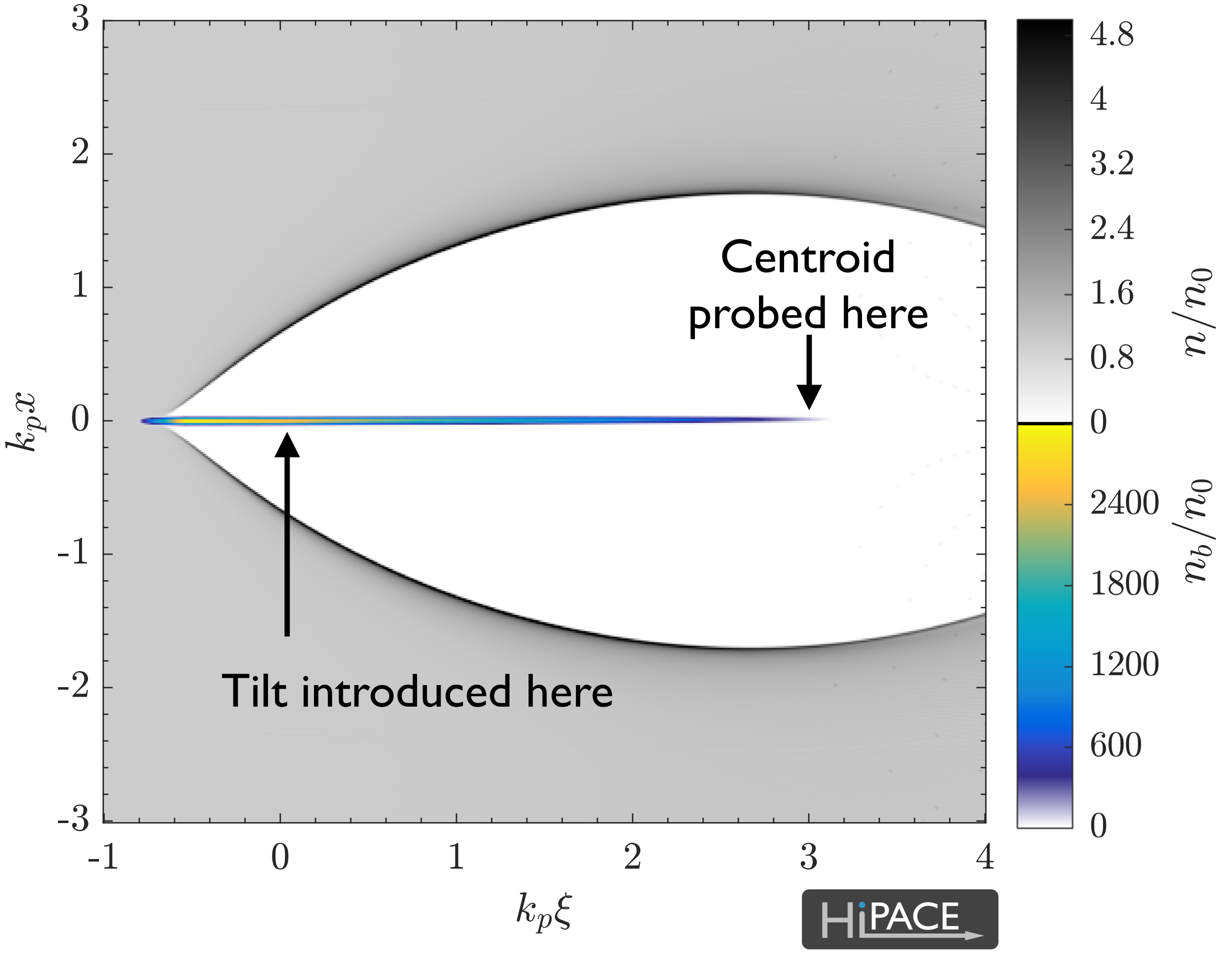
Setup as in C. Huang, *et al.* PRL **99**, 255001 (2007).



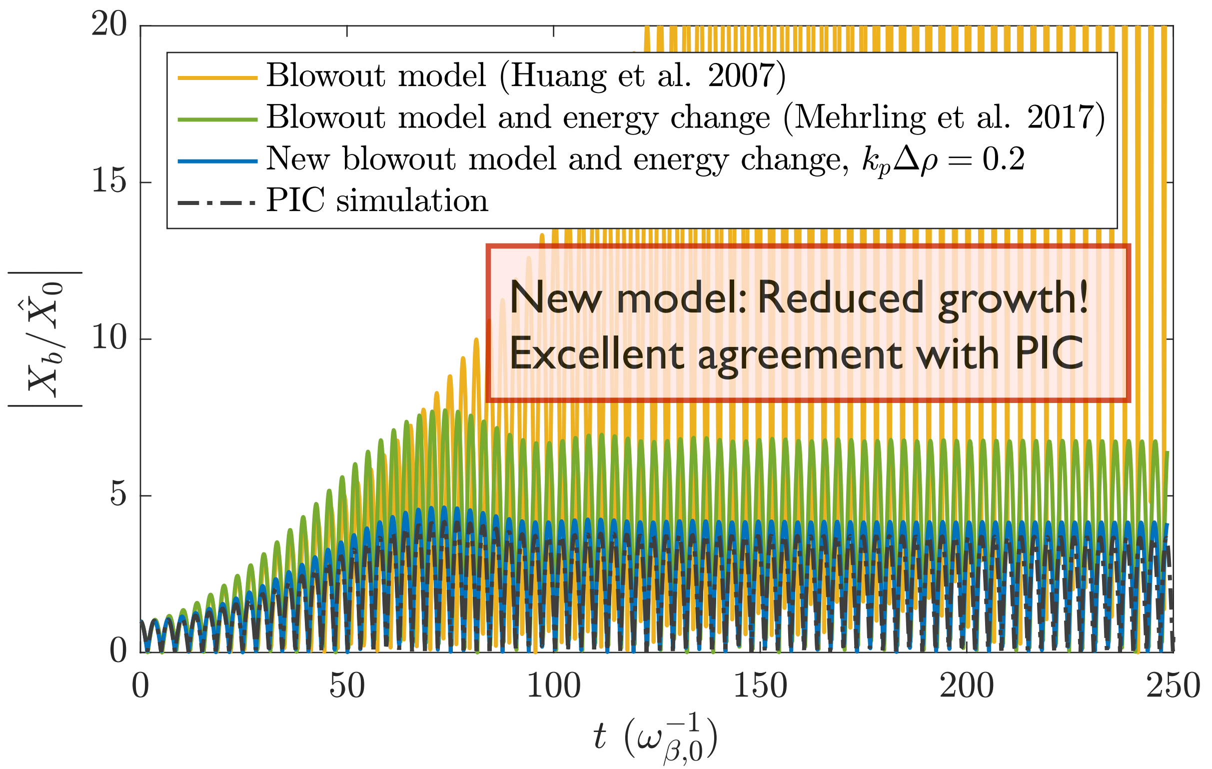

**New description of hosing in the blowout**

$$\boxed{\frac{\partial^2 X_c}{\partial \xi^2} + \frac{k_p^2}{2}\left[c_c(\xi) X_c - c_b(\xi) X_b\right] = 0}$$

*Mehrling et al., in preparation.

$$\frac{\partial^2 X_b}{\partial t^2} + \lambda(\xi, t)\frac{\partial X_b}{\partial t} + \Omega^2(\xi, t)(X_b - X_c) = 0$$

**T. Mehrling et al., PRL 118, 174801 (2017).



## Summary and concluding remarks

- Maxwell's equations → phase space moments → channel centroid equation
- Predictions by new channel centroid equation are in excellent agreement with PIC
- The central idea in Maxwell's moment equations of matter (1867) still resonates in the modern era, with applications to low temperature plasmas and to PWFA
- For further details see: *Robson, Mehrling and Osterhoff, accepted for publication in Europ. J. Phys. (2017)*

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## Thank you for your attention!

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